Trivalent Theories

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Presupposition Reading Group

Trivalent logics are built on three truth-values

- Classical truth-values: 0 and 1 (alt.: F and T, $\bot$ and $\top$, etc.)
- Third truth-value: 2 (alt.: $\#$, N, $\star$, etc.)

There are different ways of constructing trivalent logics that agree with bivalent logics for all cases not involving 2.

- Weak Kleene
- Strong Kleene
- Middle Kleene

We will start with trivalent semantics for the propositional language $\mathcal{L}$, and extend it to the language with quantifiers $\mathcal{L}^{(\mathsf{G})\mathsf{Q}}$. The syntax of these languages are given in the previous two handouts.

In all three trivalent logics, (some) simple propositions have trivalent meaning. If we represent the source of trivalency as a prefix π, as we did in $\mathcal{L}_{\mathsf{SAT}}$, ${}_{\pi}\mathtt{p_i}$ will denote (with respect to a $\mathcal{L}$-valuation v,

$$\begin{cases} 1 & \text{if } v(\pi) = 1 \text{ and } v(\mathtt{p_i}) = 1 \\ 0 & \text{if } v(\pi) = 1 \text{ and } v(\mathtt{p_i}) = 0 \\ 2 & \text{if } v(\pi) = 0 \end{cases}$$

Equivalently, we could simply redefine valuations as functions from atomic propositions to $\{0, 1, 2\}$ (which corresponds to the von Neumann ordinal 3) so that the language stays $\mathcal{L}$. This is more convenient, so let's assume so and call such valuations $\mathcal{L}^3$-valuations.

Either way, the differences among the logics come from how the logical connectives treat 2.

1 Weak Kleene

The idea behind Weak Kleene is that 2 represents *undefinedness*, or some other kind of anomaly in the system. Whenever it is present, it renders everything anomalous.

For any $\mathcal{L}^3$-valuation v,

- $[\![\neg\varphi]\!]_{\mathsf{WK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 0 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 1 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 2 \end{cases}$

- $[\![(\varphi \wedge \psi)]\!]_{\mathsf{WK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(v) = 1 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(v) = 2 \\ 0 & \text{otherwise} \end{cases}$
- $[\![(\varphi \vee \psi)]\!]_{\mathsf{WK}}(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(v) = 0 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(v) = 2 \\ 1 & \text{otherwise} \end{cases}$
- $[\![(\varphi \to \psi)]\!]_{\mathsf{WK}}(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(v) = 0 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{WK}}(v) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(v) = 2 \\ 1 & \text{otherwise} \end{cases}$

In terms of presupposition projection, all connectives are presupposition holes.
This becomes clearer once we consider the semantics-pragmatics interface. Let us represent common ground in terms of a set V of $\mathcal{L}^3$-valuations (qua possible worlds). Then we can adopt the following bridging principle, first proposed by Stalnaker 1973.

(1) **Bridging Principle**: the presupposition of $\varphi \in \mathcal{L}$ is satisfied with respect to V iff for each $v \in V$, $[\![\varphi]\!]_{\mathsf{WK}}(v) \neq 2$.

With this, we can also lift the semantics to an (eliminative) dynamic semantics:

(2) Whenever the presupposition of $\varphi \in \mathcal{L}$ is satisfied with respect to V, $[\![\![\varphi]\!]\!]_{\mathsf{WK}}(V) = \{\, v \in V \mid [\![\varphi]\!]_{\mathsf{WK}}(v) = 1 \,\}$.

Weak Kleene doesn't allow filtering, e.g.,

(3) The presupposition of $(\varphi \wedge \psi)$ is satisfied with respect to V
iff for each $v \in V$, $[\![(\varphi \wedge \psi)]\!]_{\mathsf{WK}}(v) \neq 2$
iff for each $v \in V$, $[\![\varphi]\!]_{\mathsf{WK}} \neq 2$ and $[\![\psi]\!]_{\mathsf{WK}} \neq 2$.

This renders it unfit for a theory of presupposition projection.

2 Strong Kleene

Strong Kleene regards 2 as *uncertainty*. What is represented as 2 could be 0 or 1, but we don't know which. But sometimes we can ignore such uncertainties and determine truth-values of the overall formulas.
For any $\mathcal{L}^3$-valuation v,

- $[\![\neg\varphi]\!]_{\mathsf{SK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 0 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \end{cases}$

- $[\![(\varphi \wedge \psi)]\!]_{\mathsf{SK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 1 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(v) = 0 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 1, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2) \end{cases}$

- $[\![(\varphi \vee \psi)]\!]_{\mathsf{SK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(v) = 1 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 0 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_{\mathsf{SK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 0, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2) \end{cases}$

- $[\![(\varphi \rightarrow \psi)]\!]_{\mathsf{SK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(v) = 1 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 0 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_{\mathsf{SK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 0, \\ & \quad \text{or } [\![\varphi]\!]_{\mathsf{SK}}(v) = 2 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(v) = 2) \end{cases}$

When applied to presupposition projection, Strong Kleene predicts some filtering. For example,

(4) The presupposition of $(\varphi \wedge \psi)$ is satisfied with respect to V
iff for each $v \in V$, $[\![(\varphi \wedge \psi)]\!]_{\mathsf{SK}}(v) \neq 2$
iff for each $v \in V$,
  a. if $[\![\varphi]\!]_{\mathsf{SK}}(v) = 1$, then $[\![\psi]\!]_{\mathsf{SK}}(v) \neq 2$; and
  b. if $[\![\psi]\!]_{\mathsf{SK}}(v) = 1$, then $[\![\varphi]\!]_{\mathsf{SK}}(v) \neq 2$; and
  c. it's not the case that $[\![\varphi]\!]_{\mathsf{SK}}(v) = [\![\psi]\!]_{\mathsf{SK}}(v) = 2$.

(5) The presupposition of $(\varphi \vee \psi)$ is satisfied with respect to V
iff for each $v \in V$, $[\![(\varphi \vee \psi)]\!]_{\mathsf{SK}}(v) \neq 2$
iff for each $v \in V$,
  a. if $[\![\varphi]\!]_{\mathsf{SK}}(v) = 0$, then $[\![\psi]\!]_{\mathsf{SK}}(v) \neq 2$; and
  b. if $[\![\psi]\!]_{\mathsf{SK}}(v) = 0$, then $[\![\varphi]\!]_{\mathsf{SK}}(v) \neq 2$; and
  c. it's not the case that $[\![\varphi]\!]_{\mathsf{SK}}(v) = [\![\psi]\!]_{\mathsf{SK}}(v) = 2$.

This predicts conditional filtering of the following type.

(6) a. Nathan used to smoke Marlboros and he stopped smoking.
  b. Either Nathan was never a smoker, or he stopped smoking.

However, Strong Kleene makes symmetric predictions, and therefore predicts order-insensitivity. This is arguably wrong for conjunction, although it might be a good prediction for disjunction.

(7) a. Nathan stopped smoking, and he used to smoke Marlboros, when he was younger.
  b. Either Nathan stopped smoking, or he was never a smoker.

But:

(8) Donald Trump: We cannot let China continue to screw our economy, and that's what they are doing!

3 Middle Kleene

Peters 1979 proposes Middle Kleene (as Krahmer 1998 calls it). It shares the core intuition with Strong Kleene to regard 2 as uncertainty, but it evaluates a formula from left-to-right, which brings in order sensitivity.
For any $\mathcal{L}^3$-valuation v,

- $[\![\neg\varphi]\!]_{\mathsf{MK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \\ 2 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 2 \end{cases}$

- $[\![(\varphi \wedge \psi)]\!]_{\mathsf{MK}}(v) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 1 \\ 2 & \text{if either } [\![\varphi]\!]_{\mathsf{MK}}(v) = 2, \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 2 \\ 0 & \text{otherwise} \\ & (\text{i.e., } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 0) \end{cases}$

- $[\![(\varphi \vee \psi)]\!]_{\mathsf{MK}}(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 0 \\ 2 & \text{if either } [\![\varphi]\!]_{\mathsf{MK}}(v) = 2, \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 2 \\ 1 & \text{otherwise} \\ & (\text{i.e., } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 1) \end{cases}$

- $[\![(\varphi \rightarrow \psi)]\!]_{\mathsf{MK}}(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 0 \\ 2 & \text{if either } [\![\varphi]\!]_{\mathsf{MK}}(v) = 2, \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 2 \\ 1 & \text{otherwise} \\ & (\text{i.e., } [\![\varphi]\!]_{\mathsf{MK}}(v) = 0 \text{ or } [\![\varphi]\!]_{\mathsf{MK}}(v) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(v) = 1) \end{cases}$

But this also renders disjunction asymmetric.

4 Remarks

4.1 Strong Kleene as lingua franca

It's a nice feature of trivalent logics that they provide a general recipe for constructing trivalent denotations.
Beaver & Krahmer 2001 point out that Week Kleene and Middle Kleene can be defined in terms of Strong Kleene.

(9) Middle Kleene in terms of Strong Kleene

a. $[\![(\varphi \wedge \psi)]\!]_{\mathsf{MK}} = [\![((\varphi \wedge \psi) \vee (\varphi \wedge \neg\varphi))]\!]_{\mathsf{SK}}$
b. $[\![(\varphi \vee \psi)]\!]_{\mathsf{MK}} = [\![((\varphi \vee \psi) \wedge (\varphi \vee \neg\varphi))]\!]_{\mathsf{SK}}$
c. $[\![(\varphi \rightarrow \psi)]\!]_{\mathsf{MK}} = [\![((\varphi \vee \psi) \wedge (\varphi \vee \neg\varphi))]\!]_{\mathsf{SK}}$

That is, Middle Kleene is just like Strong Kleene, except that it always projects the presupposition of the left argument.
Weak Kleene is symmetric and projects the presuppositions of both arguments.

(10) Weak Kleene in terms of Strong Kleene

a. $[\![(\varphi \wedge \psi)]\!]_{\mathsf{WK}} = [\![((\varphi \wedge \psi) \vee ((\varphi \wedge \neg\varphi) \vee (\psi \wedge \neg\psi)))]\!]_{\mathsf{SK}}$
b. $[\![(\varphi \vee \psi)]\!]_{\mathsf{WK}} = [\![((\varphi \vee \psi) \wedge ((\varphi \vee \neg\varphi) \wedge (\psi \vee \neg\psi))]\!]_{\mathsf{SK}}$
c. $[\![(\varphi \to \psi)]\!]_{\mathsf{WK}} = [\![((\varphi \vee \psi) \wedge ((\varphi \vee \neg\varphi) \wedge (\psi \vee \neg\psi)))]\!]_{\mathsf{SK}}$

4.2 Presupposition accommodation

Beaver & Krahmer 2001 make use of the A-operator to enable presupposition accommodation (in the semantics).
Define $\mathcal{L}^{\mathsf{A}}$ as the smallest set such that:

- $\mathcal{L} \subseteq \mathcal{L}^{\mathsf{A}}$
- If $\varphi \in \mathcal{L}^{\mathsf{A}}$, then $\mathsf{A}(\varphi) \in \mathcal{L}^{\mathsf{A}}$.

The A-operator turns the semantics bivalent. No matter what φ is, $\mathsf{A}(\varphi)$ denotes 1 iff φ denotes 1 and 0 otherwise. E.g., for Strong Kleene,

(11) $$[\![\mathsf{A}(\varphi)]\!]_{\mathsf{SK}} = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}} = 1 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}} = 0 \text{ or } [\![\varphi]\!]_{\mathsf{SK}} = 2 \end{cases}$$

4.3 Partial Propositional Logic as a trivalent logic

Recall that Partial Propositional Logic from two weeks ago is very similar to (Propositional) Satisfaction Theory $\mathcal{L}_{\mathsf{SAT}}$. Its syntax is the same as $\mathcal{L}_{\mathsf{SAT}}$, but formulas denote partial functions from $\mathcal{L}$-valuations to classical truth-values, rather than functions over sets of $\mathcal{L}$-valuations.

- $[\![[{}_{\pi}\mathtt{p_i}]\!]] = \lambda v\colon \|\pi\|(v) = 1.\ \|\mathtt{p_i}\|(v)$
- $[\![[\neg\varphi]\!]] = \lambda v\colon v \in \mathrm{dom}([\![[\varphi]\!]]).\ \|\neg\varphi\|(v)$
- $[\![[(\varphi \wedge \psi)]\!]] = \lambda v\colon v \in \mathrm{dom}([\![[\varphi]\!]])$ and if $[\![[\varphi]\!]](v) = 1$, then $v \in \mathrm{dom}([\![[\psi]\!]]).\ \|(\varphi \wedge \psi)\|(v)$
- $[\![[(\varphi \vee \psi)]\!]] = \lambda v\colon v \in \mathrm{dom}([\![[\varphi]\!]])$ and if $[\![[\varphi]\!]](v) = 0$, then $v \in \mathrm{dom}([\![[\psi]\!]]).\ \|(\varphi \vee \psi)\|(v)$
- $[\![[(\varphi \to \psi)]\!]] = \lambda v\colon v \in \mathrm{dom}([\![[\varphi]\!]])$ and if $[\![[\varphi]\!]](v) = 1$, then $v \in \mathrm{dom}([\![[\psi]\!]]).\ \|(\varphi \to \psi)\|(v)$

This logic can be re-formalized as a trivalent logic, by re-interpreting undefinedness as the third truth-value (2):

- $[\![{}_{\pi}\mathtt{p_i}]\!]_{\partial}(v) \begin{cases} 0 & \text{if } \|\pi\|(v) = 1 \text{ and } \|\mathtt{p_i}\|(\mathrm{v}){=}1 \\ 1 & \text{if } \|\pi\|(v) = 1 \text{ and } \|\mathtt{p_i}\|(\mathrm{v}){=}0 \\ 2 & \text{if } \|\pi\|(v) = 0 \end{cases}$
- $[\![\neg\varphi]\!]_{\partial}(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_{\partial} = 1 \\ 1 & \text{if } [\![\varphi]\!]_{\partial} = 0 \\ 2 & \text{if } [\![\varphi]\!]_{\partial} = 2 \end{cases}$

- $[\![(\varphi \wedge \psi)]\!]_\partial(v) = \begin{cases} 0 & \text{if either } [\![\varphi]\!]_\partial(v) = 0\text{, or } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 0 \\ 1 & \text{if } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 1 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_\partial(v) = 2\text{, or } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 2) \end{cases}$

- $[\![(\varphi \vee \psi)]\!]_\partial(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 0 \\ 1 & \text{if either } [\![\varphi]\!]_\partial(v) = 1\text{, or } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 1 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_\partial(v) = 2\text{, or } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 2) \end{cases}$

- $[\![(\varphi \to \psi)]\!]_\partial(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 0 \\ 1 & \text{if } [\![\varphi]\!]_\partial(v) = 0\text{, or } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 1 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_\partial(v) = 2\text{, or } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 2) \end{cases}$

Unlike the three trivalent logics above, this one lacks a general recipe for predicting the projection profile of a given logical operator.
In particular, one could add symmetric disjunctions:

- $[\![(\varphi \dot\vee \psi)]\!]_\partial(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 0 \\ 1 & \text{if } [\![\varphi]\!]_\partial(v) \neq 2 \text{ and } [\![\psi]\!]_\partial(v) \neq 2\text{, and } [\![\varphi]\!]_\partial(v) = 1 \text{ or } [\![\psi]\!]_\partial(v) = 1 \\ 2 & \text{otherwise (i.e., either } [\![\varphi]\!]_\partial(v) = 2 \text{ or } [\![\psi]\!]_\partial(v) = 2) \end{cases}$

- $[\![(\varphi \ddot\vee \psi)]\!]_\partial(v) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 0 \\ 1 & \text{if either } [\![\varphi]\!]_\partial(v) = 1 \text{ and } [\![\psi]\!]_\partial(v) = 0\text{, or } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 1 \\ 2 & \text{otherwise} \\ & \text{(i.e., either } [\![\varphi]\!]_\partial(v) = 2 \text{ and } [\![\psi]\!]_\partial(v) = 0\text{, or } [\![\varphi]\!]_\partial(v) = 0 \text{ and } [\![\psi]\!]_\partial(v) = 2) \end{cases}$

This is arguably descriptively better than any of the three trivalent logics above, but it lacks explanatory power, a feature that is shared by Satisfaction Theory as well.

5 Quantification

We can apply the recipes of Weak Kleene, Strong Kleene, and Middle Kleene to quantifiers as well.

5.1 Weak Kleene

(12) a. $[\![\exists\xi\varphi]\!]_{\mathsf{WK}}(a) = \begin{cases} 2 & \text{if for any } x \in \mathcal{D}\text{, } [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \\ 0 & \text{if for each } x \in \mathcal{D}\text{, } [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{otherwise} \\ & \text{(i.e., if for each } x \in \mathcal{D}\text{, } [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad\quad \text{and for some } y \in \mathcal{D}\text{, } [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1) \end{cases}$

b. $[\![\forall\xi\varphi]\!]_{\mathsf{WK}}(a) = \begin{cases} 2 & \text{if for any } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 1 \\ 0 & \text{otherwise} \\ & \text{(i.e., if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \qquad \text{and for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0) \end{cases}$

c. $[\![\mathsf{SOME}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{WK}}(a)$

$$= \begin{cases} 2 & \text{if for any } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1 \\ 0 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for each } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0 \end{cases}$$

d. $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{WK}}(a)$

$$= \begin{cases} 2 & \text{if for any } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for each } y \in \mathcal{D} \text{ such that } [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1,\ [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1 \\ 0 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0 \end{cases}$$

e. $[\![\mathsf{NO}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{WK}}(a)$

$$= \begin{cases} 2 & \text{if for any } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) = 2 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 1 \\ 0 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \text{ and } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto x]) \neq 2 \\ & \quad \text{and for each } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{WK}}(a[\xi \mapsto y]) = 0 \end{cases}$$

Week Kleene gives rise to very strong presuppositions.

- The predicted presupposition for (13)

  (13) Every student who stopped smoking is smart.

  is everything in the model used to smoke.
- The same strong presupposition is predicted for (14).

  (14) Every smart student stopped smoking.

5.2 Strong Kleene

(15) a. $[\![\exists\xi\varphi]\!]_{\mathsf{SK}}(a) = \begin{cases} 0 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$

b. $[\![\forall\xi\varphi]\!]_{\mathsf{SK}}(a) = \begin{cases} 0 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$

c. $[\![\mathsf{SOME}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 0 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

d. $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

e. $[\![\mathsf{NO}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

Strong Kleene predicts much weaker presuppositions. Assuming the same bridging principle as before, while the context set is now a set A of extended assignments, the predicted presupposition for each case is always disjunctive in that it requires that for each $a \in A$, the denotation be either 0 or 1.

A potential advantage is that existential sentences will similarly have disjunctive presuppositions that are weaker than universal presuppositions.

- The predicted presupposition for (16)

  (16) Some students who stopped smoking are smart.

  is that either there are some smart students who used to smoke and stopped, or else every thing in the model is either used to smoke but is not a student that stopped smoking, or is not smart.

- The predicted presupposition for (17)

  (17) Some smart students stopped smoking.

  is that either there are some smart students who used to smoke and stopped, or else every thing in the model is either not a smart student or used to smoke and is still smoking.

However, the predictions for universal quantification have some problematic aspects.

- The predicted presupposition for (18)

  (18) Every student who stopped smoking is smart.

  is that either there is a student who used to smoke and stopped but is not smart, or else everything that did not used to smoke or is a student who used to smoke and stopped is smart.

- The predicted presupposition for (19)

  (19) Every smart student stopped smoking.

is that either there is a smart student who used to smoke but did not stop, or else every smart student used to smoke and stopped.

The second case is weaker than the universal presupposition that every smart student used to smoke, which is what is often assumed to be its presupposition. It appears that Beaver & Krahmer 2001 don't consider this to be a problem, because when the sentence is true, that will be an entailment (assuming the existence presupposition). But it can be an issue if we test universal quantifiers embedded in projective contexts.

George 2008 points out that the first case is problematic. Especially, the truth of the sentence is predicted to entail that everything that fails the presupposition of the restrictor is predicted to make the nuclear scope true, i.e., everything that did not used to smoke is smart. That indeed sounds problematic.

Strong Kleene also makes wrong predictions for (certain) non-monotonic quantifiers like *exactly three* (as discussed by George 2008). Since this is a symmetric quantifier, it is sufficient to consider one case.

(20) Exactly three students stopped smoking.

Although this is formally not part of the language, it is easy to add it (but I won't do it). Crucially, the predicted meaning is:

- (20) denotes 1 if every student used to smoke and exactly three of them no longer do.
- (20) denotes 0 if
  - there are four or more students who used to smoke and stopped; or
  - there are two or fewer students who either never smoked or used to smoke but stopped; or
  - there are two or fewer students who used to smoke and stopped and there are four or more students who either never smoked or used to smoke and stopped.
- (20) denotes 2 otherwise.

Given the experimental data in Chemla 2009, the universal entailment is problematic. Also, the falsity condition is counter-intuitive. For example, the sentence is predicted to be 2, rather than false, if there are two students who used to smoke and stopped, one never smoked, and all the other students have always been smokers.

5.3 Middle Kleene

(21) a. $$[\![\exists\xi\varphi]\!]_{\mathsf{MK}}(a) = [\![\exists\xi\varphi]\!]_{\mathsf{SK}}(a) = \begin{cases} 0 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

b. $$[\![\forall\xi\varphi]\!]_{\mathsf{MK}}(a) = [\![\exists\xi\varphi]\!]_{\mathsf{SK}}(a) = \begin{cases} 0 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

c. $[\![\mathsf{SOME}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{MK}}(a)$

$$= \begin{cases} 0 & \text{if either for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0, \\ & \quad \text{or for some } y \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto y]) = 1 \text{ and} \\ & \qquad \text{for each } z \in \mathcal{D}, [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto z]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

d. $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{MK}}(a)$

$$= \begin{cases} 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if either for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0, \\ & \quad \text{or for some } y \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto y]) = 1 \\ & \qquad \text{and for each } z \in \mathcal{D} \text{ such that } [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto z]) = 1, [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto z]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

e. $[\![\mathsf{NO}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{MK}}(a)$

$$= \begin{cases} 1 & \text{if either for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 0, \\ & \quad \text{or for some } y \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto y]) = 1 \text{ and} \\ & \qquad \text{for each } z \in \mathcal{D}, [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto z]) = 0 \\ 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{MK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

6 Presupposition strengthening and accommodation

Fox 2012 starts with Strong Kleene and augments it with two mechanisms, the A-operator and presupposition strengthening.

(22) a. $[\![\mathsf{SOME}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 0 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

b. $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

c. $[\![\mathsf{NO}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

6.1 Presupposition strengthening

In a trivalent logic, the presupposition amounts to the disjunction of the truth-condition and falsity-condition.

Let us for now focus on sentences that have non-trivial presuppositions in the nuclear scope but not in the restrictor, e.g., (23).

(23) a. Some students stopped smoking.
b. Every student stopped smoking.
c. No student stopped smoking.

Assuming that the restrictor is non-empty, these presuppose:

(24) a. Either some students used to smoke and stopped, or every student used to smoke but none stopped.
b. Either every student used to smoke and stopped, or some students used to smoke but did not stop.
c. Either every student used to smoke but none stopped, or some students used to smoke and stopped (=(24a)).

Now suppose that for whatever reason, one has to (globally) accommodate such a disjunctive presupposition. Fox 2012 claims that one would not simply accommodate the disjunctive presupposition as is (similarly to what Beaver 2001, van Rooij 2007, von Fintel 2008 suggests for conditional presuppositions), but go through additional pragmatic reasoning about what contexts are more plausible (but see **mandelkern:2016**, **mandelkern:2016-salt**, Singh 2007 for empirical problems of this idea; more on this next time).

In particular, notice that the proposition that every student used to smoke entails all of (24).

6.1.1 *Some and no*

There are three types of contexts where the disjunctive presupposition (24a) of (23a) is satisfied:

1. The context entails that some students used to smoke and stopped.
2. The context entails that every student used to smoke.
3. The context entails neither that some students used to smoke and stopped nor that every student used to smoke, but still entails the disjunction of these two.

Notice that in the first context, the truth of (23a) has already been established. It is reasonable to assume that in such a context (23a) cannot be uttered felicitously anyway (cf. Stalnaker 1978).

Then we are left with the second or third context. Fox 2012 argues that the third type of context is a strange one (if not completely unrealistic; see below): it is not known whether the truth-condition holds or not and it is not known whether every student used to smoke, but it is known that if one of these is not the case, the other one must be the case. That is to say, it must be known:

(25) If no student used to smoke and stopped, then every student used to smoke, and if some students didn't used to smoke, then some student used to smoke and stopped.

Fox 2012 goes on to claim that since such a context is unnatural, one tends to exclude it. Then we are left with the second type of context, which is to say that one tends to accommodate a universal presupposition that every student used to smoke.

The reasoning would be identical for the case of *no*, except that in the first type of context, the sentence (24c) will be known to be false, but in that case too, it's natural to assume that (24c) would anyway be infelicitous.

6.1.2 Every

In the case of *every*, the contexts that satisfy the disjunctive presupposition (25b) is divided into the following three:

1. The context entails that some students used to smoke but did not stopped.
2. The context entails that every student used to smoke.
3. The context entails neither that some students used to smoke and did not stopped nor that every student used to smoke, but still entails the disjunction of these two.

Again, the sentence should be infelicitous in the first type of context, because it's already known to be false. So we exclude such contexts.

The third type of context is a weird one, and is excluded.

Then we are left with the second type of context.

6.2 When the disjunctive presupposition is satisfied

Thus, according to Fox 2012, (global) accommodation involves additional pragmatic reasoning that could lead to accommodating more than the disjunctive semantic presupposition. For all three quantified statements, the result is a universal (pragmatic) presupposition.

Before discussing the second mechanism, the A-operator, which gives rise to different readings for different quantifiers, let us turn to examples that should not involve accommodation or where the third type of context is not entirely implausible.

The prediction is that in such cases, universal presuppositions should not be observed.

Assuming that polar questions $?p$ presuppose $(p \vee \neg p)$ (this is a manipulation to make sure that A above the quantifier won't matter):

(26) Did any one of these bankers acquire the fortune he deposited in the bank last week by wiping out one of the others? (Fox 2012: p. 215, (20))

The presupposition of (26) should be:

(27) Either at least one of these bankers acquired a fortune by wiping out one of the others and deposited it in the bank last week, or each of them deposited a fortune in the bank last week but none of them acquired it by wiping out one of the others.

This can be transformed to the following condition (via the equivalence of $(\varphi \vee \psi)$ and $(\neg\varphi \rightarrow \psi)$).

(28) If none of these bankers acquired the fortune he deposited in the bank last week by wiping out one of the others, they each deposited a fortune last week.

Fox 2012 claims that this can be accommodated as is.

Here is a case where the disjunctive presupposition is already satisfied in a context that does not entail the universal statement.

(29) Context: John and Bill meet for a game of poker. The rules they set for their engagement are the following. They each give Jane 100 dollars and get chips in return. The game will continue until one of them has no more chips left. The moment this

happens, the winner (the player that has 200 chips) can go to Jane and cash his chips. Fred (who knows the rules of engagement) is responsible for cleaning the room the moment the game is over. He calls Jane and asks the following question: (Fox 2012: p. 214, (18))

(30) Is either one of the two players allowed to cash the chips that he now has in his possession? (Fox 2012: p. 215, (21))

This should presuppose:

(31) Either one of John and Bill has won, hence has all the chips, and is now allowed to cash them, or still both of them have chips (and the game is not over).

This is simply satisfied, and one can read (30) without a universal presupposition that both players have chips.
Another example:

(32) CONTEXT: Two partners (Bill and Fred) started a new company based on a new algorithm that they developed. If neither partner reveals the algorithm, they will both earn millions once the company goes public. If, however, one of them shares the algorithm with Tom a well-known English businessman, before the company goes public, this partner will be getting millions from Tom but then the other partner will remain very poor.

Will one of the two partners get his millions from Tom? (Fox 2012: p. 215, (21))

6.3 The A-operator

The A-operator effectively turns the presupposition into part of the truth-condition (so when it's not the case, the sentence is simply false).

(33) $$[\![\mathsf{A}(\varphi)]\!]_{\mathsf{SK}} = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathsf{SK}} = 1 \\ 0 & \text{if } [\![\varphi]\!]_{\mathsf{SK}} = 0 \text{ or } [\![\varphi]\!]_{\mathsf{SK}} = 2 \end{cases}$$

When A applies above the quantifier, we simply lose the disjunctive presupposition, because that conjoined with the truth-condition is just the truth-condition.

(34) a. $[\![\mathsf{A}(\mathsf{SOME}^{\xi}(\varphi)(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 1 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 0 & \text{otherwise} \end{cases}$$

b. $[\![\mathsf{A}(\mathsf{EVERY}^{\xi}(\varphi)(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 0 & \text{otherwise} \end{cases}$$

c. $[\![\mathsf{A}(\mathsf{NO}^{\xi}(\varphi)(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

If ψ has a non-trivial presupposition p:

- (34a) entails that there is some φ that is p

- (34b) entails that every φ is p
- (34c) entails that every φ is p

This generates a pattern that seems to match what Chemla 2009 found.
Recall that Chemla's experiments used an inference task, so it's not clear if the judgments were based on presuppositions or at-issue meanings.
Continuing to zoom in on projection from the nuclear scope, A can apply below the quantifier as well.

(35) a. $[\![\mathsf{SOME}^{\xi}(\varphi)(\mathsf{A}(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 0 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) \neq 1 \\ 1 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

b. $[\![\mathsf{EVERY}^{\xi}(\varphi)(\mathsf{A}(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) \neq 1 \\ 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

c. $[\![\mathsf{NO}^{\xi}(\varphi)(\mathsf{A}(\psi))]\!]_{\mathsf{SK}}(a)$
$$= \begin{cases} 1 & \text{if for each } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) \neq 1 \\ 0 & \text{if for some } x \in \mathcal{D}, [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

If φ has no presupposition, none of these have presuppositions (ignoring the existence presupposition of *every*), and for the presupposition p of ψ,

- (35a) entails some φ that is p
- (35b) entails every φ is p
- (35c) has no entailment about p

This accounts means that *no*, unlike *every*, has a reading that does not entail universal projection pattern. That is in harmony with the results of Chemla 2009, where universal inferences were more robust with *every* than with *no*.
However, this theory does not derive existential inferences for *no*, which Zehr et al. 2016 claim are possible inferences.

6.4 Inter-speaker variation

Fox 2012 assumes that the judgments are not very stable because speakers can different preferences over the parses.

- It seems reasonable to assume that local applications of A are generally less preferred to non-local applications, but one might prefer to not apply A at all. Such speakers should perceive universal presuppositions, whenever they go through pragmatic strengthening via accommodation.
- If one prefers to apply A above the quantifier, they should get universal entailments for *every* and *no*, but not for *some*

- Finally, speakers who do not have issues applying A below the quantifier should have a universal entailment for *every*, an existential entailment for *some*, and nothing for *no*.

6.5 Restrictors

Fox 2012 does not discuss presuppositions triggered in the restrictor. Consider the following examples.

(36) a. Some students who stopped smoking like semantics.
b. Every student who stopped smoking likes semantics.
c. No student who stopped smoking likes semantics.

Since *some* and *no* are symmetric quantifiers, we get the same predictions as in the case of projection out of the nuclear scope.

The predicted disjunctive semantic presuppositions of these sentences are:

(37) a. Either some students like semantics and used to smoke but stopped, or every student that likes semantics used to smoke and continued to smoke.
b. Either every student who used to smoke and stopped likes semantics, or some students who don't like semantics used to smoke but stopped.
c. Either every student that likes semantics used to smoke and continued to smoke, or some students like semantics and used to smoke and stopped.

Pragmatic strengthening should give rise to the inference that every student used to smoke for all three cases. That sounds wrong.

In addition, the symmetric predictions for *some* and *no* do not seem to be correct. To make the intuitions more concrete, consider:

(38) a. Some red triangles have the same color as the circle that they are connected to.
b. No red triangles have the same color as the circle that they are connected to.

(39) a. Some figures that have the same color as the circle that they are connected to are red triangles.
b. No figures that have the same color as the circle that they are connected to are red triangles.

According to Fox's theory, all of these should have the same disjunctive semantic presupposition:

(40) Either some red triangles are connected to (exactly) one circle and the circle is red, or all red triangles are connected to (exactly) one circle and the circle is not red.

It's intuitive that (39b) has a robust universal inference (either through pragmatic strengthening or the application of A above the quantifier) but it is counter-intuitive that (40b) should have the same universal inference.

In fact, (40b) is unlikely to even have an existential inference that some red triangles are connected to (exactly) one circle (which is what Zehr et al. 2016 discuss for sentences like (39b)).

(Note that these problems are independent of the possibility that *some* and *no* have existence presuppositions about their restrictors)

6.6 Non-monotonic quantifiers

Recall that Strong Kleene predicts strange meaning for non-monotonic quantifiers.

(41) Exactly three students stopped smoking.

- (41) denotes 1 if every student used to smoke and exactly three of them no longer do.
- (41) denotes 0 if
  - there are four or more students who used to smoke and stopped; or
  - there are two or fewer students who either never smoked or used to smoke but stopped; or
  - there are two or fewer students who used to smoke and stopped and there are four or more students who either never smoked or used to smoke and stopped.
- (41) denotes 2 otherwise.

As pointed out above, the falsity condition is strange, e.g., (41) is predicted to be 2, rather than 0, if there are two students who used to smoke and stopped, one never smoked, and all the other students have always been smokers.
Under Fox's account:

- It's perhaps not too far-fetched to assume that pragmatic strengthening results in a universal presupposition in this case too, although it's not entirely clear if that should be the case. It should also be considered if there is evidence that the big disjunction of the four cases (one true case, and three false cases) can be satisfied.
- If A is applied above the quantifier, the sentence will entail the universal inference that every student used to smoke.
- If A is applied below the quantifier, it will entail that (at least) three students used to smoke.

This pattern is similar to *no*. However, in the results reported in Chemla 2009, sentences with *exactly three* gave rise to universal inferences less robustly than *no*.
Furthermore, given the symmetric semantics of *exactly three*, the following sentences are predicted to have the same universal inference that every red triangle is connected to (exactly) one circle.

(42) a. Exactly three red circles have the same color as the circle that they are connected to.
b. Exactly three figures that have the same color as the circle that they are connected to are red triangles.

Finally, not all non-monotonic quantifiers seem to have the same projection profile.

(43) Every student but one stopped smoking.

In contexts with four (relevant) students, this sentence will have the same contextual meaning as (41). But it seems to me that this has a universal inference more robustly than (41).

7 Scalar implicatures?

The experimental literature on presupposition projection in quantified sentences often ignores the potential effect of scalar implicatures.
Consider, e.g., (44).

(44) Some students stopped smoking.

If (44) has a 'not all' scalar implicature, then that can lead to a universal presupposition (as predicted by Spector & Sudo 2017).
Cf. (45), which seems to have a universal presupposition.

(45) Some, but not all, students stopped smoking.

I believe no attempts have been made to investigate to what extent the universal inference of (44) correlates with the scalar implicature.

8 Relevant sets and disappointment

8.1 Relevant sets

George also starts with Strong Kleene, and introduces *relevant sets* for quantificational cases.

(46) For a quantificational sentence of the form "every student VP",
any set containing all students is a relevant set for $[\![\texttt{every}]\!]$ and its first argument $[\![\texttt{student}]\!]$

(47) a. Let f be a function of type $\langle\sigma_1, \langle\sigma_2, \ldots \langle\sigma_n, \tau\rangle\rangle\rangle$. X is a relevant set for f and $a_1, \ldots, a_{m-1}$ (for $m \leqslant n$, possibly $m = 1$) iff for any b_m and b'_m that agree on X and for every $c_{m+1} \ldots c_n$,

$$f(a_1)\cdots(a_{m-1})(b_m)(c_{m+1})\cdots(c_n) = f(a_1)\cdots(a_{m-1})(b'_m)(c_{m+1})\cdots(c_n).$$

b. b and b' agree on X iff for all $x \in X$, $b(x) = b'(x)$.

(48) For "every student", $m = n = 2$.

a. X is a relevant set for $[\![\texttt{every}]\!]$ and $[\![\texttt{student}]\!]$ iff for any two nuclear scopes S and S' that agree on X, $[\![\texttt{every}]\!]([\![\texttt{student}]\!])(S) = [\![\texttt{every}]\!]([\![\texttt{student}]\!])(S')$.

b. X is a relevant set for $[\![\texttt{every}]\!]$ and $[\![\texttt{student}]\!]$ iff $X \supseteq [\![\texttt{student}]\!]$.

(49) For "every", $m = 1$, $n = 2$.

a. X is a relevant set for $[\![\texttt{every}]\!]$ iff for any two restrictors R and R' that agree on X and for every nuclear scope S, $[\![\texttt{every}]\!](R)(S) = [\![\texttt{every}]\!](R')(S)$.

b. X is a relevant set for $[\![\texttt{every}]\!]$ iff X is the set of all individuals.

If there's a unique minimal relevant set, we say *the relevant set*.
George 2008 proposes that the projection algorithm applies the A-operator on a predicate P whose relevant set is X, if for at least one $x \in X$, $P(x) = 1$.
This yields the following meaning (note that applying A to the restrictor won't matter here; so omitted):

(50) $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{G}}(a)$

$$= \begin{cases} 0 & \text{if either for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and} \\ & \qquad \text{for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 1 \text{ and } [\![\mathsf{A}(\psi)]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 0 \\ & \quad \text{or for no } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 1 \text{ and} \\ & \qquad \text{for some } y \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

Compare the Strong Kleene meaning:

(51) $[\![\mathsf{EVERY}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{SK}}(a)$

$$= \begin{cases} 0 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \\ 1 & \text{if for each } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 0 \text{ or } [\![\psi]\!]_{\mathsf{SK}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

(52) "Every student stopped smoking" denotes 2 if no student ever smoked.
Assuming that there are students:

a. Every student used to smoke and stopped iff the sentence denotes 1.

b. If there is a student x that used to smoke and didn't stop (i.e., continued to smoke), then the sentence is not 1.

  (i) If there is another student y that used to smoke and stopped, then x makes the sentence denotes 0 (the first falsity condition).

  (ii) If there is no such student y, then x makes the sentence denotes 0 (the second falsity condition).

  So either way, x, who used to smoke and is still smoking, renders the sentence 0.

c. If no student both used to smoke and continued and some student x never smoked, then the sentence is not 1.

  (i) if there is a student y that used to smoke and stopped, then x makes the sentence false (the first falsity condition)

  (ii) if there is no such student y, then that means that no student used to smoke. In this case the sentence denotes 2.

Thus overall, the universal quantifier gives rise to an existential presupposition with respect to the nuclear scope.

For *some*, it makes no difference.

(53) For ''some student", $m = n = 2$. X is a relevant set for $[\![\texttt{some}]\!]$ and $[\![\texttt{student}]\!]$ iff for any two nuclear scopes S and S' that agree on X, $[\![\texttt{some}]\!]([\![\texttt{student}]\!])(S) = [\![\texttt{some}]\!]([\![\texttt{student}]\!])(S')$. X is a relevant set for $[\![\texttt{some}]\!]$ and $[\![\texttt{student}]\!]$ iff $X \supseteq [\![\texttt{student}]\!]$.

(54) $[\![\mathsf{SOME}^{\xi}(\varphi)(\psi)]\!]_{\mathsf{G}}(a)$

$$= \begin{cases} 0 & \text{if for no } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 1 \text{ and} \\ & \qquad \text{for each } y \in \mathcal{D} \text{ such that } [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 1,\ [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto y]) = 0 \\ 1 & \text{if for some } x \in \mathcal{D},\ [\![\varphi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 1 \text{ and } [\![\psi]\!]_{\mathsf{G}}(a[\xi \mapsto x]) = 1 \\ 2 & \text{otherwise} \end{cases}$$

This is the same as the Strong Kleene meaning, because the relevant set matters only when there is an individual that turns both the restrictor and nuclear scope true, but that's when the sentence is true.

8.2 Disappointment

George 2008 introduces concern for disappointment to render *and* asymmetric, as well as to derive universal presuppositions for *every* and *no*.

(55) Let f be an n-ary function. b_m has a disappointing presupposition with respect to $f(a_1)\cdots(a_{m-1})$ iff

a. for every b'_m that is presupposition equivalent to b_m, for each $c_{m+1},\ldots,c_n$, $f(a_1)\cdots(a_{m-1})(b'_m)(c_{m+1})\cdots(c_n) \neq 1$; and

b. for some $b'_m, c_{m+1},\ldots,c_n$, $f(a_1)\cdots(a_{m-1})(b'_m)(c_{m+1})\cdots(c_n) = 1$.

(56) b and b' are presupposition equivalent iff for each x, $b(x) = 2$ iff $b'(x) = 2$.

(55a) says that based on b_m's presupposition and its preceding material, we can rule out the hope for truth; (55b) ensures that b_m's presupposition is the culprit.

George proposes to augment the presuppositions of Strong Kleene (with relevant sets for quantifiers) with disappointment presuppositions. Essentially, every argument presupposition that it is not disappointing in the above sense.

8.2.1 Conjunction

The crucial assumption is that the first conjunct is the first argument (we compute disappointment from left to right).

- If the first conjunct is 2, we can rule out the truth, so the disappointment presupposition is that it cannot be 2.
- If the first conjunct is 0 or 1, there is no disappointment presupposition.
- If the second conjunct is 2, we have processed the first conjunction already. If the first conjunct is 0, then it's anyway 0. If the first conjunct is 1, then whether the second conjunct is 2 or not matters. Therefore, we get the conditional presupposition in this case, if the first conjunct is 1, then the second conjunct should be 0 or 1.

8.2.2 Quantifiers

In "Every student stopped smoking", $[\![\texttt{stopped smoking}]\!]$ will be disappointing with respect to $[\![\texttt{every}]\!]([\![\texttt{student}]\!])$ if one or more students never smoked, because such students are bound to return 2 for every presuppositionally equivalent nuclear scope, so will prevent the sentence from ever being 1.

Similarly, for "No student stopped smoking", if one or more students never smoked, they'll yield 2 for every presuppositionally eqauivalent nuclear scope, so we won't get 1 for the entire sentence, as the truth of the entire sentence requires making sure that every student being 0 for the nuclear scope.

On the other hand, for "Some student stopped smoking", the nuclear scope will never be disappointing (in a model with some students that have smoked before), because even when some students never smoked, the sentence can still be true by finding another student that has smoked before and also turn the nuclear scope true.

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