Internet Appendix for Inference in a Synchronization Game with Social Interactions

Áureo de Paula *

*Department of Economics, University of Pennsylvania, Philadelphia, PA 19104-6209. E-mail: aureo@sas.upenn.edu

Proof of Theorem 1

Consider a player $i \in I$. Let the stopping strategies for $I - \{i\}$ be given by the following profile of stopping times $\tau_{-i} = (\tau_s)_{s \in I - \{i\}}$. Given Assumption **??**, according to Theorem 4 in Fakeev (1970), there exists a solution for the optimal stopping time. Let the individual i's best response function $b_i(\cdot)$ map a stopping time profile τ_{-i} onto one such optimal stopping solution. Given this, consider $b(\cdot)$ defined as the following mapping $\tau = (\tau_s)_{s \in I} \mapsto b(\tau) = (b_i(\tau_{-i}))_{i \in I}$. A Nash Equilibrium is then simply a fixed point for the mapping $b(\cdot)$. To establish the existence of such a result we use the Knaster-Tarski Fixed Point Theorem, reproduced below from Aliprantis and Border (1999), p.6:

> **Knaster-Tarski Fixed Point Theorem**: Let $(X, \geq)$ be a partially ordered set with the property that every chain in X has a supremum. Let $f : X \to X$ be increasing, and assume that there exists some a in X such that $a \leq f(a)$. Then the set of fixed points of f is nonempty and has a maximal fixed point.

In the following discussion we consider the set of stopping time profiles and identify two stopping times that are $\mathbb{P}$-almost everywhere identical. We proceed by steps:

<u>Step 1</u>: (Partial order) The set of stopping times endowed with the relation $\geq$ defined as: $\tau \geq \upsilon$ if and only if $\mathbb{P}(\tau(\omega) \geq \gamma(\omega)) = 1$ is partially ordered. In other words, $\geq$ is reflexive, transitive and anti-symmetric.

<u>Step 2</u>: (Every chain has a supremum) Given a set of stopping times T, we should be able to find a stopping time $\overline{\tau}$ such that 1. $\overline{\tau} \geq \tau, \forall \tau \in T, \mathbb{P}$-a.s. and 2. if $\upsilon \geq \tau, \mathbb{P}$-a.s., $\tau \in T$ then $\upsilon \geq \overline{\tau}, \mathbb{P}$-a.s.. If T is countable $\sup_{\tau \in T} \tau$ is a stopping time and satisfies conditions 1 and 2 (see Karatzas and Shreve, Lemma 1.2.11). If not, first notice that, since the only structure that matters for this property is the ordering in $\overline{\mathbb{R}}_+$, we can always assume that the stopping times take values on $[0, 1]$ (otherwise, pick an increasing mapping from $\overline{\mathbb{R}}_+$ onto $[0, 1]$). Let $\mathcal{C}$ be the collection of all countable subsets $C \subset T$. For each such C, define:

$$l_C = \sup_{\tau \in C} \tau \text{ and } \upsilon = \sup_{C \in \mathcal{C}} \mathbb{E}(l_C) < \infty$$

By the previous reasoning, l_C is a stopping time. Then, there is a sequence $\{C_n\}_n \subset \mathcal{C}$ such that $\upsilon = \lim_{n \to \infty} \mathbb{E}(l_{C_n})$. Now define $\overline{C} = \cup_{n=1}^{\infty} C_n \in \mathcal{C}$. To show that $l_{\overline{C}}$ satisfies condition 1., first notice that $\overline{C} \in \mathcal{C}, \upsilon \geq \mathbb{E}(l_{\overline{C}})$. On the other hand, since $C_n \subset \overline{C}$, $\mathbb{E}(l_{\overline{C}}) \geq \mathbb{E}(l_{C_n}) \to_n \upsilon$. These two imply that $\upsilon = \mathbb{E}(l_{\overline{C}})$.

For an arbitrary $\tau \in T$, set $\overline{C}_\tau = \{\tau\} \cup \overline{C} \in \mathcal{C}$. Now, $l_{\overline{C}_\tau} \geq l_{\overline{C}}$. This renders $\upsilon \geq \mathbb{E}(l_{\overline{C}_\tau}) \geq \mathbb{E}(l_{\overline{C}}) = \upsilon \Rightarrow \mathbb{E}(l_{\overline{C}_\tau} - l_{\overline{C}}) = 0 \Rightarrow l_{\overline{C}_\tau} = l_{\overline{C}}, \mathbb{P}$-a.s. This and $l_{\overline{C}_\tau} \geq \tau, \mathbb{P}$-a.s. in turn imply that $l_{\overline{C}} \geq \tau, \mathbb{P}$-a.s.

To see that 2 is satisfied, notice that, if $\upsilon \geq \tau, \forall \tau \in T$, in particular, $\upsilon \geq \tau, \forall \tau \in \overline{C}$. This implies that $\upsilon \geq \sup_{\tau \in \overline{C}} \tau = l_{\overline{C}}$.

<u>Step 3</u>: ($\exists a$ such that $a \leq f(a)$) Pick a as the profile of stopping times that are identically zero.

<u>Step 4</u>: ($b(\cdot)$ is increasing) This is the case if each individual best response function $b_i(\cdot)$ is increasing. By the version of Itô's Lemma for twice differentiable functions (see Revuz and Yor (1999), p. 224, remark 3), and the fact that $u_i(x,t) = e^{-\gamma_i t} g_i(x)$ is twice differentiable (since $g_i(\cdot)$ is twice differentiable), $e^{-\gamma_i t} g_i(x)$ obeys the following stochastic differential equation (given a profile of stopping times τ_{-i}):

$$d[e^{-\gamma_i s} g_i(x_s^i)] = \underbrace{e^{-\gamma_i t}[g_i'(x_t^i)\alpha^i(x_t^i, \theta_t, t) + \frac{1}{2}\sigma^{i2}(x_t^i, \theta_t, t) g_i''(x_t^i) - \gamma_i g_i(x_t^i)]}_{\equiv \mu^i(x^i, \theta_t^i, t)} dt + \\ + \underbrace{e^{-\gamma_i t} g_i'(x_t^i)\sigma^i(x_t^i, \theta_t, t)}_{\equiv \beta^i(x_t^i, \theta_t^i, t)} dW_t^i$$

where the $\mu(\cdot,\cdot,\cdot)$ and $\beta(\cdot,\cdot,\cdot)$ denote the drift and dispersion coefficients of $e^{-\gamma_i t} g_i(x_t^i)$. If $g_i(\cdot)$ is increasing and convex and if $\alpha_i(\cdot,\cdot,\cdot)$ and $\sigma_i(\cdot,\cdot,\cdot)$ are decreasing in θ, the above drift is decreasing in θ.

Now consider a profile of stopping times τ_{-i} and υ_{-i} such that τ_{-i} dominates $\upsilon_{-i}, \mathbb{P}$- a.s. Moving from one profile to another will impact θ and this will have effects on both the drift and the dispersion coefficients of $e^{-\gamma_i t} g_i(x_t^i)$.

The effect on the dispersion coefficient does not affect the objective function of an individual agent. This obtains from the fact that $g'(\cdot)$ is bounded and the Bound on Volatility Assumption. These assumptions deliver that, for each $t < \infty$:

$$\mathbb{E}[\int_0^t (e^{-\gamma s} g'(x_s)\sigma(x_s,\theta_s,s))^2 ds] < K\mathbb{E}[\int_0^t (e^{-\gamma s}\sigma(x_s,\theta_s,s))^2 ds] < \infty$$

for some $K \in \mathbb{R}$. This in turn implies that $z_t = \int_0^t (e^{-\gamma s} g'(x_s)\sigma(x_s,\theta_s,s))dW_s^i$ is a martingale (see Karatzas and Shreve (1991), p.139) and by the Optional Sampling Theorem, $\mathbb{E}[\int_0^\tau (e^{-\gamma s} g'(x_s)\sigma(x_s,\theta_s,s))dW_s^i] = 0, \forall \tau$ where τ is an $(\mathcal{F}_t)$-stopping time (see Karatzas and Shreve (1991), p.19).

Given τ_{-i} and υ_{-i}, we know that $\theta_t^{i,\tau} \leq \theta_t^{i,\upsilon}, \mathbb{P}$-almost surely, $\forall t$ (where $\theta_t^{i,\tau}$ and $\theta_t^{i,\upsilon}$ aggregate the stopping decisions for the profiles τ_{-i} and υ_{-i}) we will have $\mu(x,\theta_t^{i,\upsilon},t) \leq \mu(x,\theta_t^{i,\tau},t), \mathbb{P}$-almost surely, $\forall x, t$. Letting $y_t^{i,\tau}$ be the process given by

$$dy_t^{i,\tau} = \mu^i(x_t^i,\theta_t^{i,\tau},t)dt + \beta(x_t^i,\theta_t^{i,\tau},t)dW_t^i$$

and $y_t^{i,\upsilon}$ be the process given by

$$dy_t^{i,\upsilon} = \mu^i(x_t^i,\theta_t^{i,\upsilon},t)dt + \beta(x_t^i,\theta_t^{i,\tau},t)dW_t^i$$

using a slight variation of Proposition 5.2.18 in Karatzas and Shreve (1991), we get:

$$\mathbb{P}[y_t^{i,\tau} \geq y_t^{i,\upsilon}, \forall 0 \leq t < \infty] = 1$$

Again, a slight variation of the proof of this proposition can be repeated using this fact and focusing on $y_t^{i,\tau} - y_s^{i,\tau} - (y_t^{i,\upsilon} - y_s^{i,\upsilon}), t \geq s$ instead of simply $y_t^{i,\tau} - y_t^{i,\upsilon}$. This is enough to achieve the following result:

$$\mathbb{P}[(y_t^{i,\tau} - y_s^{i,\tau}) - (y_t^{i,\upsilon} - y_s^{i,\upsilon}) \geq 0, \forall 0 \leq s \leq t < \infty] = 1$$

This suffices to show that it is not profitable for agent i to stop earlier when the profile is τ_{-i} than when the profile is υ_{-i}. Suppose not. Then, let $A = \{b_i(\tau_{-i}) < b_i(\upsilon_{-i})\}$. According to Lemma 1.2.16 in Karatzas and Shreve (1991), $A \in \mathcal{F}_{b_i(\tau_{-i})} \cap \mathcal{F}_{b_i(\upsilon_{-i})}$. By the above result we can then see that $\mathbb{E}\{\mathbb{I}_A[y_{b_i(\upsilon_{-i})}^{i,\tau} - y_{b_i(\tau_{-i})}^{i,\tau}]\} \geq \mathbb{E}\{\mathbb{I}_A[y_{b_i(\upsilon_{-i})}^{i,\upsilon} - y_{b_i(\tau_{-i})}^{i,\upsilon}]\}$. The RHS expression in this inequality is positive because $A \in \mathcal{F}_{b_i(\tau_{-i})} \cap \mathcal{F}_{b_i(\upsilon_{-i})} = \mathcal{F}_{b_i(\tau_{-i}) \wedge b_i(\upsilon_{-i})}$ which implies that the agent would do better by picking $b_i(\tau_{-i}) \wedge b_i(\upsilon_{-i})$ if the RHS were negative. But this would contradict the fact that $b_i(\upsilon_{-i})$ is a best response. So, if $A \neq \emptyset$, delaying the response by choosing $b_i(\upsilon_{-i}) \vee b_i(\tau_{-i})$ would improve the agent's payoff given that the remaining agents are playing τ_{-i}. $\blacksquare$

The following proposition displays a partial differential equation that can be used to obtain the density $g(\cdot)$ in general. It explores the close association between continuous time stochastic processes and partial differential equations. As in Section 3, let $z_k(\alpha^i, \sigma, \gamma, C, \Delta\alpha, t)$ be the optimal threshold levels defined in Proposition 2. Here, $G(t, \mathbf{x})$ is the probability that the players will abandon the activity *after* time t when the vector of initial conditions is given by $\mathbf{x}$. We will maintain the analysis conditional on $\mathbf{x}$, but this can also be integrated out using the distributions F_0^i introduced in Section 2. The density $g(\cdot, \mathbf{x})$ can then be obtained as $-\partial G(\cdot, \mathbf{x})/\partial t$. The following result then holds:

Proposition 1 *Let $G(t, \mathbf{x}) = \mathbb{P}[\tau_i^* > t, i \in I | \mathbf{x}_0 = \mathbf{x}]$. Then G is the unique solution to*

$$\begin{aligned}
&\partial G/\partial t = [\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma) + \mathcal{L}_1((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha, t)]G \text{ in } \mathcal{C}_{t=0}, t > 0 \\
&G(0, \mathbf{x}) = \mathbb{P}(\tau_i^* < \infty, i \in I), \mathbf{x} \in \mathcal{C}_{t=0} \\
&G(t, \mathbf{x}) = 0, \mathbf{x} \in \partial\mathcal{S}_{t=0} \text{ and } t \geq 0
\end{aligned}$$

where $\mathcal{S}_{t=0} = \{\mathbf{x} \in \mathbb{R}_+^I : \exists i \text{ such that } x^i \geq z_1(\alpha^i, \sigma, \gamma, C, \Delta\alpha, t = 0)\}, \mathcal{C}_{t=0} = \mathcal{S}_{t=0}^c$,

$$\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)f = \sum_{i\in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sigma^2 \sum_{i\in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho\sigma^2 \sum_{\substack{i,j\in I \\ i\neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

which is the infinitesimal generator for the I-dimensional diffusion representing the latent utility vector process with killing time at $\tau_{\mathcal{S}} : \{\mathbf{x}_t : t \leq \tau_{\mathcal{S}}\}$ and

$$\mathcal{L}_1((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha, t)f = -\sum_{i\in I} \frac{dz_1}{dt}(\alpha^i, \sigma, \gamma, C, \Delta\alpha, t)\frac{\partial f}{\partial x_i}.$$

Proof. Notice that (for $t \in [0, \tau_{\mathcal{S}}]$) the vector process with the latent utilities can be represented as the following diffusion process with killing at time $\tau_{\mathcal{S}}$:

$$dx_t^i = \alpha^i x_t^i dt + \sigma x_t^i dW_t^i, \qquad i \in I$$

Let $\mathcal{S} = \mathcal{S}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha, t) = \{\mathbf{x} \in \mathbb{R}_+^I : \exists i \text{ such that } x^i \geq z_1^i(t) \equiv z_1(\alpha^i, \sigma, \gamma, C, \Delta\alpha, t)\}$ and denote by $\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)$ the infinitesimal generator associated with the above diffusion (where the argument reminds the reader of the dependence of the operator on the parameters). In other words, $\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)$ is the following differential operator:

$$\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma)f = \sum_{i\in I} \alpha^i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sigma^2 \sum_{i\in I} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \rho\sigma^2 \sum_{\substack{i,j\in I \\ i\neq j}} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

for f in the appropriate domain (see Karatzas and Shreve (1991), p. 281).

Denote by $\tau_{\mathcal{S}} \equiv \inf\{t : \mathbf{x}_t \in \mathcal{S}\} = \inf\{t : \exists i \text{ such that } x^i \geq z_1^i(t) \equiv z_1(\alpha^i, \sigma, \gamma, C, \Delta\alpha, t)\} = \inf\{t : \exists i \text{ such that } \hat{x}^i \equiv x^i - (z^i(t) - z^i(0)) \geq z^i(0)\}$. Let $G(t, \mathbf{x})$ be the probability that the diffusion will reach $\mathcal{S}$ after t. In other words, $G(t, \mathbf{x}) = \mathbb{P}[\tau_{\mathcal{S}} > t | \mathbf{x}_0 = \mathbf{x}]$ and represents the survival function for the exit time distribution of the first deserter. Following Gardiner (2002), Subsection 5.4.2, this probability can be conveniently written as the solution to the following (parabolic) partial differential equation (Kolmogorov backward equation):

$$G_t = [\mathcal{A}((\alpha^i)_{i\in I}, \rho, \sigma) + \mathcal{L}((\alpha^i)_{i\in I}, \rho, \sigma, \Delta\alpha, t)]G$$

in $\mathcal{C}_{t=0}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha), \quad t > 0$ with the following conditions:

$$\begin{aligned} &G(0, \mathbf{x}) = \mathbb{P}[\tau_{\mathcal{S}} < \infty | \mathbf{x}_0 = \mathbf{x}], \quad \mathbf{x} \in \mathcal{C}_{t=0}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha) \\ &G(t, \mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}_{t=0}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha) \text{ and } t \geq 0 \end{aligned}$$

where the boundary condition follows since $\partial\mathcal{S}_{t=0}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha) \subset \mathcal{S}_{t=0}((\alpha^i)_{i\in I}, \sigma, \gamma, C, \Delta\alpha)$ and because 0 is an absorbing boundary for $x^i, i \in I$.

Uniqueness is obtained in Theorem 4, Section 7.1.2 in Evans (1998). ■

Under certain conditions, namely $\alpha^i - \sigma^2/2 > 0$ for some i, $\mathbb{P}(\tau_i^* < \infty, i \in I) = 1$. If not, it can be obtained from another partial differential equation (a proposition similar to the one just displayed can be stated for this case) or directly estimated from the data.

References

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