USING THE COMPLEX WKB METHOD FOR STUDYING THE EVOLUTION OF INITIAL PULSES OBEYING THE NONLINEAR SCHRÖDINGER EQUATION

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Using the complex WKB method, we found an asymptotic solution of the associated Zakharov—Shabat problem in the limit of a small coefficient h → 0 multiplying the derivative of the potential that has a single hump. The obtained formulas can be used in describing the evolution of optical pulses of such shape obeying the nonlinear Schrödinger equation (NSE). Several examples are considered.

INTRODUCTION

When studying a perturbation propagating in a weakly nonlinear medium which obeys the nonlinear Schrödinger equation (NSE) [1]

$$iU_{,t}+U_{,xx}+U|U|^2=0 \tag{1}$$

(where $U = U(x, t)$, $U_{,t} = \partial U/\partial t$, $U_{,x} = \partial U/\partial x$), one has to consider the influence of the shape of the initial pulse on the spontaneous soliton generation process (SSGP).

In papers [2, 3] the dynamics of rectangular and truncated-exponential pulses was studied using numerical methods applied to the NSE.

In some physical applications, Eq. (1) contains a small factor h multiplying the derivatives. This factor has the meaning of the ratio of the characteristic length (or time) of the pulse to the characteristic length (or time) of modulations of the medium:

$$ihU_{,t}+\frac{h^2}{2}U_{,xx}+U|U|^2=0. \tag{2}$$

The presence of the small parameter h allows one to look for an asymptotic solution of Eq. (2) in h → 0 limit. One of the approaches commonly used in constructing such solutions is the complex WKB method [4]. However, so far all attempts to construct a WKB solution directly from Eq. (2) have failed.

On the other hand, according to the inverse scattering problem approach [5], the information concerning the evolution of a pulse obeying Eq. (2) can be extracted from the solution of the associated Zakharov—Shabat linear spectral problem:

$$\begin{cases} h\psi_{1,x}=-i\lambda\psi_1+q\psi_2, \\ h\psi_{2,x}=i\lambda\psi_2-q^*\psi_1, \end{cases} \tag{3}$$

where $q = q(x) = iU^*(x, 0)$. The number of points λ_k belonging to the discrete spectrum equals the number of solitons generated in the SSGP. $\mathrm{Im}\lambda_k$ defines the soliton's amplitude and width, and $\mathrm{Re}\lambda_k$ defines its velocity [5].

Exact solutions of Eq. (3) for some classes of piecewise smooth potentials q(x) were considered in papers [6, 7]. In addition to finding the points of the discrete spectrum, Lugin and Shapovalov [6] and Shapovalov and Yurchenko [7] obtained expressions relating the area $S=\int_{-\infty}^{\infty}|q(x)|dx$ to the number of points in the spectrum.

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The complex WKB method allows one to obtain approximate solutions of Eq. (3) for any local potential with a single maximum. This may be very important for physical applications.

The purpose of this paper is to find the asymptotic behavior of the spectrum for the Zakharov–Shabat problem (3) as $h \to 0$ near the fundamental state using an expansion in x in the neighborhood of the maximum of $|q|$, and, therefore, to evaluate the number and the parameters of solitons generated in the SSGP for $U(x, 0) = iq^*(x)$.

1. Consider a Zakharov–Shabat system for the spectrum (3), where q(x) is a smooth function with a single maximum satisfying the following conditions:

$$\int_{-\infty}^{\infty} |q(x)|\, dx < \infty,$$

$q(x) \neq 0$ for some compact Ω, $h \in [0, 1]$.

Setting

$$\begin{cases} \psi_1 = \sqrt{q}\,\psi, \\ \psi_2 = \dfrac{1}{q}(h\psi_{1,x} + i\lambda\psi_1), \end{cases} \tag{4}$$

we obtain, instead of the system (3), a second-order differential equation

$$h^2\psi_{,xx} + \left(h^2 \frac{q_{,xx}}{2q} - \frac{3}{4}h^2\left(\frac{q_{,x}}{q}\right)^2 - i\lambda h\frac{q_{,x}}{q} + \lambda^2 + |q|^2\right)\psi = 0. \tag{5}$$

Let us look for solutions of Eq. (5) up to the terms of order $O(h^{3/2})$ in the form of a WKB approximation:

$$\psi = \Phi\exp(iS/h),$$
$$S = S(x) = S_1 + iS_2, \quad S_2 \geq 0.$$
$$\left[h^2\Phi_{,xx} + 2\Phi_{,x}S_{,x}h + \left(h^2\frac{q_{,xx}}{2q} - \frac{3}{4}h^2\left(\frac{q_{,x}}{q}\right)^2 - \right.\right.$$
$$\left.\left. - i\lambda h\frac{q_{,x}}{q} + \lambda^2 + |q|^2 + iS_{,xx}h - S_{,x}^2\right)\Phi\right]\exp(iS/h) = 0. \tag{6}$$

Then in the domain where $S_2 > 0$, $\exp(iS/h)$ is of the order $O(h^\infty)$, so that, to construct an approximate solution of Eq. (5), one should search for $\psi(x)$ only in the neighborhood of the set

$$\Gamma = \{x \in R : \operatorname{Im} S = 0\}. \tag{7}$$

As shown in [4], in some neighborhood of Γ the functions S(x) and $\Phi(x)$ can be represented by an expansion in half-integer powers of $S_2(x)$. Therefore, we expand $\Phi(x)$ and λ as follows:

$$\Phi = \sum_{\kappa=-N}^{M} \Phi^{(\kappa)} h^{\kappa/2}, \quad \lambda = \sum_{\kappa=0}^{M} \lambda_\kappa h^\kappa, \tag{8}$$

where $\Phi^{(k)} = O(S_2^{k/2})$.

In accordance with [4], let us introduce the following notion of asymptotic equivalence of functions. Let $f(x) \geq 0$ and g(x) be smooth functions. We denote $g = O_f(h^a)$ if

$$\left(\frac{\partial^\kappa}{\partial x^\kappa} g\right)\exp(-f/h) = O(h^{a-\kappa/2})$$

when $a - k/2 \geq 0$ on any compact $\Omega \subset R$.

Then the evident relation

$$t^a \exp(-t/h) = O(h^a)$$

implies that $\Phi^{(k)} h^{k/2} = O_{S_2}(1)$, $k = -N, -N+1, \ldots, 0$.

Let us introduce the notation

$$\Phi_0 \equiv \sum_{\kappa=-N}^{0} \Phi^{(\kappa)} h^{\kappa/2}.$$

Substituting Eq. (8) in Eq. (6), we obtain if one wants to construct an approximate solution of Eq. (5) up to terms of order $O(h^{3/2})$, one has to solve the following equations:

$$\Phi_0(-S_x^2 + \lambda_0^2 + |q|^2) = O_{S_2}(h^{3/2}),$$
$$2\Phi_{0,x} S_x + \Phi_0 S_{,xx} - \Phi_0 \lambda_0 \frac{q_{,x}}{q} - 2i\lambda_0\lambda_1\Phi_0 - ih\Phi_{0,xx} = O_{S_2}(h^{1/2}), \tag{9}$$

where we took into account the fact that q, $q_{,x}$, $q_{,xx}$ are of order $O_{S_2}(h^0)$, and the $ih\Phi_{0,xx}$ term remains by virtue of the relations [4]:

$$\frac{\partial}{\partial x} O_{S_2}(h^a) \to O_{S_2}(h^{a-1/2}),$$
$$h\Phi_{0,xx} = O_{S_2}(h^0).$$

Using the well-known notation [4], we can rewrite Eq. (9) in the form

$$H(S_{,x}, x) = O_{S_2}(h^{3/2}), \tag{10a}$$
$$\hat{\Pi}\Phi_0 - i\omega\Phi_0 = O_{S_2}(h^{1/2}), \tag{10b}$$
$$H(p, x) = p^2 - |q|^2 - \lambda_0^2, \tag{10c}$$

where

$$\hat{\Pi} = H_{,p}(S_{,x}, x)\frac{\partial}{\partial x} + \frac{1}{2} H_{,pp}(S_{,x}, x) S_{,xx} - i\frac{h}{2} H_{,pp}(S_{,x}, x)\frac{\partial^2}{\partial x^2} + G(x),$$
$$G(x) = -\lambda_0 \frac{q_{,x}}{q}, \quad \omega = 2\lambda_0\lambda_1,$$

Eq. (10a) is the Hamilton–Jacobi equation with the Hamiltonian (10c), while Eq. (10b) is a transfer equation.

In accordance with [4], let us construct a zero-dimensional Lagrange manifold in the $R^2_{p,x}$ phase space [8]:

$$\Lambda^0 = ((p, x) \in R^2_{p,x} / p = P_0, \ x = X_0),$$

This mainfold satisfies the following conditions:

1) the projection of Λ^0 on the x-plane of the $R^2_{p,x}$ phase space coincides with Γ (see Eq. (7));

2) on Λ^0, the Hamilton equations are satisfied:

$$\left.\frac{\partial H}{\partial x}\right|_{(P_0, X_0)} = -\frac{dP_0}{d\tau} = 0,$$

$$\left.\frac{\partial H}{\partial p}\right|_{(P_0, X_0)} = \frac{dX_0}{d\tau} = 0; \quad (11)$$

3) $H(P_0, X_0) = 0$, i.e., it lies on the zero level line of $H(p, x)$.

According to [4], the solution (8) can be constructed only if the matrix

$$\mathcal{H}_{\text{var}}|_{\Lambda^0} = \left.\begin{pmatrix} -H_{,xp} & -H_{,xx} \\ H_{pp} & H_{,px} \end{pmatrix}\right|_{\Lambda^0} \quad (12)$$

is nondegenerate, and if all solutions of the variational system

$$\frac{\partial}{\partial \tau}\begin{pmatrix} W \\ Z \end{pmatrix} = \mathcal{H}_{\text{var}}|_{\Lambda^0} \cdot \begin{pmatrix} W \\ Z \end{pmatrix}$$

are bounded for $\forall \tau \in (-\infty, +\infty)$, or, which is equivalent, that $\mathcal{H}_{\text{var}}|_{\Lambda^0}$ is diagonalizable and all its eigenvalues are pure imaginary. In this case, the point Λ^0 is called a nondegenerate, linearly stable fixed point of $H(p, x)$.

Now, following [4], we shall obtain a solution of Eqs. (10a), (10b).

The point Λ^0 for $H(p, x)$ in the form (10c) can be found from Eq. (11):

$$H_{,p} = 0 \Rightarrow P_0 = 0,$$
$$H_{,x} = 0 \Rightarrow \left.\frac{\partial V}{\partial x}\right|_{x=X_0} = 0, \quad (13)$$

where

$$V(x) = |q(x)|^2.$$

The requirement that the matrix (12) be nondegenerate and its eigenvalues pure imaginary, leads to the inequality

$$V_{,xx}|_{x=X_0} < 0 .$$

In addition,

$$\kappa = i\beta, \ \beta = \pm\sqrt{-2V^0_{,xx}}, \quad (14)$$

where $V^0 \equiv V(X_0)$. Following [4], we shall consider only the upper sign in Eq. (14).

The condition (3) results in

$$\lambda_0 = \pm i\sqrt{V_0},$$

where we shall choose the plus sign. Expanding $S(x)$ and $V(x)$ in powers of x near X_0 in Eq. (10a) and limiting ourselves to terms of order $(x - X_0)^2$, we find

$$S(x) = \frac{S^0_{,xx}}{2}(x - X_0)^2, \quad (15)$$

where

$$S^0_{,xx} = i\sqrt{\frac{-V^0_{,xx}}{2}}.$$

According to [4], the solution of Eq. (10b) is

$$\omega_n = \beta(n+1/2) + i(1/2\, H_{,px}(P_0, X_0) - G(X_0)),$$
$$\Phi_{0n} = \chi_n \bar{\Lambda}^n 1,$$

where X_n are complex numbers, and

$$\bar{\Lambda} = \sqrt{h}\left(\frac{\partial}{\partial x} - \frac{2}{h}\,\mathrm{Im}\, S_{,xx}(x - X_0)\right).$$

In our case it gives

$$\omega_n = 2\lambda_0\lambda_{1n} = \sqrt{-2V^0_{,xx}}\,(n+1/2) - \sqrt{V^0}\,\frac{q^0_{,x}}{q^0},$$
$$\Phi_{0n} = \chi_n \bar{\Lambda}^n 1,$$
$$\bar{\Lambda} = \sqrt{h}\left(\frac{\partial}{\partial x} - \frac{1}{h}\sqrt{-2V^0_{,xx}}\,(x - X_0)\right). \tag{16}$$

It is easy to show that $\begin{pmatrix}\Psi_{1n}\\ \Psi_{2n}\end{pmatrix}$ from Eq. (4), where

$$\psi_n = \Phi_{0n}\exp(iS/h),$$

and Φ_{0n} and $S(x)$ are defined by Eqs. (15) and (16), is a solution of the system (3), also up to terms of order $O(h^{3/2})$, if

$$\lambda_n = i\left(\sqrt{V^0} - h\sqrt{\frac{-V^0_{,xx}}{2V^0}}\,(n+1/2) + \frac{q^0_{,x}}{2q^0}\,h\right). \tag{17}$$

Using the complex WKB method and expanding $q(x)$ up to terms of order $(x - X_0)^M$, one can construct a solution of Eq. (3) up to terms of order $O(h^{M/2})$. In doing so, we can see that λ_n from Eq. (17) is a spectral parameter for the solution of Eq. (3) up to terms of order $O(h^2)$, i.e., if one limits oneself to the terms of order $(x - x_0)^3$ in the expansion of $q(x)$ near X_0.

In the formal limit $\mathrm{Im}\lambda_n \to 0$, we can find the total number of spectral points $N = n + 1$:

$$N = \mathrm{ent}\left(\frac{2V^0}{h\sqrt{-2V^0_{,xx}}} + \frac{1}{2}\right). \tag{18}$$

2. Let us consider an application of the above results to particular initial potentials.

2.1. An oscillator potential

$$q(x) = -a^2x^2 + b,$$

where $a, b \in R$.

The above expressions (15), (16), (17), and (18), applied to

$$V(x) = (q(x))^2$$

give the following result:

$$S = ia\sqrt{b/2}\,x^2,$$
$$\bar{\Lambda} = \sqrt{h}\left(\frac{\partial}{\partial x} - \frac{2a}{h}\sqrt{2b}\,x\right).$$

$$\lambda_n = i\,(b - ah\sqrt{2/b}\,(n+1/2)),$$
$$N = \mathrm{ent}\left(\frac{b^{3/2}}{ah2^{1/2}} + \frac{1}{2}\right).$$

2.2. A soliton-like potential

$$q(x) = 2\eta\,\frac{\exp(-2i\varkappa x)}{\mathrm{ch}\,(2\eta x)}\,.$$

The expressions (15), (16), (17), and (18) yield

$$\begin{gathered}\Lambda^0 = (0,\,0),\quad S(x) = 2i\eta^2 x^2,\\ \overline{\Lambda} = \sqrt{h}\left(\frac{\partial}{\partial x} - \frac{8}{h}\eta^2 x\right),\\ \lambda_n = i\,(2\eta - 2h\eta\,(n+1/2) - i\varkappa h),\quad N = \mathrm{ent}(1/h + 1/2).\end{gathered} \tag{19}$$

Notice that if we set h = 1 in Eq. (19), we obtain the well-known eigenvalue for a soliton-like potential:

$$\lambda_0 = \varkappa + i\eta,\quad N = 1.$$

2.3. In [9] an example of a real pulse is given from which one can write

$$q(x) = A\,(x - x^{(0)})^m \exp(-\alpha\,(x - x^{(0)})), \tag{20}$$

α, $A \in C$, $\alpha = a + ib$. The dynamics of the pulse (20) has been studied in [2].

Let us find approximate solutions of Eq. (3) up to terms of order $O(h^{3/2})$:

$$V(x) = |A|^2 (x - x^{(0)})^{2m} \exp(-2a\,(x - x^{(0)})).$$

The manifold Λ^0 (see Eq. (13)) is defined by

$$P_0 = 0,\quad X_0 = m/a + x^{(0)},$$

where X_0 is the maximum of V(x):

$$\begin{gathered}V^0 = |A|^2 (m/a)^{2m} \exp(-2m),\quad V^0_{,x} = 0,\\ V^0_{,xx} = |A|^2 (m/a)^{(2m-2)} \exp(-2m)\,(-2m) < 0.\end{gathered}$$

Then the functions

$$\begin{gathered}\psi_{1n} = \chi_1 q^{1/2} \exp(-\mathrm{Im} S_{,xx}(x - X_0)^2/2h)\,\overline{\Lambda}1,\\ \psi_{2n} = \frac{1}{q}\,(h\psi_{1n,x} + i\lambda_n \psi_{1n}),\end{gathered}$$

where

$$\begin{gathered}\overline{\Lambda} = \sqrt{h}\left(\frac{\partial}{\partial x} - \frac{|A|}{h}(m/a)^{m-1} e^{-m}\, m^{1/2}\,(x - X_0)\right),\\ \mathrm{Im} S_{,xx} = |A|\,(m/a)^{m-1} e^{-m} m^{1/2},\end{gathered}$$

satisfy the Zakharov–Shabat system (3) if

$$\lambda=\lambda_n=bh/2+i(|A|(m/a)^m e^{-m}-hm^{1/2}(a/m)(n+1/2)).$$

The number of spectral points is

$$N=\text{ent}\left(\frac{|A|(m/a)^{m+1}e^{-m}}{hm^{1/2}}+\frac{1}{2}\right),$$

which corresponds to the number of generated solitons when an initial pulse of the form (20) evolves according to the NSE (2).

An approximate expression for the number of solitons for q, x/q << 1 [5] and taking into account the presence of a small parameter h has the following form:

$$N_0=\text{ent}\left(\frac{S}{\pi h}+\frac{1}{2}\right), \tag{21}$$

where S is the surface area of the initial pulse.

Let us introduce an effective surface area of the initial pulse such that $N = N_0$. Then

$$S_{\text{eff}}=|A|\pi(m/a)^{m+1}e^{-m}m^{-1/2}+\delta h\pi,\ \ \delta\in[0,\ 1].$$

For an area $|q(x)|$, we obtain

$$S=\int_{-\infty}^{\infty}|q(x)|\,dx=\frac{|A|}{(m+1)}m!.$$

Comparing S and S_{eff}, for $\delta = 0$ we find

$$\frac{S_{\text{eff}}}{S}=\frac{\pi m^{m+1}}{m^{1/2}\,m!\,e^m}.$$

The above expressions for the number of generated solitons and their parameters in the SSGP are meaningful for single-hump-shaped initial pulses with a small area, so that N given by Eq. (18) is sufficiently small that one can search for a solution of Eq. (3) near the fundamental state. For larger N, Eq. (18) turns into Eq. (21) [5].

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