EFFECT OF INITIAL PULSE SHAPE MODULATION ON SPONTANEOUS SOLITON FORMATION IN THE NSE MODEL

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An initial pulse with fairly steep fronts whose evolution is described by the nonlinear Schrödinger equation, splits into soliton-like pulses (spontaneous soliton formation). The number of solitons formed in this process can be estimated by the number of spectral points of the associated linear Zakharov–Shabat problem for the initial pulse. Exact solutions of the Zakharov–Shabat problem are constructed for some classes of initial piecewise-continuous pulses by using the Darboux method. This allows us to estimate the effect of the shape of the initial pulse on the number of formed solitions and their parameters.

INTRODUCTION

The nonlinear Schrödinger equation (NSE)

$$iu_{,t} + \frac{1}{2} u_{,xx} + |u|^2 u = 0 \tag{1}$$

(where $u = u(x, t)$, $u_{,t} = \partial u/\partial t$, $u_{,x} = \partial u/\partial x$) describes the evolution of the envelope of a wave train in a coordinate system moving with the group velocity of the carrier wave in a weakly nonlinear and strongly dispersive medium. It has a wide range of applications in nonlinear physics (see, for example, [1-3]).

Exact analytic solutions of Eq. (1) were obtained by the method of the inverse scattering problem [1-3] for a narrow class of nonreflecting potentials in the form of an infinite phased line [4]. The dynamics of a bound initial pulse with fairly steep fronts is characterized by its decomposition into soliton-like pulses [spontaneous soliton formation (SSF)], which was considered in [5-8]. The SSF can be characterized qualitatively by the number of formed solitons and their parameters. According to the theory of the inverse scattering problem [1], the number of solitons and their parameters can be estimated from the spectrum of the associated linear Zakharov–Shabat problem for a two-component function $\psi(x, k)$ with the potential $u(x) \equiv u(x, t)|_{t=0}$, viz.,

$$L\psi(x, \kappa) = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} i\frac{\partial}{\partial x} + \begin{pmatrix}0 & u^*(x)\\ -u(x) & 0\end{pmatrix}\right]\begin{pmatrix}\psi^{(1)}\\ \psi^{(2)}\end{pmatrix} = \kappa\psi, \tag{2}$$

where u^* is the complex conjugate to u.

From this point of view, the following problem is of considerable importance: indicate all the potentials having preset points of the discrete spectrum.

The solution of this problem is fraught with considerable mathematical difficulties. Even in the simpler case of the linear Schrödinger equation, the analogous problem has not been solved yet, although some progress has been made towards its solution [9]. Under these conditions, it would be interesting to find the discrete spectrum of Eq. (2) for as wide a class of the initial potentials $u(x)$ as possible. Besides, for the purpose of approximation, it is important to obtain a solution of the spectral problem for piecewise-continuous potentials, which can be assembled from pieces for which an exact solution can be obtained.

Exact solutions have been found for some classes of potentials [5]. These classes can be extended by using the methods of generation of exactly solvable potentials. One of the effective generation methods is the Darboux method [12] which allows us to obtain new solutions of problem (2) from known initial solutions.

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The present work aims at obtaining a new class of potentials with a preset spectrum on the basis of the Darboux method by using the known exact solutions including piecewise-smooth solutions.

1. BASIC CONCEPTS AND NOTATION

Let us use the inverse scattering problem to obtain a relation between the spectral points of the Zakharov–Shabat problem (2) and the solitons formed in a SSF process.

The Jost function f(x, k) [1] for rapidly decreasing potentials u(x) $\left(\int_{-\infty}^{+\infty}|u(x)|^2<\infty\right)$, which is defined by the asymptotic form

$$f(x,\kappa)\to e^{i\kappa x}\begin{pmatrix}0\\1\end{pmatrix},\quad x\to+\infty,$$

has the following asymptotic form for $x\to-\infty$:

$$f(x,\kappa)\to e^{i\kappa x}\begin{pmatrix}0\\1\end{pmatrix}a(\kappa)+e^{-i\kappa x}\begin{pmatrix}1\\0\end{pmatrix}b^*(\kappa),\ \operatorname{Im}\kappa=0.$$

The coefficients $a^{-1}(k)$ and $r(k) = b(k)a^{-1}(k)$ have the meaning of the amplitudes of forward and backward scattering. The zeros of the function a (k) in the upper half-plane of k are the points of the discrete spectrum of problem (2). In the nonreflecting case [r(k) = 0], the zeros and the analytical properties of a (k) define this function in the form

$$a(\kappa)=\prod_{j=1}^{n}\left(\frac{\kappa-\lambda_j}{\kappa-\lambda_j^*}\right)^{l_j},\tag{3}$$

where λ_j is a zero of a (k) and l_j its multiplicity (j = 1, ..., n).

It was found from the inverse scattering problem and computer simulation of SSF in the NSE model [5-8] that the evolution of the initial pulse u(x) is characterized qualitatively by the number of formed solitons and their parameters. The number N of solitons is determined by the number of zeros of the function a (k) by taking their multiplicity into account. The parameter Im λ_i characterizes the amplitude and width of the i-th soliton, while Re λ_i determines its velocity.

Expression (3) generalizes the formulas used in [5-8] to the case of multiple zeros of the function a (k).

We shall give here the required information concerning the Darboux method. Following [12], we shall write the equation of the Zakharov–Shabat spectral problem (2) in the form

$$\Phi_{,x}=V\Phi,\ V=i\kappa\begin{pmatrix}-1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&q(x)\\-q^*(x)&0\end{pmatrix},\tag{4}$$
$$q(x)=iu^*(x),$$

where $\Phi(x,\kappa)=\begin{pmatrix}\varphi_1(x,\kappa)&\psi_1(x,\kappa)\\\varphi_2(x,\kappa)&\psi_2(x,\kappa)\end{pmatrix}$ is the fundamental matrix of the solutions of system (4). Let $\Phi_0(x, k)$ be a solution of system (4) with the potential $q_0(x)$, and

$$f^{(i)}(x,\lambda_i)=\begin{pmatrix}f_1(x,\lambda_i)\\f_2(x,\lambda_i)\end{pmatrix},\ i=1,\dots,n\tag{5}$$

is the solution of system (4) with the potential $q_0(x)$ for $k = \lambda_i$, which is fixed arbitrarily. Then

$$\Phi=P\Phi_0\tag{6}$$

is the fundamental matrix with the potential

$$q(x)=q_0(x)+2iB_n,\tag{7}$$

where

$$P=\sum_{j=0}^{n} Q_j \lambda^j,\quad Q_j=\begin{pmatrix} A_{j+1} & B_{j+1} \\ C_{j+1} & D_{j+1}\end{pmatrix},\ j=0,\ 1,\ \ldots,\ n-1, \tag{8}$$

$Q_n = E$ is the unit matrix,

$$A_j=-\Delta^{-1}\Delta_j(\alpha),\quad C_j=-\Delta^{-1}\Delta_j(\beta),$$
$$B_j=-\Delta^{-1}\Delta_{n+j}(\alpha),\quad D_j=-\Delta^{-1}\Delta_{n+j}(\beta),$$
$$\Delta=\begin{vmatrix} \alpha_1\ \lambda_1\alpha_1\ldots\lambda_1^{n-1}\alpha_1\ \beta_1\lambda_1\beta_1\ldots\lambda_1^{n-1}\beta_1 \\ \alpha_2\ \lambda_2\alpha_2\ldots\lambda_2^{n-1}\alpha_2\ \beta_2\lambda_2\beta_2\ldots\lambda_2^{n-1}\beta_2 \\ \cdots \\ \alpha_{2n}\lambda_{2n}\alpha_{2n}\ldots\lambda_{2n}^{n-1}\alpha_{2n}\beta_{2n}\lambda_{2n}\beta_{2n}\ldots\lambda_{2n}^{n-1}\beta_{2n}\end{vmatrix},$$

$\lambda_{n+i}=\lambda_i^*$, $\alpha_i=f_1^{(i)}(x,\lambda_i)$, $\alpha_{n+1}=\beta_i^*$, $\beta_i=f_2^{(i)}(x,\lambda_i)$, $\beta_{n+i}=-\alpha_i^*$, $i=1, 2, \ldots, n$. The expression for $\Delta_i(\delta)$ can be obtained from Δ by replacing the i-th column by a column of the form

$$(\lambda_1^n\delta_1,\ \lambda_2^n\delta_2,\ \ldots,\ \lambda_{2n}^n\delta_{2n})^T.$$

We shall call $q_0(x)$, $f^{(i)}(x, \lambda_i)$, and λ_i the initial potential, initial function, and initial value, respectively (initial data). It can be easily shown that like the initial potential, the new potential (7) will also decrease at infinity.

2. SPECTRAL PROBLEM FOR THE NEW POTENTIAL

Let the initial potential $q_0(x)$ be a piecewise-continuous function which decreases quite rapidly at infinity. We assume that the fundamental matrix $\Phi_0(x, k)$ of solutions and the functions $\varphi^{(\alpha)}(x, \xi_\alpha)$ of the discrete spectrum corresponding to the eigenvalues ξ_α, $\alpha = 1, \ldots, M$ are also known. Then the following statement is valid.

Statement 1. The function $a(k)$ for solutions with the piecewise-continuous potential (7) is defined completely by the zeros $a_0(k)$ of the solution Φ_0 as well as by the initial values λ_i, λ_j:

$$a(\kappa)=\prod_{i,j} a_0(\kappa)\frac{(\kappa-\lambda_i)(\kappa-\lambda_j^*)}{(\kappa-\lambda_i^*)(\kappa-\lambda_j)}, \tag{9}$$

where λ_i ($i = 1, \ldots, m$) is the initial value corresponding to the initial solution $f^{(i)}(x)$ with the asymptotic form

$$f^{(i)}(x)\to e^{-i\lambda_i x}\begin{pmatrix}1\\0\end{pmatrix},\quad x\to+\infty,$$
$$f^{(i)}(x)\to c_i e^{i\lambda_i x}\begin{pmatrix}0\\1\end{pmatrix},\quad x\to-\infty, \tag{10}$$

and λ_j ($j = m + 1, \ldots, m + p$) is the initial value corresponding to the initial spectral function. Here $p \le M$, $m + p \le n$.

Corollary. The number of zeros of $a(k)$ is defined as

$$N=N_0+m-p, \tag{11}$$

where N_0 is the number of zeros $a_0(k)$ taking their multiplicity into consideration.

Statement 1 for $n = 1$ (point transformation) is proved by a direct substitution of the asymptotic forms of the relevant functions into formulas (5)-(8) by taking into account the continuity of the functions $\Phi_0(x, k)$ and $f^{(1)}(x, \lambda_1)$ at the discontinuity points of the potential. For an arbitrary n, formulas (5)-(8) can be obtained by a successive application of point transformations.

3. EXAMPLES OF GENERATION OF POTENTIALS

We shall consider below some new classes of potential, which are obtained by using formulas (5)-(8) for n = 1, and for which the solution of the spectral problem (4) is known. New potentials (solid curves) obtained against the background of initial potentials (dashed lines) are shown in Figs. 1-4. For each case, an analog of formula (11) is indicated, in which the number of spectral points is expressed in terms of the relevant number for the initial potential.

1. The initial potential is chosen in the form of a rectangle:

$$q_0(x)=\begin{cases}0, & -\infty<x<a,\\ q_0, & a\leqslant x\leqslant b,\\ 0, & b<x<+\infty.\end{cases} \tag{12}$$

The number of points in the spectrum [zeros of the function a (k)] for it is defined by the following expression [5]:

$$N_0=S/\pi+1/2, \tag{13}$$

where S is the area of the rectangle.

1.1. We choose the initial potentials on [a , b] in the form

$$f^{(1)}(\lambda_1)=f(\lambda_1)=\begin{pmatrix}\cos\eta x\\ -1/q_0(\varkappa\cos\eta x+\eta\sin\eta x)\end{pmatrix}$$

(here and below in this section, $\eta = |q_0|^2 + \lambda_1^2$, $\lambda_1 = i\kappa$). Then the Darboux transformation (7) gives

$$q(x)=q_0(x)\left(\frac{\eta^2-2\varkappa^2\cos^2\eta x-2\eta\varkappa\cos\eta x\sin\eta x}{\eta^2+2\varkappa^2\cos^2\eta x+2\eta\varkappa\cos\eta x\sin\eta x}\right). \tag{14}$$

Figure 1 shows the potentials corresponding to $\kappa = 1/2$. Formula (11) gives $N = N_0 - 1$.

1.2. We shall choose $f^{(1)}(\lambda_1)$ in the form of a discrete spectrum function. This gives

$$q(x)=q_0\left(\frac{\eta^2-\varkappa^2+\alpha\cos 2\eta x+\beta\sin 2\eta x}{|q_0|^2-\alpha\cos 2\eta x-\beta\sin 2\eta x}\right), \tag{15}$$

where $\alpha = \kappa^2\cos 2\eta b - \kappa\eta\sin 2\eta b$, $\beta = \kappa^2\sin 2\eta b + \kappa\eta\cos 2\eta b$. Formula (11) gives $N = N_0 - 1$. For the case presented in Fig. 2, we have $\lambda = i\kappa$, $\kappa = 0.19268$.

1.3. It is interesting to consider the case when the initial value is $\lambda_1 = i|q_0|$. For such a value, function (5) can be taken in the form

$$f(\lambda_1)=\begin{pmatrix}x\\ (-i|q_0|x+1)\,q_0^{-1}\end{pmatrix}.$$

This gives

$$q(x)=\begin{cases}2q_0\,\mathrm{ch}^{-1}(2|q_0|(x-a)+\theta_1), & -\infty<x<a,\\ -q_0\dfrac{2|q_0|^2x^2-2|q_0|x-1}{2|q_0|^2x^2-2|q_0|x+1}, & a\leqslant x\leqslant b,\\ 2q_0\,\mathrm{ch}^{-1}(2|q_0|(x-b)+\theta_2), & b<x<\infty,\end{cases} \tag{16}$$

where $\theta_1 = \ln(a|q_0|(1 - |q_0|a)^{-1}$; $\theta_2 = \ln(b|q_0|(1 - |q_0|b)^{-1})$. For any a, b, and q_0, $N = N_0 + 1$. The solution of this type, which as if "protrudes" against the background of a slowly varying function, is known as an exulton in the literature [10]. However, in contrast to [10] where the potential q(x) is not localized in space, q(x) differs from zero in a bounded region in the present case. Figure 3 shows such a potential by way of an example.

Remark. The boundaries in Fig. 1 are chosen so that any component of the initial function vanishes at the points $x = a$ and $x = b$, the potential (7) also being equal to zero for $x \overline{\in} [a, b]$ as well as the initial potential (in the case when the boundaries are

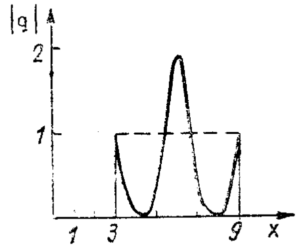

Fig. 1

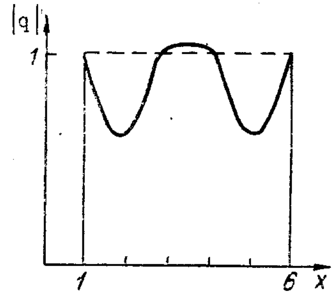

Fig. 2

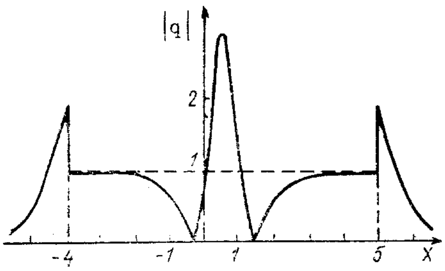

Fig. 3

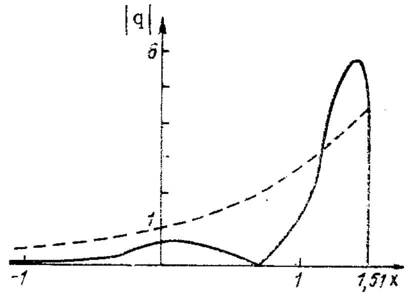

Fig. 4

specified, the initial function and the initial value (λ_i) must be chosen accordingly). Otherwise, the condition of continuity of the initial function would give nonzero "tails" of the form $A[\cosh(Bx + C)]^{-1}$ (see, for example, Fig. 3). In the case 1.2, the condition of equality to zero for one of the components is naturally satisfied.

2. We shall choose the initial potential in the form of a truncated exponential function

$$q_0(x) = \begin{cases} \rho e^{\alpha x}, & -\infty < x \leqslant x_0, \quad \alpha = a - ib, \\ 0, & x_0 < x < +\infty. \end{cases} \tag{17}$$

The spectrum for this potential can be determined from the equation $J_\nu(z_0) = 0$ [5, 6], where $z_0 = \rho a^{-1} e^{a x_0}$ is the area $\int_{-\infty}^{+\infty} |q_0(x)|\, dx$, $\nu = -\dfrac{1}{2} + ib/2a - i\xi/a$, and $J_\nu(z)$ is a first-order Bessel function. We shall choose the initial values in the form

$$\lambda_1 = \lambda' + i\lambda'', \ \lambda' = b/2, \ \lambda'' = a,$$

$$f(\lambda_1) = \begin{pmatrix} -i\,(\rho/a)^{1/2+ib/2a}\, e^{\left(ax - \frac{\alpha}{2}x\right)} J_{1/2}(\rho/a\, e^{ax}) \\ (\rho/a)^{1/2+ib/2a}\, e^{\alpha x/2} J_{3/2}(\rho/a\, e^{ax}) \end{pmatrix}.$$

Here $q(x) = q_0(x)(2z^2 - 3 + 3\cos 2z + 2z\sin 2z) \times (2z^2 + 1 - \cos 2z - 2z\sin 2z)^{-1}$. Where $\rho = 1$, $a = 1$, $x_0 = 1.51$, $N = N_0 + 1$ (Fig. 4).

The new classes of potentials obtained in this work can be regarded as initial pulses propagating in a cubically nonlinear medium in accordance with NSE (1).

Obviously, the shape of some of the pulses encountered in applications differs from that considered in the present work. As an example, we can mention that the shape of the laser pulse (whose dynamics was considered in [6]) presented in [11] is different. In such cases, the initial pulse can be approximated by a piecewise-continuous pulse assembled from pieces for which the solution of problem (2) is known. The required spectrum and, hence, the number of solitons in SSF as well as their parameters can be obtained by joining the solutions.

LITERATURE CITED

1. V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, Solitons: Method of Inverse Scattering Problem [in Russian], Nauka, Moscow (1980).
2. R. K. Bullough and P. J. Caydvet (eds.), Solitons, Springer-Verlag, Berlin (1980).
3. A. Newell, Solitons in Mathematics and Physics [Russian translation], Mir, Moscow (1989).
4. M. A. Sall', Teor. Mat. Fiz., 53, 227-237 (1982).
5. É. V. Lugin and A. V. Shapovalov, Izv. Vyssh. Uchebn. Zaved., Fiz., No. 2, 36-39 (1989).
6. V. A. Donchenko, M. V. Kabanov, et al., Opt. Atmos., 1, 67-72 (1988).
7. V. A. Donchenko, M. V. Kabanov, et al., Opt. Atmos., 2, 1286-1290 (1989).
8. A. A. Nalivaiko and A. V. Shapovalov, Izv. Vyssh. Uchebn. Zaved., Fiz., No. 7, 107-108 (1990).
9. A. G. Ushveridze, Elem. Chastitsy Atom. Yadra, 20, 1185-1245 (1989).
10. L. N. Labzovskii (ed.), Physics on the Verge of New Discoveries [in Russian], Izd. LGU, St. Petersburg (1990).
11. B. A. Grigor'ev, Pulsed Heating by Radiation Part I. Characteristics of Pulsed Irradiation and Radiant Heating [in Russian], Nauka, Moscow (1984).
12. G. Neugebauer and R. Meinel, Phys. Lett., A100, 467-470 (1984).

CREATION OF EXCITED LEPTONS IN COLLIDING ELECTRON—POSITRON BEAMS

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The generalized model assuming the existence of internal substructures in intermediate (virtual) leptons is used to calculate the differential cross section of the electroweak annihilation of unpolarized, longitudinally polarized and transversely polarized electrons and positrons with the formation of a lepton—antilepton pair and a photon in the final state. It is shown that there is a real possibility of experimental investigation of the internal substructure of particles (if such a structure does exist at all) in colliding beams with particle energies $E \geq 100$ BeV. The parameters of the model under consideration are calculated.

In view of the considerable progress made in the field of constructing high-energy (especially colliding-beam) accelerators, it will soon become possible to carry out a more precise verification of the standard model of electroweak interaction and to study its possible generalization based on an extension of Higgs' and lepton sectors of the theory, introduction of right currents and currents of the second kind, association of the electromagnetic properties and mass to different types of neutrinos, introduction of anomalous particle interactions, etc.

The increased energy of particle beams in the proposed experiments on new-generation accelerators will perhaps require the introduction of a new physical theory which may differ significantly from the standard $SU(2) \times U(1)$ electroweak theory of Glashow, Weinberg, and Salam [1-3] which has been confirmed experimentally at present. The most natural way of leaving the framework of the standard model is to introduce an internal substructure of weak gauge bosons $W^{\pm}$, Z^0, and (or) the fundamental fermions, i.e., leptons and quarks. Indeed, theoretical attempts to generalize the standard theory in this direction were made earlier also [4-6] in order to explain the observed anomalies in the emission of $l\bar{l}\gamma$, formed during the annihilation of an e^+e^- pair through an intermediate Z^0-boson or a γ-quantum.

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