Satisfaction Theory

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24 April 2026
Presupposition Reading Group

We will discuss a propositional fragment of Satisfaction Theory today (we'll discuss quantifiers in a couple of weeks).
Before Satisfaction Theory itself, let's review Propositional Logic.

1 Propositional Logic $\mathcal{L}$

1.1 Syntax

- The set of all atomic formulas: $\mathcal{A} = \mathtt{p}_1, \mathtt{p}_2, \cdots$.
- The set of all formulas $\mathcal{L}$ (in BNF): $\varphi ::= \mathtt{p_i} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi)$.

1.2 Semantics

- Each function $v : \mathcal{A} \to \{0, 1\}$ is a model for $\mathcal{L}$, also called an $\mathcal{L}$*-valuation*.
- $\mathcal{L}$-valuations can be seen as possible worlds (in the propositional setting).
- We define $\|\phi\|$ as a function from $\mathcal{L}$-valuations to $\{0, 1\}$.

$$\|\mathtt{p_i}\|(v) = v(\mathtt{p_i})$$

$$\|\neg\varphi\|(v) = 1 - \|\varphi\| = \begin{cases} 1 & \text{if } \|\varphi\| = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\|(\varphi \wedge \psi)\|(v) = \begin{cases} 1 & \text{if } \|\varphi\| = \|\psi\| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\|(\varphi \vee \psi)\|(v) = \begin{cases} 0 & \text{if } \|\varphi\| = \|\psi\| = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\|(\varphi \to \psi)\|(v) = \begin{cases} 0 & \text{if } \|\varphi\| = 1 \text{ and } \|\psi\| = 0 \\ 1 & \text{otherwise} \end{cases}$$

1.3 Lifting

Satisfaction Theory encodes presuppositions as definedness conditions on sets of $\mathcal{L}$-valuations.
In preparation for Satisfaction Theory, let's consider a version of $\mathcal{L}$ that operates on sets of $\mathcal{L}$-valuations.

- Each formula is interpreted as a (total) function over sets of $\mathcal{L}$-valuations.
- For any $\varphi \in \mathcal{L}$, $[\![\varphi]\!](V) = \{ v \in V \mid \|\varphi\|(v) = 1 \}$.

With this 'lifted' semantics, $\mathcal{L}$ is a kind of dynamic semantics. But it's not irreducibly so, as $\|\varphi\|$ can also be defined in terms of $[\![\varphi]\!]$ (cf. Van Benthem 1986, Rothschild & Yalcin 2016, 2017).

- For any $\varphi \in \mathcal{L}$, $\|\varphi\|(v) = 1$ iff $[\![\varphi]\!](\{ v \}) = \{ v \}$.

Satisfaction Theory is properly dynamic.

2 Propositional Satisfaction Theory $\mathcal{L}_{\mathsf{SAT}}$

2.1 Syntax

We will represent presuppositions as (bivalent) formulas π of $\mathcal{L}$ that are pre-subscripted to atomic formulas.

- This means that we assume that presuppositions are always attributed to atomic formulas.
- Consequently, the logical connectives are assumed to not trigger presuppositions.
- For natural language applications, you might want to drop this assumption, at least for certain connectives.
- We assume that formulas without presuppositions have tautological presuppositions ($\top$).

The syntax of $\mathcal{L}_{\mathsf{SAT}}$ is simple:

- If $\pi \in \mathcal{L}$ and $\mathtt{p_i} \in \mathcal{A}$, then ${}_\pi\mathtt{p_i} \in \mathcal{L}_{\mathsf{SAT}}$.
- If $\varphi, \psi \in \mathcal{L}_{\mathsf{SAT}}$, then $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in \mathcal{L}_{\mathsf{SAT}}$.
- Nothing else is a formula of $\mathcal{L}_{\mathsf{SAT}}$.

2.2 Semantics

Propositional Satisfaction Theory operates on sets of $\mathcal{L}$-valuations (qua possible worlds).

- Presuppositions are interpreted as definedness conditions.
- Each formula is interpreted as a partial function over sets of $\mathcal{L}$-valuations.

Atomic formulas of course involve no filtering.

- $V \in \mathrm{dom}([\![{}_\pi\mathtt{p_i}]\!])$ iff for each $v \in V$, $\|\pi\|(v) = 1$.
  Whenever $V \in \mathrm{dom}([\![{}_\pi\mathtt{p_i}]\!])$, $[\![{}_\pi\mathtt{p_i}]\!](V) = \{ v \in V \mid \|\mathtt{p_i}\|(v) = 1 \}$.
- In Heim-Kratzer notation for partial functions:

$$[\![{}_\pi\mathtt{p_i}]\!] = \lambda V\!: \text{ for each } v \in V, \|\pi\|(v) = 1. \ \{ v \in V \mid \|\mathtt{p_i}\|(v) = 1 \}$$

Complex formulas do involve filtering.

- $\llbracket \neg\varphi \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket).\ V - \llbracket\varphi\rrbracket(V)$
- $\llbracket (\varphi \wedge \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ \llbracket\psi\rrbracket(\llbracket\varphi\rrbracket(V))$
- $\llbracket (\varphi \vee \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\neg\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ \llbracket\varphi\rrbracket(V) \cup \llbracket\psi\rrbracket(\llbracket\neg\varphi\rrbracket(V))$
- $\llbracket (\varphi \rightarrow \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ V - \llbracket\neg\psi\rrbracket(\llbracket\varphi\rrbracket(V))$

Note that the output can be stated in terms of $[\![\cdot]\!]$:

- $\llbracket \neg\varphi \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket).\ [\![\neg\varphi]\!](V)$
- $\llbracket (\varphi \wedge \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ [\![(\varphi \wedge \psi)]\!](V)$
- $\llbracket (\varphi \vee \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\neg\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ [\![(\varphi \vee \psi)]\!](V)$
- $\llbracket (\varphi \rightarrow \psi) \rrbracket = \lambda V\colon\ V \in \mathrm{dom}(\llbracket\varphi\rrbracket)$ and $\llbracket\varphi\rrbracket(V) \in \mathrm{dom}(\llbracket\psi\rrbracket).\ [\![(\varphi \rightarrow \psi)]\!](V)$

Heim 1982, 1983 hoped that the her Satisfaction Theory gave a principled account of presupposition projection, but the possibility of this formulation shows that the theory actually stipulates how presuppositions project in each case.
We will discuss attempts to derive the correct projection patterns in several weeks.

2.3 The semantics-pragmatics interface

Satisfaction Theory jibes well with the Stalnakerian theory of pragmatics.

- Stalnaker 1973, 1974, 1978 propose that the aspect of conversational context that matters for presupposition (and assertion) is the *common ground* (see also Stalnaker 1998, 2002 for refinements).
- The common ground is the set of propositions that discourse participants mutually believe to be true.

  (1) A set A of agents *mutually believe* proposition p to be true iff
  for each $a_i \in A$, a_i believes p to be true; and
  for each $a_j \in A$, a_i believes that a_j believes p to be true; and
  for each $a_k \in A$, a_i believes that a_j believes that a_k believes p to be true,; and
  for each $a_l \in A$, . . .

- For formal purposes, we can simply work with the grand conjunction of the common ground, which can be modeled as a set of possible worlds (or $\mathcal{L}$-valuations). This set of possible worlds is called the *context set*.

In order to deal with quantifiers, Heim 1982, 1983 enriches the context set with assignment functions. We'll come back to this when we discuss presupposition projection through quantifiers.

2.4 Natural language applications

The conjunction above captures asymmetric presupposition filtering:

(2) a. John read *War & Peace* when he was young, and he is reading it again.
    b. John is reading *War & Peace* again, and he read it when he was young.

In developing a compositional semantics for a natural language, it'll suffice to have a translation rule like (3).

(3) For any pair of sentences S_1 and S_2 such that $\mathsf{S}_1 \rightsquigarrow \varphi$ and $\mathsf{S}_2 \rightsquigarrow \psi$,

$$\mathsf{S}_1 \text{ and } \mathsf{S}_2 \rightsquigarrow (\varphi \wedge \psi)$$

Similarly for conditionals.

(4) a. If John is a carpenter, then Bill, too, is a carpenter.
    b. If Bill, too, is a carpenter, then John is a carpenter.

(5) For any pair of sentences S_1 and S_2 such that $\mathsf{S}_1 \rightsquigarrow \varphi$ and $\mathsf{S}_2 \rightsquigarrow \psi$,

$$\text{if } \mathsf{S}_1\text{, then } \mathsf{S}_2 \rightsquigarrow (\varphi \rightarrow \psi)$$

However, it's a general consensus that natural language conditionals are not material implication and involve modality. So an analysis like (5) is at best tentative.

Alternatively, the intermediate formal language can be dropped and natural language expressions can be directly mapped onto the model theoretical objects the corresponding expressions of the formal language denote (Montague 1970).

2.5 Disjunction

The above disjunction is an asymmetric disjunction, but natural language disjunction seems to allow for symmetric filtering.

(6) a. Either John never smoked, or he stopped smoking.
    b. Either John stopped smoking, or he never smoked.

How can we capture this? The following won't allow filtering at all.

- $[\![(\varphi \dot{\vee} \psi)]\!] = \lambda V\colon\ V \in \text{dom}([\![\varphi]\!]) \text{ and } V \in \text{dom}([\![\psi]\!]).\ [\![\varphi]\!](V) \cup [\![\psi]\!](V)$

How about:

- $[\![(\varphi \ddot{\vee} \psi)]\!] = \lambda V\colon \begin{array}{c} V \in \text{dom}([\![\varphi]\!]) \text{ and } [\![\neg\varphi]\!](V) \in \text{dom}([\![\psi]\!]) \\ \text{or} \\ V \in \text{dom}([\![\psi]\!]) \text{ and } [\![\neg\psi]\!](V) \in \text{dom}([\![\varphi]\!]) \end{array}.\ \begin{cases} [\![\varphi]\!](V) \cup [\![\psi]\!]([\![\neg\varphi]\!](V)) & \text{if } V \in \text{dom}([\![\varphi]\!]) \\ [\![\psi]\!](V) \cup [\![\varphi]\!]([\![\neg\psi]\!](V)) & \text{otherwise} \end{cases}$

Can we filter in both directions at the same time?

(7) Either Nathan, too, met no man from France, or no woman is also from France and met no man from France.

3 Partial Propositional Logic

We actually do not need full dynamic semantics to capture the essence of Satisfaction Theory: we can simply make Propositional Logic partial in the following way:

- $[\![\![{}_{\pi}\mathtt{p_i}]\!]\!] = \lambda v\colon \|\pi\|(v) = 1.\ \|\mathtt{p_i}\|(v)$
- $[\![\![\neg\varphi]\!]\!] = \lambda v\colon v \in \mathrm{dom}([\![\![\varphi]\!]\!]).\ \|\neg\varphi\|(v)$
- $[\![\![(\varphi \wedge \psi)]\!]\!] = \lambda v\colon v \in \mathrm{dom}([\![\![\varphi]\!]\!])$ and if $[\![\![\varphi]\!]\!](v) = 1$, then $v \in \mathrm{dom}([\![\![\psi]\!]\!]).\ \|(\varphi \wedge \psi)\|(v)$
- $[\![\![(\varphi \vee \psi)]\!]\!] = \lambda v\colon v \in \mathrm{dom}([\![\![\varphi]\!]\!])$ and if $[\![\![\varphi]\!]\!](v) = 0$, then $v \in \mathrm{dom}([\![\![\psi]\!]\!]).\ \|(\varphi \vee \psi)\|(v)$
- $[\![\![(\varphi \rightarrow \psi)]\!]\!] = \lambda v\colon v \in \mathrm{dom}([\![\![\varphi]\!]\!])$ and if $[\![\![\varphi]\!]\!](v) = 1$, then $v \in \mathrm{dom}([\![\![\psi]\!]\!]).\ \|(\varphi \rightarrow \psi)\|(v)$

We can require the same thing as Satisfaction Theory in the pragmatics.

(8) **Bridging Principle**: An utterance of a sentence S such that S $\rightsquigarrow$ φ against context V requires that each $v \in V$, $v \in \mathrm{dom}([\![\![\varphi]\!]\!])$.

The two symmetric disjunctions can be defined as well.

- $[\![\![(\varphi \dot{\vee} \psi)]\!]\!] = \lambda v\colon v \in \mathrm{dom}([\![\![\varphi]\!]\!])$ and $v \in \mathrm{dom}([\![\![\psi]\!]\!]).\ \|(\varphi \vee \psi)\|(v)$
- $[\![\![(\varphi \ddot{\vee} \psi)]\!]\!] = \lambda v\colon \begin{array}{c} v \in \mathrm{dom}([\![\![\varphi]\!]\!]) \text{ and if } [\![\![\varphi]\!]\!](v) = 0, \text{ then } v \in \mathrm{dom}([\![\![\psi]\!]\!]) \\ \text{or} \\ v \in \mathrm{dom}([\![\![\psi]\!]\!]) \text{ and if } [\![\![\psi]\!]\!](v) = 0, \text{ then } v \in \mathrm{dom}([\![\![\varphi]\!]\!]) \end{array}.\ \|(\varphi \vee \psi)\|(v)$

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