Lévy processes and fluctuation theory

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This course is an introduction to the theory of Lévy processes, and in particular their fluctuations. We will cover: the definition and construction of Lévy processes; some basic distributional properties; local times and excursions of Markov processes (sketch); fluctuation theory of Lévy processes.

The beginning of the course is heavily based on Kyprianou [Kyp14]. The fluctuation theory mostly follows Bertoin [Ber96].

1. Lévy processes

Let us begin by recalling the definition of two familiar processes, a Brownian motion and a Poisson process.

A real-valued process $B = \{B_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion if the following hold:

- The paths of $B$ are $\mathbb{P}$-almost surely continuous.
- $\mathbb{P}(B_0 = 0) = 1$.
- For $0 \leq s \leq t$, $B_t - B_s$ is equal in distribution to $B_{t-s}$.
- For $0 \leq s \leq t$, $B_t - B_s$ is independent of $\{B_u : u \leq s\}$.
- For each $t > 0$, $B_t$ is equal in distribution to a normal random variable with variance $t$.

A process valued on the non-negative integers $N = \{N_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:

- The paths of $N$ are $\mathbb{P}$-almost surely right continuous with left limits.
- $\mathbb{P}(N_0 = 0) = 1$.
- For $0 \leq s \leq t$, $N_t - N_s$ is equal in distribution to $N_{t-s}$.
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- For $0 \leq s \leq t$, $N_t - N_s$ is independent of $\{N_u : u \leq s\}$.
- For each $t > 0$, $N_t$ is equal in distribution to a Poisson random variable with parameter $\lambda t$.

There are clearly a lot of similarities between these two definitions, and these lead us to the definition of a Lévy process.

**Definition 1.1.** A process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:

- [Càdlàg paths] The paths of $X$ are $\mathbb{P}$-almost surely right continuous with left limits.
- [Initial condition] $\mathbb{P}(X_0 = 0) = 1$.
- [Stationary increments] For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to $X_t - s$.
- [Independent increments] For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

Certain facts follow immediately from this definition. For instance, the property of independent increments implies that any Lévy process is a spatially-homogeneous Markov process; indeed, Lévy processes even possess the strong Markov property, but we will postpone discussion of this.

The class of Lévy processes is rather rich, and the reader may already know plenty of examples which fall into it. To better appreciate this, we first need to discuss infinitely divisible distributions, which are in bijection with Lévy processes.

**Definition 1.2.** We say that a real-valued random variable $U$ possesses an infinitely divisible distribution if for each $n = 1, 2, \ldots$ there exists a collection of i.i.d. random variables $U_{1,n}, \ldots, U_{n,n}$ such that

$$U \overset{d}{=} U_{1,n} + \cdots + U_{n,n},$$

where $\overset{d}{=} \text{ represents equality in distribution}.$

The fundamental theorem about infinitely divisible distributions is formulated in terms of characteristic exponents, which we now define. Recall that, if $U$ is a random variable, then its characteristic function (Fourier transform) $h: \mathbb{R} \to \mathbb{C}$ is given by

$$h(\theta) = \mathbb{E}[e^{i\theta U}], \quad \theta \in \mathbb{R}.$$ 

We will begin with a technical lemma.

**Lemma 1.3** (and definition). Let $U$ be an infinitely divisible random variable, and $h$ its characteristic function. Then:

(i) The function $h$ is continuous and non-zero, and $h(0) = 1$. 

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(ii) There exists a unique continuous function \( f : \mathbb{R} \to \mathbb{C} \), such that \( e^{f(\theta)} = h(\theta) \) for all \( \theta \in \mathbb{R} \) and \( f(0) = 0 \). We will denote the function \( f \) by \( \log h \), and refer to it as the characteristic exponent of \( U \).

We make some remarks on the above lemma. Firstly, the converse of (i) is not true, as is shown by the example of the Binomial distribution. Secondly, the function \( \log h \) in (ii) is not formed by composing \( h \) with some branch of the logarithm function (i.e., by cutting the complex plane), and one should be careful not to assume that \( h(\theta_1) = h(\theta_2) \) implies that \( \log h(\theta_1) = \log h(\theta_2) \).

We do not offer a proof of Lemma 1.3, but one may be found in [Sat99, Lemma 7.6]. It is perhaps not too surprising—one may visualise the continuous path drawn out by \( h \) as lifting to a continuous path on the Riemann surface of the log function.

It’s worth remarking on the following fact, whose proof is immediate. Suppose that \( U_1 \) and \( U_2 \) are two independent infinitely divisible random variables with characteristic functions \( h_1 \) and \( h_2 \) and characteristic exponents \( f_1 \) and \( f_2 \). Then, the random variable \( U = U_1 + U_2 \) has an infinitely divisible distribution, and if we denote by \( h \) and \( f \) the characteristic function and exponent of \( U \), then \( h = h_1h_2 \) and \( f = f_1 + f_2 \).

The relationship between infinitely divisible distributions and Lévy processes is given by the following lemma, of which we will only prove one direction. The remaining direction is given in [Sat99, Theorem 7.10].

**Lemma 1.4.** If \( X \) is a Lévy process, then for any \( t \geq 0 \), the random variable \( X_t \) possesses an infinitely divisible distribution. Conversely, if \( U \) is an infinitely divisible random variable, then there exists a (unique in distribution) Lévy process \( X \) such that \( X_1 \sim U \).

**Proof (of easy direction).** Let \( X \) be a Lévy process and \( t \geq 0 \). Then, for any \( n = 1, 2, \ldots, \)

\[
X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}),
\]

and the summands on the right-hand side are i.i.d. by the properties of stationary independent increments. Thus, \( X_t \) possesses an infinitely divisible distribution. \( \square \)

We now examine the characteristic exponents of the one-dimensional distributions of Lévy processes. Take a Lévy process \( X \), and write \( \Psi \) for the characteristic exponent of \( X_t \), that is,

\[
e^{\Psi(t)} = \mathbb{E}[e^{i\theta X_t}], \quad \theta \in \mathbb{R}.
\]

Applying (1.1) we obtain that for \( m, n \in \mathbb{N} \),

\[
m\Psi_1(\theta) = \Psi_m(\theta) = n\Psi_{m/n}(\theta),
\]

which is to say that for rational \( t > 0 \),

\[
\Psi_t(\theta) = t\Psi_1(\theta).
\]
We now wish to extend this to all $t > 0$. Recall that $X$ is almost surely right-continuous; by applying bounded convergence in (1.2), the same holds for $t \mapsto e^{\Psi(t)}$, for each fixed $\theta \in \mathbb{R}$. Thus, $e^{\Psi(t)} = e^{t \Psi(1)}$ holds for all real $t > 0$ and $\theta \in \mathbb{R}$, that is, 
$$E[e^{i \theta X_t}] = e^{t \Psi(\theta)}, \quad \theta \in \mathbb{R},$$
where $\Psi := \Psi_1$ is the characteristic exponent of $X_1$. This leads us to:

**Definition 1.5.** Let $X$ be a Lévy process. We refer to $\Psi$, the characteristic exponent of $X_1$, as the **characteristic exponent of the Lévy process $X$.**

The main representation theorem for Lévy processes, which may of course be viewed as a theorem about infinitely divisible distributions, is the following.

**Theorem 1.6 (Lévy–Khintchine formula).** Let $X$ be a Lévy process with characteristic exponent $\Psi$. Then, there exist (unique) $a \in \mathbb{R}$, $\sigma \geq 0$, and a measure $\Pi$, with no atom at zero, satisfying $\int_{\mathbb{R}} 1 \wedge x^2 \Pi(dx) < \infty$, such that
$$\Psi(\theta) = ia\theta - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i \theta x} - 1 - i \theta x 1_{[-1,1]}(x)) \Pi(dx). \quad (1.4)$$
Conversely, given any admissible choices $(a, \sigma, \Pi)$, there exists a Lévy process $X$ with characteristic exponent given by the above formula.

The first part of this theorem is rather technical, and we will not address it; see [Sat99, Theorem 8.1]. The second part amounts to constructing a Lévy process out of its characteristics, and we will prove it shortly as part of the more detailed Lévy–Itô decomposition.

The names and approximate meanings of the characteristics in the Lévy–Khintchine formula are as follows. The term $a$ is known as the **centre** of $X$, and incorporates any deterministic drift term; $\sigma$ is the **Gaussian coefficient**, and represents the volatility of a Brownian component, if present; and the so-called **Lévy measure** $\Pi$ represents the size and intensity of the jumps of $X$.

### 2. Examples of Lévy processes

As part of our thesis on the variety of Lévy processes, we list a few examples. Several of these will be used later in the text; in particular, the characteristic exponents of compound Poisson processes and linear Brownian motion will be essential in §3, and stable processes will remain a prominent example throughout.

#### 2.1. Poisson processes

For each $\lambda > 0$, denote by $\mu_\lambda$ the Poisson distribution, that is, a measure concentrated on $k = 0, 1, 2, \ldots$ such that $\mu_\lambda(\{k\}) = e^{-\lambda} \lambda^k / k!$. The characteristic function of this distribution satisfies
$$\sum_{k \geq 0} e^{i \theta k} \mu_\lambda(\{k\}) = e^{\lambda(e^{i \theta} - 1)} = \left[ e^{\frac{\lambda}{n}(e^{i \theta} - 1)} \right]^n.$$
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The right-hand side is the characteristic function of the sum of \( n \) independent Poisson distributions, each with parameter \( \lambda/n \), from which it follows that \( \mu_\lambda \) is infinitely divisible. In the Lévy–Khintchine decomposition we see that \( \sigma = 0; \Pi = \lambda \delta_1 \), the Dirac measure supported on \{1\}; and \( a = -\lambda \).

A Poisson process \( \{N_t : t \geq 0\} \) is a Lévy process whose distribution at time \( t > 0 \), is Poisson with parameter \( \lambda t \). From the above calculations we have

\[
\mathbb{E}(e^{i\theta N_t}) = e^{\lambda t(e^{i\theta} - 1)},
\]

and hence its characteristic exponent is given by \( \Psi(\theta) = \lambda(e^{i\theta} - 1) \) for \( \theta \in \mathbb{R} \).

2.2. Compound Poisson processes

Suppose now that \( N \) is a Poisson random variable with parameter \( \lambda > 0 \) and that \( \{\xi_i : i \geq 1\} \) is an sequence of i.i.d. random variables (independent of \( N \)), whose common
law $F$ having no atom at zero. By first conditioning on $N$, we have for $\theta \in \mathbb{R}$,

$$
\mathbb{E}(e^{i\theta \sum_{i=1}^{N} \xi_i}) = \sum_{n \geq 0} \mathbb{E}(e^{i\theta \sum_{i=1}^{n} \xi_i}) e^{-\lambda n} \frac{n!}{n!}
= \sum_{n \geq 0} \left( \int_{\mathbb{R}} e^{i\theta x} F(dx) \right)^n e^{-\lambda n} \frac{n!}{n!}
= e^{\lambda \int_{\mathbb{R}} (e^{i\theta x} - 1) F(dx)}. \tag{2.1}
$$

We see from (2.1) that distributions of the form $\sum_{i=1}^{N} \xi_i$ are infinitely divisible with triple $a = -\lambda \int_{|x|<1} x F(dx)$, $\sigma = 0$ and $\Pi(dx) = \lambda F(dx)$. When $F$ is simply a Dirac mass at the point 1, we have the Poisson process considered in the previous section.

Suppose now that $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity $\lambda$, independent of the sequence $\{\xi_i : i \geq 1\}$. We say that the process $\{X_t : t \geq 0\}$ defined by

$$
X_t = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,
$$

is a **compound Poisson process** with jump distribution $F$. If we write

$$
X_t = X_s + \sum_{i=N_s+1}^{N_t} \xi_i,
$$

and recall that $N$ has independent stationary increments, we see that $X_t$ is the sum of $X_s$ and an independent copy of $X_{t-s}$. Right continuity and left limits of the process $N$ also ensure right continuity and left limits of $X$. Thus compound Poisson processes are Lévy processes. From the calculations in the previous paragraph, for each $t \geq 0$ we may substitute $N_t$ for the variable $N$ to discover that the Lévy–Khintchine formula for a compound Poisson process takes the form $\Psi(\theta) = \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1) F(dx)$. Note in particular that the Lévy measure of a compound Poisson process is always finite with total mass equal to the rate $\lambda$ of the underlying process $N$.

If a drift of rate $c \in \mathbb{R}$ is added to a compound Poisson process so that now

$$
X_t = \sum_{i=1}^{N_t} \xi_i + ct, \quad t \geq 0
$$

then it is straightforward to see that the resulting process is again a Lévy process. The associated infinitely divisible distribution is nothing more than a shifted compound Poisson distribution with shift $c$. The Lévy-Khintchine exponent is given by

$$
\Psi(\theta) = \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1) F(dx) + ic\theta.
$$

If further the shift is chosen to centre the compound Poisson distribution, that is $c = \lambda \int_{\mathbb{R}} x F(dx)$, then

$$
\Psi(\theta) = \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) F(dx), \tag{2.2}
$$

and the resulting process has mean zero at all times. We call the corresponding Lévy process a **compensated compound Poisson process**.
2.3. Linear Brownian motion

Consider the probability law

\[ \mu_{s,\gamma}(dx) := \frac{1}{\sqrt{2\pi s^2}} e^{-(x-\gamma)^2/2s^2} \, dx, \quad x \in \mathbb{R}, \]

where \( \gamma \in \mathbb{R} \) and \( s > 0 \). This is the Gaussian distribution with mean \( \gamma \) and variance \( s^2 \).

It is well known that

\[
\int_{\mathbb{R}} e^{i\theta x} \mu_{s,\gamma}(dx) = e^{-\frac{1}{2}s^2\theta^2 + i\theta \gamma} = \left[ e^{-\frac{1}{2}(\frac{\theta}{\sqrt{n}})^2 + i\theta \gamma} \right]^n,
\]
from which is follows that $\mu_{a,\gamma}$ is an infinitely divisible distribution, this time with $a = -\gamma$, $\sigma = s$ and $\Pi = 0$.

The characteristic exponent $\Psi(\theta) = -s^2\theta^2/2+i\theta\gamma$ is that of a linear Brownian motion, namely

$$X_t := sB_t + \gamma t, \quad t \geq 0,$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion. Since (as we have already discussed) $B$ has stationary independent increments and continuous paths, it is simple to deduce that $X$ does as well, making it another example of a Lévy process.

2.4. Gamma processes

Let $\alpha, \beta > 0$ and define the probability measure

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} dx$$

supported on $[0, \infty)$. This is called the Gamma($\alpha, \beta$) distribution; note that when $\beta = 1$ this is the exponential distribution.

It may be shown that $\mu_{\alpha,\beta}$ is infinitely divisible, and its characteristics can be calculated using the following lemma.

**Lemma 2.1** (Frullani integral). For all $\alpha, \beta > 0$ and $z \in \mathbb{C}$ such that $\text{Re } z \leq 0$ we have

$$(1 - z/\alpha)^{-\beta} = \exp \left\{ - \int_0^\infty (1 - e^{zx}) \beta x^{-1} e^{-\alpha x} dx \right\}.$$

(The proofs of these statements form Exercises 5 and 6.)

The corresponding Lévy process is called a gamma process.

Suppose now that $X = \{X_t : t \geq 0\}$ is a gamma process. Stationary independent increments tell us that for all $0 \leq s < t < \infty$, $X_t = X_s + \tilde{X}_{t-s}$ where $\tilde{X}_{t-s}$ is an independent copy of $X_{t-s}$. This increment is strictly positive with probability one, since it is gamma distributed; that is, $X_t > X_s$ almost surely. Hence a gamma process is an example of a Lévy process with almost surely non-decreasing paths. Another such Lévy process is a compound Poisson process where the jump distribution $F$ is concentrated on $(0, \infty)$. However, there are two key differences between these processes. Firstly, the Lévy measure of a gamma process has infinite total mass, unlike the Lévy measure of a compound Poisson process, whose mass is necessarily finite (and equal to the arrival rate of jumps). Secondly, while a compound Poisson process with positive jumps does have paths which are almost surely non-decreasing, it does not have paths that are almost surely strictly increasing.

Lévy processes whose paths are almost surely non-decreasing (or simply non-decreasing for short) are called subordinators, and we will discuss them again in §4.6.
Figure 5: Graphs of strictly stable processes for varying choices of $\alpha$. Plots (a) through (c) show symmetric stable processes, while (d) shows a stable process which only jumps upwards. Note the scale in plot (a).
2.5. Stable processes

The reader may already be aware of the so-called Brownian scaling property, which states that if \( B \) is a Brownian motion and \( c > 0 \), then the scaled stochastic process \((cB_{t/c^2} : t \geq 0)\) is equal in distribution to \( B \). It is interesting to ask which Lévy processes satisfy a similar property, and these are the subject of this section. A Lévy process \( X \) is stable if, there exist \( \alpha \in (0,2] \) and \( f : (0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) such that for any \( c > 0 \)

\[
(cX_{tc^{-\alpha}} + f(c,t) : t \geq 0) \overset{d}{=} X
\]

and strictly stable if there exists \( \alpha \in (0,2] \) such that for any \( c > 0 \)

\[
(cX_{tc^{-\alpha}} : t \geq 0) \overset{d}{=} X. \tag{\alpha\text{-ss)}
\]

Clearly there is a lot embedded in the above statements. Let’s go back to basics and give a more typical presentation of these classes of processes.

Stable processes are those Lévy processes whose one-dimensional distributions are stable distributions. A random variable \( Y \) is said to have a stable distribution if, for any \( n \geq 1 \), it observes the distributional equality

\[
Y_1 + \cdots + Y_n \overset{d}{=} a_n Y + b_n, \tag{2.3}
\]

where \( Y_1, \ldots, Y_n \) are independent copies of \( Y \), \( a_n > 0 \) and \( b_n \in \mathbb{R} \). By subtracting \( b_n/n \) from each of the terms on the left-hand side of (2.3) one sees in particular that this definition implies that any stable random variable is infinitely divisible. It turns out that necessarily \( a_n = n^{1/\alpha} \) for some \( \alpha \in (0,2] \) (see [Fel71, §VI.1]), and we refer to the parameter \( \alpha \) as the index of \( Y \). A smaller class of distributions is that of strictly stable distributions. A random variable \( Y \) is said to have a strictly stable distribution if it observes (2.3) but with \( b_n = 0 \). In that case, we necessarily have

\[
Y_1 + \cdots + Y_n \overset{d}{=} n^{1/\alpha} Y. \tag{2.4}
\]

The case \( \alpha = 2 \) corresponds to zero mean Gaussian random variables and has essentially been dealt with in §2.3, so we exclude it in the remainder of the discussion.

A stable random variable \( Y \) with index \( \alpha \in (0,1) \cup (1,2) \) has characteristic exponent of the form

\[
\Psi (\theta) = -c|\theta|^\alpha \left( 1 - i\beta \tan \frac{\pi \alpha}{2} \text{sgn} \theta \right) + i\theta \eta, \tag{2.5}
\]

where \( \beta \in [-1,1], \eta \in \mathbb{R} \) and \( c > 0 \); \( Y \) is strictly stable if and only if \( \eta = 0 \).

A stable random variable with index \( \alpha = 1 \) has characteristic exponent of the form

\[
\Psi (\theta) = -c|\theta|\left( 1 + i\beta \frac{2}{\pi} \text{sgn} \theta \log|\theta| \right) + i\theta \eta, \tag{2.6}
\]

where \( \beta \in [-1,1], \eta \in \mathbb{R} \) and \( c > 0 \); \( Y \) is strictly stable if and only if \( \beta = 0 \).

(The parameters in (2.5) and (2.6) are unique, and every choice of them gives the characteristic exponent of a stable distribution.)
Since we have said that stable random variables are infinitely divisible, let us make the connection with the Lévy–Khintchine formula. We define $\sigma = 0$ and

$$\Pi(dx) = \begin{cases} 
   c_1 x^{1-\alpha} dx, & x \in (0, \infty), \\
   c_2 |x|^{1-\alpha} dx, & x \in (-\infty, 0), 
\end{cases}$$

(2.7)

where $c_1, c_2 \geq 0$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$ if $\alpha \in (0, 1) \cup (1, 2)$ and $c_1 = c_2$ if $\alpha = 1$. The relationship between $c$ and $c_1, c_2$ is given by $c = -\Gamma(-\alpha) \cos(\pi \alpha/2) (c_1 + c_2)$, if $\alpha \in (0, 1) \cup (1, 2)$; and $c = \pi/2 (c_1 + c_2)$, if $\alpha = 1$. (See [Sat99, proof of Theorem 4.10].)

The choice of $a \in \mathbb{R}$ also depends on $\alpha$.

Unlike the previous examples, the distributions that lie behind these characteristic exponents are heavy tailed, in that, if $Y$ is a stable random variable with index $\alpha$, then

$$E[|Y|^\gamma] = \gamma \int_0^\infty t^{\gamma-1} P(|Y| > t) \, dt \begin{cases} < \infty, & \gamma < \alpha, \\
= \infty, & \gamma \geq \alpha. 
\end{cases}$$

In particular, if $Y$ is not Gaussian, then it does not have finite variance; and if $\alpha \leq 1$, then $Y$ has no mean. The value of the parameter $\beta$ gives control over the asymmetry in the Lévy measure.

**2.6. Other examples, and concluding remarks**

There are many more known examples of infinitely divisible distributions (and hence Lévy processes). Of the many known proofs of infinitely divisibility for specific distributions, most of them are non-trivial, often requiring intimate knowledge of special functions. A brief list of such distributions might include generalised inverse Gaussian, variance gamma, truncated stable, tempered stable (CGMY), generalised hyperbolic, Meixner, Pareto, $F$-distributions, Gumbel, Weibull, log-normal, Student $t$-distributions, Lamperti stable, and $\beta$- and $\theta$-processes (see [Kyp14, §1.2.7] or [Sat99, Remark 8.2] for an extensive list of references.)

Despite being able to identify a large number of infinitely divisible distributions and hence associated Lévy processes, it is not clear at this point what the paths of Lévy processes look like. The task of giving a mathematically precise account of this lies ahead in the next section. In the meantime let us make the following informal remarks concerning paths of Lévy processes.

Exercise 2 shows that a linear combination of a finite number of independent Lévy processes is again a Lévy process. It turns out that one may consider any Lévy process as an independent sum of a Brownian motion with drift and a countable number of independent compound Poisson processes with different jump rates, jump distributions and drifts. The superposition occurs in such a way that the resulting path remains almost surely finite at all times and, for each $\varepsilon > 0$, the process experiences at most a countably infinite number of jumps of magnitude $\varepsilon$ or less with probability one and an almost surely finite number of jumps of magnitude greater than $\varepsilon$, over all fixed finite time intervals. If in the latter description there is always an almost surely finite number
of jumps over each fixed time interval then it is necessary and sufficient that one has
the linear independent combination of a Brownian motion with drift and a compound
Poisson process. Depending on the underlying structure of the jumps and the presence of
a Brownian motion in the described linear combination, a Lévy process will either have
paths of bounded variation on all finite time intervals or paths of unbounded variation
on all finite time intervals.

We include some computer simulations to give a rough sense of how the paths of Lévy
processes look. Figures 1 and 2 depict the paths of Poisson process and a compound
Poisson process, respectively. Figures 3 and 4 show the paths of a Brownian motion
and the independent sum of a Brownian motion and a compound Poisson process, re-
spectively. Figure 5 depicts four sample paths of strictly stable processes with different
parameter values. All stable processes experience an infinite number of jumps over any
finite time horizon, but when \( \alpha < 1 \), the paths are almost surely of bounded variation,
whereas when \( \alpha \geq 1 \), they are of unbounded variation. The reader is cautioned however
that, ultimately, computer simulations can only depict a finite number of jumps in any
given path. Figures 1–4 were very kindly produced by Antonis Papapantoleon.

### 3. Lévy–Itô decomposition

We will prove the converse part of Theorem 1.6, the Lévy–Khintchine formula, by means
of constructing a Lévy process with a given Lévy triple \((a, \sigma, \Pi)\). To this end, let us first
recall, from the examples we saw in the previous section, that the characteristic exponent

\[
i b \theta - \frac{1}{2} \sigma^2 \theta^2
\]

belongs to a linear Brownian motion with volatility \( \sigma \) and drift \( b \), while the characteristic exponent

\[
\lambda \int_{\mathbb{R}} (e^{i \theta x} - 1) \mu(dx)
\]

belongs to a compound Poisson process with jump distribution \( \mu \) and jump rate \( \lambda \).

Recall furthermore that the sum of two independent Lévy processes \( X \) and \( Y \) with
characteristic exponents \( \Psi \) and \( \Phi \) is a Lévy process with characteristic exponent \( \Psi + \Phi \);
see Exercise 2.

With this in mind, let \( \Psi \) be the function given in Theorem 1.6 and we give the
following decomposition, which appears to make sense at least formally.

Let \( A_\varepsilon = [-1, 1] \setminus (-\varepsilon, \varepsilon) \) be a closed annulus around zero with inner radius \( \varepsilon \). We have:

\[
\Psi(\theta) = i a \theta - \frac{1}{2} \sigma^2 \theta^2 \\
+ \Pi(\mathbb{R} \setminus [-1, 1]) \int_{\mathbb{R} \setminus [-1, 1]} (e^{i \theta x} - 1) \frac{\Pi(dx)}{\Pi(\mathbb{R} \setminus [-1, 1])} \\
+ \lim_{\varepsilon \downarrow 0} \left[ \int_{A_\varepsilon} (e^{i \theta x} - 1) \Pi(dx) - i \theta \int_{A_\varepsilon} x \Pi(dx) \right].
\] (3.1)
3. Lévy–Itô decomposition

We therefore claim that a Lévy process with triple \((a, \sigma, \Pi)\) may be obtained by summing three independent simpler Lévy processes: the first term represents a linear Brownian motion; the second term represents a compound Poisson process of ‘large’ jumps (note that this process need not have a mean); and the third term represents a limit of compound Poisson processes compensated via deterministic drift to have zero mean. It is far from obvious that the final limit gives a characteristic exponent of an infinitely divisible random variable, or that this has meaning in terms of convergence of stochastic processes; this will be the main content of the proof.

3.1. Poisson random measures

It is clear by now that compound Poisson processes will play an important role in the proof of the Lévy–Itô decomposition. We will therefore make a short digression into the theory of Poisson random measures (which are closely related to Poisson point processes, or simply Poisson processes) in general state spaces. This theory will also be crucial to the excursion theory developed toward the end of the course, so it is worth discussing it in some detail.

To motivate our development, and provide a simple example, consider a compound Poisson process with drift, \(X\), which has jump distribution \(\mu\); that is,

\[
X_t = \delta t + \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,
\]

where \(N\) is a Poisson process with intensity \(\lambda\), and the \(\xi_i\) are i.i.d. random variables with common law \(\mu\). Let us denote the arrival times of the Poisson process by \(T_i\), for \(i \geq 1\), and write \(L = \{(T_i, \xi_i) : i \geq 1\}\) for the (random) set of ‘time-space jump points’.

A sample path of \(X\) is visualised in Figure 6, together with the set \(L\) of its jumps.

Define a measure \(N\) on the Borel sets of \([0, \infty) \times \mathbb{R}\) by

\[
N(A) = \#(L \cap A) = \sum_{i=1}^{\infty} 1_A(T_i, \xi_i). \tag{3.2}
\]

\(N\) is a random measure, and the random point set \(L\) is its support. In fact, \(N\) will be our first example of a Poisson random measure; here is the definition.

**Definition 3.1.** Let \((S, \mathcal{S}, \eta)\) be a \(\sigma\)-finite measure space, and \((\Omega, \mathcal{F}, P)\) a probability space. A function \(N: \Omega \times S \to \{0, 1, 2, \ldots, \infty\}\) is called a discrete random measure on \(S\) if, for every \(A \in \mathcal{S}\), \(N(\cdot, A)\) is a random variable; and for every \(\omega \in \Omega\), \(N(\omega, \cdot)\) is a measure. For convenience, we will usually write a random measure \(N\) without the first argument.

A random measure \(N\) is called a Poisson random measure on \(S\) with intensity measure (or mean measure) \(\eta\), if the following are satisfied:

(i) if \(A_1, \ldots, A_n\) are pairwise disjoint elements of \(\mathcal{S}\), then \(N(A_1), \ldots, N(A_n)\) are independent;
Figure 6: The path of the compound Poisson process with drift, $X$ (above), the set of points $L$ generated by its jumps (below), and an evaluation of the Poisson random measure $N$ (below, dashed shape).
(ii) for each $A \in \mathcal{S}$, $N(A)$ is distributed as a Poisson random variable with parameter $\eta(A)$.

With respect to the second point, we consider a Poisson distribution with parameter 0 (resp., $\infty$) to be equal to 0 (resp., $\infty$) with probability one.

(We give notice here that, in cases where $\mathcal{S}$ is a topological space, we will usually assume that $\mathcal{S}$ is its Borel $\sigma$-algebra; the general theorems we give are valid for any measurable structure, however.)

Our first example is as follows.

**Lemma 3.2.** The measure $N$ defined in (3.2) is a Poisson random measure on $[0, \infty) \times \mathbb{R}$ with intensity measure $\lambda \text{Leb} \times \mu$.

We do not offer a proof of this result since it would be so similar to that of the forthcoming Theorem 3.3, which we prove in full.

**Theorem 3.3.** Let $(S, \mathcal{S}, \eta)$ be a $\sigma$-finite measure space. Then there exists a Poisson random measure $N$ on $S$ with intensity measure $\eta$.

**Proof.** Suppose first that $\eta$ is finite. Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting independent random variables $\mathbb{N}$ and $\xi_i$, $i \geq 1$ such that $\mathbb{N}$ has a Poisson distribution with parameter $\eta(S)$, and

$$P(\xi_i \in A) = \eta(A)/\eta(S), \quad A \in \mathcal{S}, \quad i \geq 1.$$  

Then, let

$$N(A) = \sum_{i=1}^{\mathbb{N}} 1_A(\xi_i), \quad A \in \mathcal{S}. \quad (3.3)$$

This $N$ is certainly a random measure on $(S, \mathcal{S})$. Take disjoint measurable sets $A_1, \ldots, A_k$; we must show that these are independent and each $N(A_i)$ is Poisson distributed.

Fixing a set $A \in \mathcal{S}$, under the conditional law $P(\cdot \mid N = n)$, the random variable $N(A)$ is the number of ‘independently thrown’ points $\{\xi_i : i = 1, \ldots, n\}$ which happen to land inside $A$; that is, it is Binomial$(n, p)$-distributed, where $p = \eta(A)/\eta(S)$. The generalisation of this is to say that, again under this conditional law, the tuple $(N(A_1), \ldots, N(A_k))$ has multinomial distribution, in that

$$P(N(A_1) = n_1, \ldots, N(A_k) = n_k \mid N = n) = \frac{n!}{n_0! n_1! \cdots n_k!} \prod_{i=0}^{k} \left( \frac{\eta(A_i)}{\eta(S)} \right)^{n_i},$$

where $n_0 = n - (n_1 + \cdots + n_k)$ and $A_0 = S \setminus \bigcup_{i=1}^{k} A_i$. We then sum over $n$, finding

$$P(N(A_1) = n_1, \ldots, N(A_k) = n_k)$$

$$= \sum_{n \geq n_1 + \cdots + n_k} e^{-\eta(S)} \frac{(\eta(S))^{n}}{n!} \sum_{n_0}^{n} \frac{n!}{n_0! n_1! \cdots n_k!} \prod_{i=0}^{k} \left( \frac{\eta(A_i)}{\eta(S)} \right)^{n_i}$$

$$= \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{(\eta(A_i))^{n_i}}{n_i!} \sum_{n_{i+1} + \cdots + n_k} e^{-\eta(A_0)} \frac{\eta(A_0)^{n - (n_1 + \cdots + n_k)}}{(n - (n_1 + \cdots + n_k))!}$$

$$= \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{(\eta(A_i))^{n_i}}{n_i!}.$$
This shows that the $N(A_i)$ are mutually independent and each is \(\text{Poisson}(\eta(A_i))\) distributed.

Suppose instead that $\eta(S) = \infty$. Since we know that $\eta$ is $\sigma$-finite, we can break $S$ up into disjoint subsets $S_1, S_2, \ldots$ such that $S = \bigcup_{i=1}^{\infty} S_i$ and $\eta(S_i) < \infty$. We then set $\eta_i = \eta(\cdot \cap S_i)$, which is a finite measure on $S_i$ (with the $\sigma$-algebra induced from $S$). These measures thus induce Poisson random measures $N_i$ on $S_i$, which we place on one probability space so that they are independent, and we then define

$$N(A) = \sum_{i=1}^{\infty} N_i(A \cap S_i) \quad A \in \mathcal{S}.$$ 

We claim that $N$ is a random measure on $S$ with intensity $\eta$; this is not difficult to show. We must also show that it is a Poisson random measure. Take $A \in \mathcal{S}$; then the random variables $N_i(A)$ are independent and Poisson distributed. Then $N(A) = \sum_{i=1}^{\infty} N_i(A)$, so by the additive property of Poisson distributions, this random variable is Poisson distributed with rate $\sum_{i=1}^{\infty} \eta(A \cap S_i) = \eta(A)$. Now take disjoint sets $A_1, \ldots, A_k$. We need to show that $N(A_1), \ldots, N(A_k)$ are independent, but this follows because the random variables in $\{N_i(A_j) : i = 1, 2, \ldots ; j = 1, \ldots, k\}$ are independent.

We remark that if $\eta$ is also diffuse (that is, for all $x \in S$, $\eta\{x\} = 0$) then the measure $N$ may be constructed as the counting measure of an at most countable random set of points (i.e., the support of $N$); this is already suggested in the above proof by (3.3). Such a random set is called a Poisson point process (or simply a Poisson process). For a detailed exposition of these processes, see Kingman [Kin93]. On the other hand, we warn that when some authors refer to a ‘Poisson point process on $S$’, they are referring to a random subset of $[0, \infty) \times S$ which is the support of a Poisson random measure on $[0, \infty) \times S$ with intensity $\text{Leb} \times \eta$, for some $\sigma$-finite measure $\eta$ on $S$.

### 3.2. Functionals of Poisson random measures

In this section, we are interested in the law of integrals

$$\int_S f(x) \, N(dx),$$

for measurable functions $f : S \to \mathbb{R}$. When this integral has meaning (i.e., converges absolutely) and $\eta$ is diffuse, it is equal to the sum

$$\sum_{x \in L} f(x),$$

where $L$ is the support of the measure $N$. (When $\eta$ is not diffuse, one must modify the sum to include multiplicities, since $N$ will, with positive probability, give mass greater than 1 to a single point.)

**Theorem 3.4.** Let $f : S \to \mathbb{R}$ be a measurable function. Then:
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(i) The integral
\[ X_f := \int_S f(x) \, N(dx) \]
is almost surely absolutely convergent if and only if
\[ \int_S 1 \wedge |f(x)| \eta(dx) < \infty. \]

(ii) If \( X_f \) is a.s. absolutely convergent, then
\[ E[e^{i\beta X_f}] = \exp\left\{ -\int_S (1 - e^{i\beta f(x)}) \, \eta(dx) \right\}, \quad \beta \in \mathbb{R}. \] (3.4)

(iii) Furthermore, \( E|X_f| < \infty \) and
\[ E[X_f] = \int_S f(x) \, \eta(dx) \] (3.5)
provided that
\[ \int_S |f(x)| \, \eta(dx) < \infty, \]
and
\[ E[X_f^2] = \int_S (f(x))^2 \, \eta(dx) + \left( \int_S f(x) \, \eta(dx) \right)^2 < \infty \]
provided that
\[ \int_S |f(x)| \, \eta(dx) < \infty \quad \text{and} \quad \int_S (f(x))^2 \, \eta(dx) < \infty. \]

In the context of Poisson point processes, (3.4) is often called the exponential formula, and (3.5) the compensation formula. Both are occasionally called master formulas.

Using the theorem above, we have the following result, part of which is a converse to Lemma 3.2. (The remarks about filtrations may seem rather odd at this point—why not just make \((\mathcal{F}_t)_{t \geq 0}\) the natural filtration generated by \(X\)? But all will become clear in the next section, where we need a collection of martingales all adapted to the same filtration.)

**Lemma 3.5.** Suppose that \( N \) is a Poisson random measure on \([0, \infty) \times \mathbb{R} \), with intensity measure \( \text{Leb} \times \Pi \), where \( \Pi \) has no atom at zero. Define \( \hat{\mathcal{F}}_t = \sigma\{N([0,s] \times C) : s \leq t, C \text{ Borel}\} \), and let \((\mathcal{F}_t)_{t \geq 0}\) be the natural enlargement\(^1\) of \((\hat{\mathcal{F}}_t)_{t \geq 0}\). Let \( B \subset \mathbb{R} \) be a Borel subset of \( \mathbb{R} \) such that \( 0 < \Pi(B) < \infty \).

---

\(^1\)The natural enlargement of \((\hat{\mathcal{F}}_t)\) is given as follows. One first sets \( \hat{\mathcal{F}}_t = \sigma\{A \cup N : A \in \hat{\mathcal{F}}_t, N \subset M \text{ for some event } M \text{ s.t. } P(M) = 0\} \). This is called \(P\)-regularisation. One then defines \(\mathcal{F}_t = \bigcap_{s \geq t} \hat{\mathcal{F}}_s\). The filtration \(\mathcal{F}_t\) is now right-continuous and contains all \(P\)-nullsets.
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(i) The process

\[ X_t = \int_{[0,t] \times B} x \, N(du, dx), \quad t \geq 0, \]

is a compound Poisson process with rate \( \Pi(B) \) and jump distribution \( \Pi(\cdot \cap B)/\Pi(B) \). Indeed, for \( s < t \), \( X_t - X_s \) is even independent of \( \mathcal{F}_s \) - a so-called \( (\mathcal{F}_t)_{t \geq 0} \)-Lévy process.

(ii) If, in addition, \( \int_B |x| \, \Pi(dx) < \infty \), then the compensated Poisson process

\[ M_t = \int_{[0,t] \times B} x \, N(du, dx) - t \int_B x \, \Pi(dx), \quad t \geq 0, \]

is a zero-mean \( \mathcal{P} \)-martingale with respect to \( (\mathcal{F}_t)_{t \geq 0} \).

(iii) If furthermore \( \int_B x^2 \, \Pi(dx) < \infty \), then \( M \) is a square-integrable martingale, which is to say that \( E[M^2_t] < \infty \) for all \( t \geq 0 \); indeed,

\[ E[M^2_t] = t \int_B x^2 \, \Pi(dx), \quad t \geq 0. \]

Proof. (i) The fact that \( X_t \) is well-defined for all \( t \) follows from Theorem 3.4 (since \( \int_{[0,t] \times B} 1 \wedge |x| \, dt \, \Pi(dx) \leq t \Pi(B) < \infty \)). One may show that \( X \) has càdlàg paths by applying the dominated convergence theorem to functions of the form \( 1_{[0,t] \times B^c} \), path-by-path on a set of full measure.

Now, we calculate the increment from \( s \) to \( t \) to be

\[ X_t - X_s = \int_{(s,t] \times B} xN(du, dx), \quad 0 \leq s < t < \infty, \]

and this is independent of \( \mathcal{F}_s \) by the independence properties of Poisson random measures, since \( (s,t] \times B \) is disjoint from \( [0,s] \times C \) for any Borel \( C \). Finally, \( N \) restricted to \( [0,t-s] \times B \) has the same law as \( N \) restricted to \( (s,t] \times B \), which shows that the increment has the same law as \( X_{t-s} \).

Thus, \( X \) is certainly a Lévy process. We need to show that it is a compound Poisson process. Applying the exponential formula (3.4), we obtain

\[ E[e^{i\theta X_t}] = \exp\left\{ t \int_B (e^{i\theta x} - 1) \, \Pi(dx) \right\}, \quad t \geq 0, \quad \theta \in \mathbb{R}, \]

and from (2.1) this is precisely the characteristic function of a compound Poisson process with rate \( \Pi(B) \) and jump distribution \( \Pi(\cdot \cap B)/\Pi(B) \).

(ii) Under the conditions of this part of the lemma, it follows from Theorem 3.4(iii) that \( X_t \) is integrable and has expectation

\[ E[X_t] = t \int_B x \, \Pi(dx). \quad (3.6) \]
Therefore certainly $E[M_t] = 0$; we must also show it is a martingale. $M$ is definitely adapted to $(\mathcal{F}_t)_{t \geq 0}$, and $M_t$ is integrable for each $t$ because the same holds for $X_t$. To show the martingale property, let $s < t$ and calculate, using the Markov property of $X$,

$$E[M_t - M_s \mid \mathcal{F}_s] = E[M_{t-s}] = E\left[\int_{[0,t-s] \times B} xN(du, dx) - (t - s) \int_B x\Pi(dx)\right] = 0,$$

where the final equality follows from (3.6). (Notice that we have actually shown that every zero mean $(\mathcal{F}_t)_{t \geq 0}$-Lévy process is a martingale.)

(iii) Apply Theorem 3.4(iii) and perform a short computation. □

While we are here, let us remind ourselves of our goals. We have already seen, in (2.2), that the characteristic exponent of the process $M$ appearing in the previous lemma is given by:

$$\int_B (e^{i\theta x} - 1 - i\theta x) \Pi(dx), \quad \theta \in \mathbb{R}. \quad (3.7)$$

If $\Pi$ is a Lévy measure, i.e. $\int_{\mathbb{R}} 1 \wedge x^2 \Pi(dx) < \infty$, then one may choose for $B$ any of the annuli $A_\varepsilon = [-1, 1] \setminus (-\varepsilon, \varepsilon)$ considered at the beginning of the section and obtain a square-integrable martingale $M$ with characteristic exponent given by (3.7). These are precisely the limands in (3.1). However, the set $[-1, 1]$ itself is not an acceptable choice for $B$, because $\int_{[-1,1]} |x| \Pi(dx)$ may be infinite (for example, this is the case for the stable processes defined in §2.5). The aim of this section is to find a way to extend $B$ to $[-1, 1]$ via a limiting procedure; and the final point in the above lemma gives a hint about how we can do this.

3.3. Square-integrable martingales

The last lemma made reference to square-integrable martingales. The space of these processes has a nice structure which we now discuss. Let us first give (or refresh) some definitions.

We take as given a probability measure $P$ and a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the natural conditions. This means that $(\mathcal{F}_t)_{t \geq 0}$ is equal to its natural enlargement, which was described in footnote 1 on page 17; we reiterate the definition here: the natural enlargement of $(\mathcal{F}_t)$ is given as follows. One first sets $\hat{\mathcal{F}}_t = \sigma\{A \cup N : A \in \mathcal{F}_t, N \subset M \text{ for some event } M \text{ s.t. } P(M) = 0\}$. This is called $P$-regularisation. One then defines $\mathcal{F}_t = \cap_{s \geq t} \hat{\mathcal{F}}_s$. The filtration $\mathcal{F}_t$ is now right-continuous and contains all $P$-nullsets. For a discussion of this definition, see [Bic, Definition 1.3.38].

**Definition 3.6.** A martingale $M$ is called a square-integrable martingale if $E[M_t^2] < \infty$ for every $t \geq 0$. We will write $\mathcal{M}^2$ for the collection of zero-mean, càdlàg, square-integrable martingales. For each $t \geq 0$, we will also define $\mathcal{M}_t^2 = \{(M_s, 0 \leq s \leq t) : M \in \mathcal{M}^2\}$. 19
3. Lévy–Itô decomposition

For each \( t \geq 0 \), we define a seminorm on \( \mathcal{M}^2 \) by

\[
\| M \|_t = \sqrt{E[M_t^2]},
\]

and we also define the symbol

\[
\| M \| = \sum_{n=1}^{\infty} 2^{-n}(\| M \|_n \wedge 1), \quad M \in \mathcal{M}^2.
\]

Recall that a seminorm satisfies all of the conditions of a norm except for the requirement that \( \| M \|_t = 0 \) implies \( M = 0 \). In fact, \( \| \cdot \|_t \) is about as good a seminorm as you can get: if \( M, N \in \mathcal{M}^2 \) and \( \| M - N \|_t = 0 \), then \( M \) and \( N \) restricted to \( [0, t] \) are indistinguishable as stochastic processes. (This means that \( P(M_s = N_s \text{ for all } s \in [0, t]) = 1 \).)

We should also remark that

\[
\| M \|_s = \sqrt{E[M_s^2]} \leq \sqrt{E[M_t^2]} = \| M \|_t, \quad s < t,
\]

since \( M^2 \) is a submartingale; in particular, if \((M^{(n)})_{n \geq 1}\) converges under \( \| \cdot \|_t \), then it will also converge under \( \| \cdot \|_s \) when \( s < t \); and in particular, every sequence \((M^{(n)}_s)_{n \geq 1}\) converges in \( L^2(\Omega, \mathcal{F}_s, P) \).

We warn that \( \| \cdot \| \) does not define even a seminorm on \( \mathcal{M}^2 \); however, \( d(M, N) := \| M - N \| \) is a pseudometric, that is, a function satisfying all conditions of a metric except that \( d(M, N) = 0 \) implies \( M = N \). Again, \( d \) comes with the nice property that \( d(M, N) = 0 \) implies that \( M \) and \( N \) are indistinguishable as stochastic processes on \( [0, \infty) \). The point of \( \| \cdot \| \) is that \( d(M^{(n)}, M) \to 0 \) if and only if \( \| M^{(n)} - M \|_t \to 0 \) for every \( t \geq 0 \). (This follows from a standard result of metric space theory, as \( d \) is essentially a (Hilbert cube-like) product metric on \( \prod_{n \geq 1} \mathcal{M}^2 \).

Let us remind the reader that a square-integrable martingale is not necessarily bounded in \( L^2 \) (i.e., \( \sup_{t \geq 0} E[M_t^2] < \infty \)). The theory of so-called \( L^2 \)-martingales is rather neater than that of square-integrable martingales, but unfortunately the processes \( M \) appearing in Lemma 3.5 do not fall into that class!

The following proposition foreshadows how useful the space \( \mathcal{M}^2 \) will be.

**Proposition 3.7.** The metric space \( (\mathcal{M}^2, d) \) is complete.

**Proof.** Take a sequence \((X^{(n)})_{n \geq 1}\) which is Cauchy in \( \mathcal{M}^2 \). For each \( t \geq 0 \), the sequence \((X^{(n)}_t)_{n \geq 1}\) is Cauchy in \( L^2(\Omega, \mathcal{F}_t, P) \), by much the same reasoning as in the second remark above. Denote the \( L^2 \) limit by \( \hat{X}_t \), and define a stochastic process \( \hat{X} = (\hat{X}_t : t \geq 0) \). Certainly \( \hat{X} \) is adapted to \( (\mathcal{F}_t)_{t \geq 0} \), and \( E[\hat{X}_t^2] < \infty \) for every \( t \).

We must next show that \( \hat{X} \) is a martingale, and we do this directly using the definition of conditional expectation. (This argument is weirdly low-level, but I haven’t found a better way.) Fix \( s < t \) and pick \( A \in \mathcal{F}_s \). Firstly,

\[
|E[\mathbb{1}_A(X^{(n)}_s - \hat{X}_s)]| \leq \sqrt{E[\mathbb{1}_A]} \sqrt{E[(X^{(n)}_s - \hat{X}_s)^2]} \to 0,
\]
by the Cauchy–Schwarz inequality; and the same argument shows that \( E[\mathbb{1}_A(X_t^{(n)} - \tilde{X}_t)] \to 0 \) also. Now we calculate:

\[
E[\mathbb{1}_A(\tilde{X}_t - \tilde{X}_s)] = E[\mathbb{1}_A(\tilde{X}_t - X_t^{(n)})] + E[\mathbb{1}_A(X_t^{(n)} - X_s^{(n)})] + E[\mathbb{1}_A(X_s^{(n)} - \tilde{X}_s)].
\]

The first and third terms on the right-hand side have limit zero, as we established; and the second term is identically zero by the martingale property of the process \( X^{(n)} \). We have thus shown that \( E[\tilde{X}_t | \mathcal{F}_s] = \tilde{X}_s \), i.e., \( \tilde{X} \) is a martingale.

Since \( \tilde{X} \) is a martingale, it has a càdlàg modification \( X \) (see [Bic, Proposition 2.5.13].) (This means that \( P(X_t = \tilde{X}_t) = 1 \) for every \( t \).) We now have that \( X^{(n)} \) converges to \( X \) in \( \mathcal{M}^2 \). But this is automatic, because the convergence holds in \( \mathcal{M}_m^2 \) for every \( m \geq 1 \).

We now have everything in place for the following theorem, which is the missing ingredient in the Lévy–Itô decomposition.

Recall that \( A_\varepsilon = [-1,1] \setminus (-\varepsilon,\varepsilon) \). We use the same filtration as in Lemma 3.5, i.e. define \( \tilde{\mathcal{F}}_t = \sigma \{ N([0,s] \times C) : s \leq t, C \text{ Borel} \} \), and let \( (\mathcal{F}_t)_{t \geq 0} \) be the natural enlargement of \( (\tilde{\mathcal{F}}_t)_{t \geq 0} \).

**Theorem 3.8.** Let \( \Pi \) be a Lévy measure and \( N \) a Poisson random measure on \([0,\infty) \times \mathbb{R}\) with intensity measure \( \text{Leb} \times \Pi \). For \( \varepsilon \in (0,1) \), define

\[
M^\varepsilon_t = \int_{[0,t] \times A_\varepsilon} xN(ds, dx) - t \int_{A_\varepsilon} x\Pi(dx), \quad t \geq 0,
\]

which is an element of \( \mathcal{M}^2 \). Then, there exists a martingale \( M \in \mathcal{M}^2 \) satisfying the following properties.

(i) \( M^\varepsilon \xrightarrow{\mathcal{M}^2} M \) as \( \varepsilon \downarrow 0 \). Furthermore, there exist a nullset \( Q \) and a deterministic sequence \( (\varepsilon_n)_{n \geq 1} \) with \( \varepsilon_n \downarrow 0 \), such that for every \( t \geq 0 \),

\[
\lim_{n \to \infty} \sup_{s \in [0,t]} (M^\varepsilon_n_s - M_s)^2 = 0
\]

outside of \( Q \).

(ii) \( M \) has almost surely càdlàg paths and stationary, independent increments.

In other words, \( M \) is a Lévy process which is also a martingale. Furthermore, \( M \) has characteristic exponent

\[
\int_{[-1,1]} (e^{i\theta x} - 1 - i\theta x) \Pi(dx), \quad \theta \in \mathbb{R}.
\]

**Proof.** (i) We begin by proving that \( M \) exists as a limit in \( \mathcal{M}^2 \). The first step is to show that \( (M^\varepsilon) \) is Cauchy.
Fix $0 < \eta < \varepsilon < 1$, and pick $t \geq 0$. Using Theorem 3.4 we calculate
\[
\|M^n - M^\varepsilon\|_t^2 = E\left[ (M^n_t - M^\varepsilon_t)^2 \right] \\
= E\left[ \int_{[0,t]} \int_{\eta \leq |x| < \varepsilon} x N(ds, dx) - t \int_{\eta \leq |x| < \varepsilon} x \Pi(dx) \right]^2 \\
= t \int_{\eta \leq |x| < \varepsilon} x^2 \Pi(dx),
\]
from which follows that
\[
\|M^n - M^\varepsilon\| \leq \int_{\eta \leq |x| < \varepsilon} x^2 \Pi(dx) \sum_{n=1}^{\infty} n2^{-n}.
\]
Using the condition $\int_{[-1,1]} x^2 \Pi(dx) < \infty$, it follows that when $\eta, \varepsilon$ become small, the distance $\|M^n - M^\varepsilon\| \to 0$; that is, $(M^\varepsilon)$ is Cauchy in $\mathcal{M}^2$ It therefore follows that a limit exists in $\mathcal{M}^2$, and we shall call it $M$.

We now seek to prove the stronger convergence given in the theorem, which will emerge as a fairly simple consequence of classical martingale theory. To begin with, let $t = 1$ and choose any sequence $(\varepsilon^n_0)$ which approaches zero. According to Doob’s maximal inequality \cite[Theorem 2.5.19]{Bic},
\[
\lim_{n \to \infty} E\left[ \sup_{s \in [0,1]} \left( M_s - M^0_s \right)^2 \right] \leq 4 \lim_{n \to \infty} \|M - M^0\|_1 = 0. \tag{3.9}
\]
The $L^1$ convergence of the sequence of suprema on the left-hand side of (3.9) entails the existence of a subsequence $(\varepsilon^n_1) \subset (\varepsilon^n_0)$ and a $P$-nullset $Q_1$ such that
\[
\lim_{n \to \infty} \sup_{s \in [0,1]} \left( M_s - M^1_s \right)^2 = 0,
\]
outside of $Q_1$. (This is a fact from measure theory; see \cite[Theorem 15.7]{Bau01} or \cite[Theorem 3.12]{Rud87}. This link in the PDF works at time of writing.)

To extend this idea to all $t$, we first iterate; the relation
\[
\lim_{n \to \infty} E\left[ \sup_{s \in [0,m]} \left( M_s - M^{t-1}_s \right)^2 \right] \leq 4 \lim_{n \to \infty} \|M - M^{t-1}\|_m = 0
\]
holds, and we extract a subsequence $(\varepsilon^n_m) \subset (\varepsilon^n_{t-1})$ along which almost sure convergence holds outside of some nullset $Q_m$. Then we define the diagonal sequence $\varepsilon_n = \varepsilon^n_n$, and claim that for any fixed $t \geq 0$,
\[
\lim_{n \to \infty} \sup_{s \in [0,t]} \left( M_s - M^n_s \right)^2 = 0
\]
outside of the nullset $Q = \bigcup_m Q_m$. This is easy to believe. Take $m$ to be the next integer after $t$ and use the simple bound $\sup_{s \in [0,t]} \ldots \leq \sup_{s \in [0,m]} \ldots$; then observe that $(\varepsilon_n)_{n \geq m} \subset (\varepsilon^n_n)_{n \geq m}$, and use the almost sure convergence along the sequence $(\varepsilon^n_m)$ to obtain the result.
(ii) The fact that $M$ has càdlàg paths is automatic by virtue of its being an element of $\mathcal{M}^2$. Fix $0 \leq s_1 < t_1 < s_2 < \cdots < s_n < t_n$ and $\theta_1, \ldots, \theta_n$. The almost sure convergence given in the previous parts, together with the dominated convergence theorem, allows us to make the following calculation, which is sufficient by Exercise 1.

$$E \left[ \prod_{j=1}^{n} e^{i\theta_j (M_{t_j} - M_{s_j})} \right] = \lim_{m \to \infty} E \left[ \prod_{j=1}^{n} e^{i\theta_j (M_{t_j}^{(m)} - M_{s_j}^{(m)})} \right]$$

$$= \lim_{m \to \infty} \prod_{j=1}^{n} E \left[ e^{i\theta_j M_{t_j}^{(m)} - s_j} \right]$$

$$= \prod_{j=1}^{n} E \left[ e^{i\theta_j M_{t_j}^{(m)} - s_j} \right].$$

The final claim about the characteristic exponent follows simply, since $M_t^\varepsilon$ converges to $M_1$ in $L^2$, and therefore also in probability, and indeed in distribution; and this is enough [see Sat99, Proposition 2.5(iv)] for the characteristic functions to converge! \( \square \)

Let us make a short remark, which may be of interest but which we will not need again. The convergence given in (3.8) actually tells us that the paths of $M_t^\varepsilon$ converge to the paths of $M$ almost surely in the local uniform topology on the space $D[0, \infty)$ of real càdlàg functions on $[0, \infty)$. This is a rather strong mode of convergence and in particular implies that the paths of $M_t^\varepsilon$ converge to those of $M$ almost surely in the Skorokhod topology on $D[0, \infty)$. For definitions and a broader discussion of these concepts, see [JS03, Chapter VI].

### 3.4. Lévy–Itô decomposition

We now state formally the decomposition we have proved.

**Theorem 3.9** (Lévy–Itô decomposition). Let $a \in \mathbb{R}$, $\sigma \geq 0$ and let $\Pi$ be a measure with no atom at zero satisfying $\int_{\mathbb{R}} 1 \land x^2 \Pi(dx)$. Then there exist three independent Lévy processes: a linear Brownian motion

$$X^{(1)} = \sigma B_t + at;$$

a compound Poisson process $X^{(2)}$ with rate $\Pi(\mathbb{R} \setminus [-1, 1])$ and jump distribution $\Pi(\cdot \cap \mathbb{R} \setminus [-1, 1]) / \Pi(\mathbb{R} \setminus [-1, 1])$; and a third Lévy process $X^{(3)}$, the square-integrable martingale $M$ in [Theorem 3.8] given by the limit of compensated compound Poisson processes the magnitude of whose jumps is less than 1. The process $X = X^{(1)} + X^{(2)} + X^{(3)}$ is a Lévy process with characteristic exponent

$$\Psi(\theta) = ia\theta - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - x1_{[-1,1]}(x)) \Pi(dx).$$

We remark that this theorem proves the converse part of [Theorem 1.6] the Lévy–Khintchine formula.

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We should also point out something that is not completely obvious: the random measure defined by

\[ N([0,t] \times B) = \# \{(s, \Delta X_s) : s \leq t, \Delta X_t \in B\} \]

is a Poisson random measure with intensity \( \text{Leb} \times \Pi \). This is because it may be viewed as a superposition (i.e., sum) of the Poisson random measure induced by the compound Poisson process \( X^{(2)} \), together with a collection of independent Poisson random measures with intensities \( \text{Leb} \times \Pi(\cdot \cap \{ x : \varepsilon_{n+1} \leq |x| < \varepsilon_n \}) \), for \( n \geq 1 \), which arise from the sequence of martingales \( M^{\varepsilon_n} \). This superposition is again a Poisson random measure by the same reasoning as in the proof of Theorem 3.3.

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4.1. Killed Lévy processes

We will occasionally find it useful to consider a Lévy process on \( \mathbb{R} \cup \{ \partial \} \), where \( \partial \) is a cemetery state, isolated from \( \mathbb{R} \), at which the process is absorbed, or killed. If we denote by \( \zeta = \inf \{ t \geq 0 : X_t = \partial \} \) the killing time of \( X \), it follows from the definition of a Lévy process that \( \zeta \) must have an exponential distribution with some rate \( q \geq 0 \) and be independent of the path of the Lévy process. (This is because the exponential distribution is the only continuous distribution with the memoryless property.) Here, we interpret rate \( q = 0 \) to mean that \( \zeta = \infty \) almost surely; if \( q = 0 \), we say that \( X \) is unkilled, while when \( q > 0 \), we say that \( X \) is killed.

In other words, if \( X \) is a killed Lévy process, then there exist an unkilled Lévy process \( \tilde{X} \) and an independent random variable \( \tau \sim \text{Exp}(q) \), such that \( X \) is equal in distribution to the process

\[ \tilde{X}_t \mathbb{1}_{\{t < \tau\}} + \partial \mathbb{1}_{\{t \geq \tau\}}, \quad t \geq 0. \]

The killing rate manifests in the Lévy–Khintchine formula in a simple way. If \( X \) is a killed Lévy process, then its characteristic exponent satisfies

\[ \Psi(\theta) = i a \theta - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i \theta x} - 1 - x \mathbb{1}_{[-1,1]}(x)) \Pi(dx) - q. \]

4.2. Path variation

In this section, we are interested in deciding the question of when the paths of a Lévy process have bounded variation. The answer will emerge fairly simply from the Lévy–Itô decomposition.

We give a brief recap of the notion of path variation. First consider a function \( f : [0, \infty) \to \mathbb{R} \). Given any partition \( P = \{ a = t_0 < t_2 < \cdots < t_n = b \} \) of the bounded interval \( [a, b] \) we define the variation of \( f \) over \( [a, b] \) with partition \( P \) by

\[ V_P(f, [a, b]) = \sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right|. \]
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The variation of $f$ over $[a, b]$ is given by

$$V(f, [a, b]) := \sup_{\mathcal{P}} V_{\mathcal{P}}(f, [a, b]),$$

where the supremum is taken over all partitions of $[a, b]$; if this quantity is finite, we say that $f$ is of bounded variation over $[a, b]$. Moreover, $f$ is said to be of bounded variation if $V(f, [a, b]) < \infty$ for all bounded intervals $[a, b]$. If $V(f, [a, b]) = \infty$ for all bounded intervals $[a, b]$ then we say that $f$ is of unbounded variation.

A very useful fact, which we will not prove, is that if $f$ is a càdlàg function (and therefore has at most a countable number of discontinuities), then its variation may be written

$$V(f, [a, b]) = V(f^c, [a, b]) + \sum_{x \in (a, b]} |\Delta f(x)|,$$

where $f^c(x) = f(x) - \sum_{y \leq x} \Delta f(y)$ is the continuous part of $f$ and $\Delta f(x) := f(x) - f(x^-)$ is the jump of $f$ at $x$.

When considering a stochastic process $X = \{X_t : t \geq 0\}$, we may adopt these notions in the almost sure sense. So, for example, the statement $X$ is a process of bounded variation (or has paths of bounded variation) simply means that $P$-almost every path of $X$ is of bounded variation.

For a Lévy process $X$, we give the following test for bounded variation.

**Lemma 4.1.** The Lévy process $X$ is of bounded variation if

$$\sigma = 0 \quad \text{and} \quad \int_{\mathbb{R}} 1 \wedge |x| \Pi(dx) < \infty,$$

and unbounded variation otherwise.

**Proof.** From the description of the Poisson point process of jumps given immediately after [Theorem 3.9] as well as [Theorem 3.4] we see that the sum $\sum_{s \leq t} |\Delta X_s|$ converges a.s. if and only if $\int_{\mathbb{R}} 1 \wedge |x| \Pi(dx) < \infty$. Suppose that this does hold. Using (4.1) we see that we then only have to be worried about the variation of the continuous part of $X$. The continuous part of $X^{(2)}$ is zero, and under our assumption $X^{(3)}$ is precisely an absolutely convergent jump part compensated by a deterministic drift, $-t \int_{[-1,1]} x \Pi(dx)$, so the variation of $X^{(3),c}$ is always finite. Finally, $X^{(1)}$ has finite variation if and only if $\sigma = 0$, and this completes the proof. \hfill $\square$

We end with a nice representation of bounded variation Lévy processes.

**Lemma 4.2.** Let $X$ be a Lévy process of bounded variation. Then the characteristic exponent may be written

$$\Psi(\theta) = i \theta \sigma + \int_{\mathbb{R}} (e^{i \theta x} - 1) \Pi(dx),$$

where $\sigma = a - \int_{[-1,1]} x \Pi(dx)$. Furthermore,

$$X_t = dt + \sum_{s \leq t} \Delta X_s, \quad t \geq 0.$$
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Proof. The expression for \( d \) follows from the Lévy–Khintchine representation. The expression for \( X_t \) is derived by observing that, when \( X \) is of bounded variation,

\[
X_t^{(3)} = \sum_{s \leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s| \leq 1\}} - t \int_{[-1,1]} x \Pi(dx), \quad t \geq 0.
\]

Due to (4.3), the term \( d \) is called the drift coefficient of \( X \).

4.3. Behaviour around stopping times

We now offer some results around the theme of ‘Lévy processes as Markov processes’. We will not prove anything in this section, but the results will be useful later. Proofs of all the results may be found in [Kyp14, Chapter 3].

Let \( X \) be a Lévy process, and define a filtration via \( \tilde{\mathcal{F}}_t = \sigma(X_s, s \leq t) \), for \( t \geq 0 \). This filtration is typically neither right-continuous nor complete. However, if we define \( \mathcal{F}_t \) to be the \( \mathbb{P} \)-regularisation\(^2\) of \( \tilde{\mathcal{F}}_t \), then the new filtration \( (\mathcal{F}_t)_{t \geq 0} \) is automatically right-continuous, thus, it satisfies the natural conditions. (See [Ber96, Proposition I.4].)

As we have already remarked, a direct consequence of the independent increments property of \( X \) is the (spatially homogeneous) Markov property; that is, for any fixed time \( t \geq 0 \), the process \( (X_t + s - X_t, s \geq 0) \) is independent of \( \mathcal{F}_t \) and is equal in distribution to \( X \).

Lévy processes also possess the strong Markov property, which describes the process after a stopping time; here are the definitions. A \([0, \infty]\)-valued random variable \( T \) is called a stopping time if, for every \( t \geq 0 \), the set \( \{T \leq t\} \in \mathcal{F}_t \). Since \( (\mathcal{F}_t)_{t \geq 0} \) is right-continuous, this is equivalent to \( \{T < t\} \in \mathcal{F}_t \) for every \( t \geq 0 \). For any stopping time \( T \), we may define the associated \( \sigma \)-algebra

\[
\mathcal{F}_T = \{A : \text{for all } t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}.
\]

We will first give some nice examples of stopping times. Let \( B \) be a Borel subset of \( \mathbb{R} \). The random times

\[
T_B = \inf\{t \geq 0 : X_t \in B\} \quad \text{and} \quad \tau_B = \inf\{t > 0 : X_t \in B\}
\]

are referred to as the first entrance time and first passage time of \( B \), respectively.

Proposition 4.3. If \( B \) is either open or closed, then

(i) \( T_B \) is a stopping time and \( \mathbb{P}(X_{T_B} \in \overline{B} \mid T_B < \infty) = 1 \), and

(ii) \( \tau_B \) is a stopping time and \( \mathbb{P}(X_{\tau_B} \in \overline{B} \mid \tau_B < \infty) = 1 \).

We have the following extension of the Markov property to stopping times.

Theorem 4.4 (strong Markov property). Let \( T \) be any stopping time. Under the conditional measure \( \mathbb{P}(\cdot \mid T < \infty) \), the process \( (X_{T+t} - X_T, t \geq 0) \) is independent of \( \mathcal{F}_T \) and has the same distribution as \( X \).

\(^2\)i.e., we adjoin the \( \mathbb{P} \)-nullsets by defining \( \mathcal{F}_t = \sigma\{A \cup N : A \in \tilde{\mathcal{F}}_t, \exists M \supset N : \mathbb{P}(M) = 0\} \)
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We can even say something about the behaviour of the Lévy process just before certain stopping times. Recall that the Lévy process is right-continuous with left-limits. It is certainly not left-continuous unless it is a linear Brownian motion. However, we do have the following.

**Theorem 4.5** (quasi-left-continuity). Suppose that \((T_n)\) is an \((a.s.)\) increasing sequence of stopping times such that the almost sure limit \(\lim_{n \to \infty} T_n =: T\) exists. Then, under the conditional measure \(P(\cdot \mid T < \infty)\), \(\lim_{n \to \infty} X_{T_n} = X_T\) a.s.

A closely related result applying to fixed times is the following, which states that \(X\) has ‘no fixed jumps’.

**Proposition 4.6.** Fix a time \(t \geq 0\). Then, \(X\) is continuous at time \(t\) with probability 1.

This can be easily proved without the need for Theorem 4.5:

\[
\mathbb{P}\left(\left\{ t \right\} \times \mathbb{R} \right) = 0 \quad \text{a.s.,}
\]

which implies that, with probability 1, \(X\) has no jump at the fixed time \(t\).

### 4.4. Compensation formula

This section gives a precise statement of a ‘process’ version of Theorem 3.4, which is also often called the compensation formula. We also give the specialisation to the jumps of a Lévy process, which will be rather useful. We will later use the Poisson random measure version in another context as well.

**Theorem 4.7** (Compensation formula, Poisson random measure). Let \(S\) be a Polish space and \(\mathcal{S}\) its \(\sigma\)-algebra of Borel sets. Let \(N\) be a Poisson random measure on \([0, \infty) \times S\) with intensity \(\text{Leb} \times \eta\), and write \((\mathcal{F}_t)_{t \geq 0}\) for the natural filtration associated with it. Consider a function \(\phi: \Omega \times [0, \infty) \times S \to [0, \infty)\) such that:

(i) For each fixed \(t \geq 0\), the function \((\omega, x) \mapsto \phi(\omega, t, x)\) is \(\mathcal{F}_t \times \mathcal{S}\)-measurable.

(ii) For each fixed \(x \in S\), \(\{\phi(\omega, t, x), t \geq 0, \omega \in \Omega\}\), viewed as a stochastic process, is left-continuous with probability 1.

Then,

\[
\mathbb{E}\left[\int_{[0, \infty) \times S} \phi(t, x) N(dt, dx)\right] = \mathbb{E}\left[\int_{[0, \infty) \times S} \phi(t, x) dt \eta(dx)\right].
\]

Applying this to a Lévy process, we obtain the following.

**Theorem 4.8** (Compensation formula, Lévy process). Let \(X\) be a Lévy process and \((\mathcal{F}_t)_{t \geq 0}\) the natural enlargement of its induced filtration, and write \(N\) for the Poisson process on \([0, \infty) \times \mathbb{R}\) associated with the jumps of \(X\). Let \(\phi\) be a function satisfying the conditions in Theorem 4.7. Then,

\[
\mathbb{E}\left[\int_{[0, \infty) \times \mathbb{R}} \phi(t, x) N(dt, dx)\right] = \mathbb{E}\left[\int_{[0, \infty) \times \mathbb{R}} \phi(t, x) dt \Pi(dx)\right].
\]

This result is stated and proved in [Kyp14, Theorem 4.4]. We also remark that both these theorems can be extended to predictable processes \(\phi\) via the monotone class theorem, but we will not need this.
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4.5. Existence of moments

Here we are interested in knowing when $E[g(X_t)]$ is finite, and for a certain, quite wide, class of functions, there is a simple integral test.

**Definition 4.9.** A non-negative function $g$ on $\mathbb{R}$ is called **submultiplicative** if there exists a constant $a > 0$ such that

$$g(x + y) \leq ag(x)g(y), \quad x, y \in \mathbb{R}.$$ 

A function $g$ is called **locally bounded** if it is bounded on every compact subset of its domain.

A very important fact about moments of the one-dimensional distributions of Lévy processes is that, in almost every important case, their finiteness is determined by the Lévy measure $\Pi$, as follows.

**Theorem 4.10.** Let $g$ be a locally bounded, submultiplicative measurable function on $\mathbb{R}$. Then, for any $t > 0$,

$$E[g(X_t)] < \infty \quad \text{if and only if} \quad \int_{|x| > 1} g(x) \Pi(dx) < \infty.$$ 

We point out that the functions $x \mapsto |x|; x \mapsto \exp(x^2)$ (for any $\beta \in (0, 1]$); and $x \mapsto \log_+|x|$ are submultiplicative. In addition, if $g$ is a submultiplicative function, then the function $x \mapsto g(cx + \gamma)^\alpha$ is also submultiplicative, for any choice of $c, \gamma \in \mathbb{R}$ and $\alpha > 0$; and the product of any two submultiplicative functions is submultiplicative. For a proof of these facts, see [Sat99, Proposition 25.4].

We will not prove Theorem 4.10; a proof may be found in Sato [Sat99, Theorem 25.3].

Let us remark here that this theorem gives the proof of our claim, at the end of §2.5, about the ‘heavy tails’ of stable distributions.

4.6. Subordinators

A **subordinator** is a Lévy process whose paths are non-decreasing with probability one. It is simple to give a criterion for a Lévy process $X$ to be a subordinator. The paths must be of bounded variation; $X$ may not have any negative jumps; and it must have non-negative drift. That is, we require

$$\sigma = 0, \quad \Pi(-\infty, 0) = 0, \quad \int_{(0, \infty)} 1 \wedge x \Pi(dx) < \infty, \quad \text{and} \quad d \geq 0, \quad (4.4)$$ 

where $d$ is as given in Lemma 4.2. Further, it is clear that any Lévy process satisfying these conditions has non-decreasing paths, so (4.4) characterises subordinators.

Since $\mathbb{P}(X_t < 0) = 0$, $X$ possesses every negative exponential moment, i.e. $E[e^{-\lambda X_t}]$ is finite for every $\lambda, t \geq 0$. We therefore define the **Laplace exponent** $\kappa$ of a subordinator $X$ to be the function $\kappa : [0, \infty) \to \mathbb{R}$ given by

$$E[e^{-\lambda X_t}] = e^{-\tau e(\lambda)}, \quad \lambda \geq 0.$$ 

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In the following proposition, we give the Lévy–Khintchine formula for $\kappa$, and we show that it has an analytic extension. To be precise: suppose that $A \subset B \subset C$. An analytic extension of a function $f : A \to C$ is a function $g : B \to C$, which is analytic on $B$ (this means that it is analytic on the interior of $B$ and continuous on all of $B$), such that $g|_A = f$. Under various conditions, this extension will be unique. For instance, if $A$ is an open set and such a $g$ exists, then it is unique by the identity theorem of complex analysis (see Rudin [Rud87, Corollary to Theorem 10.18].) On the other hand, if $A = \partial B$, and $\partial B$ is sufficiently smooth (say, $C^2$) then a unique analytic extension $g$ exists because it is the solution to the Dirichlet problem. (See, for instance, [Sil84, §13.24] for a discussion of precisely the case we need. This is available on Google Books at time of writing.)

**Proposition 4.11.** If $X$ is a possibly killed subordinator, then for $\lambda \geq 0$,

$$
\kappa(\lambda) = q + d\lambda + \int_{[0, \infty)} (1 - e^{-\lambda x}) \Pi(dx).
$$

Furthermore, $\kappa$ has an analytic extension to $C_r = \{ \lambda \in C : \text{Re} \lambda \geq 0 \}$, and $-\kappa(-i\theta) = \Psi(\theta)$.

**Proof.** We will show that the left- and right-hand sides of the equation

$$
\mathbb{E}[e^{i\theta X_1}] = \exp \left\{ id\theta + \int_{\mathbb{R}} (e^{i\theta x} - 1) \Pi(dx) - q \right\} 
$$

(which follows from (4.2)) are analytic for $\theta$ in the set $C_u = \{ \theta \in C : \text{Im} \theta \geq 0 \}$. Then, since they agree when $\text{Im} \theta = 0$, it follows (by the discussion above) that they are equal on $C_u$. The proof will then be complete by identifying $\theta = i\lambda$.

We begin with the right-hand side. Certainly the drift and killing terms are entire, so we focus on the integral

$$
\int_{[0, \infty)} (e^{i\theta x} - 1) \Pi(dx) = \int_{[0, 1]} (e^{i\theta x} - 1) \Pi(dx) + \int_{(1, \infty)} (e^{i\theta x} - 1) \Pi(dx).
$$

For the second integral, we note that the (complex) derivative in $\theta$ of the integrand is equal to $ixe^{i\theta x}$, which has absolute value $xe^{-x \text{Im} \theta} = o(1)$ as $x \to \infty$. This is integrable under the finite measure $\Pi(\cdot \cap (1, \infty))$, and by dominated convergence the derivative in $\theta$ of the second integral is given by $i \int_{(1, \infty)} xe^{i\theta x} \Pi(dx)$. Thus the second integral is analytic on $C_u$.

We now deal with the first integral; the above method does not work here because $\Pi$ restricted to $[0, 1]$ is not a finite measure. Instead, we have

$$
\int_{[0, 1]} (e^{i\theta x} - 1) \Pi(dx) = \int_{[0, 1]} \sum_{n \geq 1} \frac{(i\theta x)^n}{n!} \Pi(dx).
$$

(4.6)

The estimate

$$
\int_{[0, 1]} \sum_{n \geq 1} \frac{|i\theta x|^n}{n!} \Pi(dx) \leq \int_{[0, 1]} |x| \Pi(dx) \cdot \sum_{n \geq 1} \frac{|\theta|^n}{n!} < \infty
$$

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justifies us in applying Fubini’s theorem to the right-hand side of (4.6). This amounts to writing the integral as a power series, which is then analytic on all of \( \mathbb{C} \).

Finally, we need to show that
\[
E[e^{i\theta X_1}] = \int e^{i\theta x} \mathbb{P}(X_1 \in dx)
\] (4.7)
is analytic for \( \theta \in \mathbb{C} \). The complex derivative of \( e^{i\theta x} \) is \( ix e^{i\theta x} \), which, due to Theorem 4.10, is integrable under \( \mathbb{P}(X_1 \in dx) \). We can then use dominated convergence to show that the complex derivative of (4.7) is given by \( iE[X_1 e^{i\theta X_1}] \), so it is analytic.

We have elided the discussion of the continuity up to the boundary, but for both sides of (4.5) this can be proven by dominated convergence. This completes the proof. \( \square \)

Let us state and prove a rather nice result about subordinators, which has been proved (in various versions) by Kesten, Horowitz and Bertoin. This statement will have an interesting generalisation to Lévy processes once we are in possession of the Wiener–Hopf factorisation. Suppose that \( X \) is a subordinator. We define the potential measure (or the 0-potential measure) \( U \) by:
\[
U(dx) = E \left[ \int_0^\infty \mathbb{1}_{\{X_t \in dx\}} \, dt \right] = \int_0^\infty \mathbb{P}(X_t \in dx) \, dt, \quad x \geq 0,
\]
and the first-passage time
\[
\tau^+_x := \tau(x,\infty) = \inf\{t > 0 : X_t > x\},
\]
for any \( x > 0 \). Then we have the following proposition concerning the so-called overshoot and undershoot at first passage.

**Proposition 4.12.** Suppose that \( X \) is a subordinator with Lévy measure \( \Pi \) and potential measure \( U \), and choose a level \( x > 0 \). Suppose that \( f \) and \( g \) are Borel-measurable functions and that \( f(0) = 0 \). Then,
\[
E[f(X_{\tau^+_x} - x)g(x - X_{\tau^+_x})] = \int_{[0,x]} U(dy) \int_{(x-y,\infty)} \Pi(du) f(u + y - x)g(x - y).
\]

**Proof.** The proof is a typical application of the compensation formula applied to the Poisson point process of jumps associated with the Poisson random measure \( N \) (see the end of §3.4 for a discussion of this.) We make the following calculation. We use the fact that \( X \) has at most countably many jumps to write the first sum; then the elementary identity \( X_t = X_{t-} + \Delta X_t \); then the compensation formula; then the fact that Lebesgue
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The measure is diffuse and the set of jumps is countable. The calculation is as follows:

\[
\mathbb{E}\left[f(X_{\tau_x^+} - x)g(x - X_{\tau_x^-})\right] = \mathbb{E}\left[ \sum_{t > 0} f(X_t - x)g(x - X_{t^-}) \mathbbm{1}_{\{X_{t^-} \leq x, X_t > x\}} \right] \\
= \mathbb{E}\left[ \int_{(0,\infty) \times \mathbb{R}} f(x - X_{t^-} + z - x)g(x - X_{t^-}) \mathbbm{1}_{\{X_{t^-} \leq x\}} \mathbbm{1}_{\{X_{t^-} + z > x\}} N(dt, dz) \right] \\
= \mathbb{E}\left[ \int_0^\infty dt \ g(x - X_{t^-}) \mathbbm{1}_{\{X_{t^-} \leq x\}} \int_{(0,\infty)} \Pi(du) \ f(X_{t^-} + u - x) \mathbbm{1}_{\{u > x - X_{t^-}\}} \right] \\
= \int_0^\infty dt \ P(X_t \in dy, X_t \leq x) g(x - y) \int_{(x-y,\infty)} \Pi(du) f(u + y - x),
\]

which yields the statement we are trying to prove.

The result is often stated as:

\[
\mathbb{P}(X_{\tau_x^+} - x \in dv, x - X_{\tau_x^-} \in dz) = U(x - dv)\Pi(z + dv), \quad v > 0, \ z \in [0, x],
\]

and this is obtained simply by a change of variables.

Note that the assumption on \(f\) means that the proposition will not ‘capture’ the probability of the event \(\{X_{\tau_x^+} = x\}\), even if this is non-zero. This phenomenon is known as ‘creeping’. A discussion of it may be found in [Kyp14, §5.3], but we will not discuss it further here.

Let us also mention that the statement of the proposition applies, without modification, to the case of killed subordinators \(X\), where one has

\[
U(dx) = \mathbb{E}\left[ \int_0^{\zeta} \mathbbm{1}_{\{X_t \in dx\}} dt \right],
\]

with \(\zeta\) the lifetime of \(X\). The proof is not much more difficult; an explicit version can be found in [Kyp14, Theorem 5.6].

4.7. Duality lemma

In this section we discuss a simple feature of all Lévy processes which follows as a direct consequence of stationary independent increments. When the path of a Lévy process over a finite time horizon is time reversed (in an appropriate sense) the new path is equal in law to the process reflected about the origin. This property will prove to be of crucial importance in a number of fluctuation calculations later on.

**Lemma 4.13** (Duality lemma). For each fixed \(t > 0\), define the reversed process

\[
\{Y_s := X_{(t-s)^-} - X_t : 0 \leq s \leq t\}
\]

and the process,

\[
\{-X_s : 0 \leq s \leq t\}.
\]

Then the two processes have the same law under \(\mathbb{P}\), and \(-X\) is known as the dual Lévy process.
5. Wiener–Hopf factorisation

Proof. First, note that \( Y_0 = -\Delta X_t = 0 \) almost surely (due to Proposition 4.6). As can be seen from Figure 7 (which is to be understood symbolically), the paths of \( Y \) are obtained from those of \( X \) by a reflection about the vertical axis with an adjustment of the continuity at the jump times so that its paths are almost surely right continuous with left limits. For fixed \( 0 \leq s \leq t \), the distribution of \( X_{(t-s)} - X_t \) is identical to that of \( -X_s \), so \( Y_s \) has the same distribution as \( -X_s \). Since \( X \) has stationary independent increments, the same as is true of \( Y \) (prove this using Exercise 1). This then implies that the law of the process \( Y \) is determined by its one-dimensional distributions, and it follows that \( Y \) and \( (-X_s, s \leq t) \) are equal in law.

The duality lemma is well-known for random walks, which are the discrete time analogue of Lévy processes, and is justified using an identical proof. See, for example, Feller [Fel71, §XII.2].

One consequence of the duality lemma is the relationship between the running supremum, the running infimum, the process reflected in its supremum and the process reflected in its infimum. The last four objects are, respectively,

\[
\bar{X}_t = \sup_{0 \leq s \leq t} X_s, \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s,
\]

\[
\{ \bar{X}_t - X_t : t \geq 0 \} \quad \text{and} \quad \{ X_t - \underline{X}_t : t \geq 0 \}.
\]

Lemma 4.14. For each fixed \( t > 0 \), the pairs \( (\bar{X}_t, \bar{X}_t - X_t) \) and \( (X_t - \underline{X}_t, -\underline{X}_t) \) have the same distribution under \( \mathbb{P} \).

Proof. Define \( \bar{X}_s = X_t - X_{(t-s)} = -Y_s \) for \( 0 \leq s \leq t \) and write \( \bar{X}_t = \inf_{0 \leq s \leq t} \bar{X}_s \). Using right continuity and left limits of paths we may deduce that

\[
(\bar{X}_t, \bar{X}_t - X_t) = (\bar{X}_t - \bar{X}_t, -\bar{X}_t)
\]

almost surely. One may visualise this in Figure 8. By rotating the picture about by 180° one sees the almost sure equality of the pairs \( (\bar{X}_t, \bar{X}_t - X_t) \) and \( (\bar{X}_t - \bar{X}_t, -\bar{X}_t) \). Now appealing to the duality lemma we have that \( \{ \bar{X}_s : 0 \leq s \leq t \} \) is equal in law to \( \{ X_s : 0 \leq s \leq t \} \) under \( \mathbb{P} \). The result now follows.

5. Wiener–Hopf factorisation

5.1. Random walks

We will warm up by discussing the Wiener–Hopf factorisation of a random walk. Many of the ideas are the same as for Lévy processes, but the discrete nature of random walks makes definitions and proofs considerably simpler.

Suppose that \( \{ \xi_i : i = 1, 2, \ldots \} \) is a sequence of \( \mathbb{R} \)-valued i.i.d. random variables with common distribution function \( F \). Let

\[
S_0 = 0 \text{ and } S_n = \sum_{i=1}^{n} \xi_i.
\]
5. Wiener–Hopf factorisation

Figure 7: Duality of the processes \( X = \{ X_s : s \leq t \} \) and \( Y = \{ X_{(t-s)} - X_t : s \leq t \} \). The path of \( Y \) is a reflection of the path of \( X \) with an adjustment of continuity at jump times.

Figure 8: Duality of the pairs \((X_t, X_t - X_t)\) and \((X_t - X_t, -X_t)\).
5. Wiener–Hopf factorisation

The process \( S = \{S_n : n \geq 0\} \) is called a (real valued) random walk. We shall make a number of assumptions on \( F \). First,

\[
\min\{F(0, \infty), F(-\infty, 0)\} > 0,
\]

meaning that the random walk may experience both positive and negative jumps; second, \( F\{0\} = 0 \).

Fix \( 0 < p < 1 \), and let \( \Gamma \) be a geometrically distributed random variable with parameter \( p \) which is independent of the random walk \( S \). Define

\[
G_\Gamma = \max\{k = 0, 1, \ldots, \Gamma : S_k = \max_{j=0,1,\ldots,\Gamma} S_j\},
\]

and

\[
G^*_\Gamma = \min\{k = 0, 1, \ldots, \Gamma : S_k = \min_{j=0,1,\ldots,\Gamma} S_j\}.
\]

In words, \( G_\Gamma \) is the last visit of \( S \) to its final maximum over the time period \( \{0, 1, \ldots, \Gamma\} \), and \( G^*_\Gamma \) is the first visit to its final minimum. (Note that if \( F \) is diffuse, then whether we look at the first or last extremum is irrelevant.)

The next theorem perhaps the most fundamental of the statements which make up the ‘Wiener–Hopf factorisation’.

**Theorem 5.1.** \((G_\Gamma, S_{G_\Gamma})\) is independent of \((\Gamma - G_\Gamma, S_\Gamma - S_{G_\Gamma})\) and both pairs are infinitely divisible. Moreover, the latter has the same law as \((G^*_\Gamma, S_{G^*_\Gamma})\).

*Proof postponed to page 35.*

Other statements that can be made in the same spirit go further in specifying additional distributional information about the pairs \((G_\Gamma, S_{G_\Gamma})\) and \((G^*_\Gamma, S_{G^*_\Gamma})\); since we will discuss these in the context of Lévy processes, we will not try to give an extensive collection of results here. Feller [Fel71, §XII] is one of many works available.

Crucial to the proof of Theorem 5.1 are the local time and the ladder height process of the random walk \( S \).

Defining the maximum process \( M_n = \max\{S_m : 0 \leq m \leq n\} \), we let

\[
L_n = \sum_{k=1}^{n} 1_{\{M_k = S_k\}}, \quad n \geq 0,
\]

the local time of \( S \) at the maximum.

We define \( T \) to be the right-inverse of \( L \), which is to say \( T_n = \inf\{k > 0 : L_k = n\} \); it is not difficult to show that this is a stopping time. \( T \) is called the inverse local time, or the ladder time process. In fact, we may alternatively define it as part of the following construction, which defines the bivariate ladder process of \( S \) at the maximum.

---

\[\text{We defined infinite divisibility in Definition 1.2 in terms of random variables on } \mathbb{R}. \text{ In Exercise 7 we discussed the extension to } \mathbb{R}^d \text{-valued random variables; the definition is essentially identical.}\]
5. Wiener–Hopf factorisation

Define the bivariate random walk \((T, H) := \{(T_n, H_n) : n = 0, 1, 2, \ldots\}\) where \((T_0, H_0) = (0, 0)\) and otherwise for \(n = 1, 2, 3, \ldots\),

\[
T_n = \begin{cases} \min\{k > T_{n-1} : S_k \geq H_{n-1}\} & \text{if } T_{n-1} < \infty \\ \infty & \text{if } T_{n-1} = \infty \end{cases}
\]

and

\[
H_n = \begin{cases} S_{T_n} & \text{if } T_n < \infty \\ \infty & \text{if } T_n = \infty. \end{cases}
\]

That is to say, the process \((T, H)\), until becoming infinite in value, represents the times and positions of the running maxima of \(S\); the so-called ladder times and ladder heights. Note the weak inequality in the definition of \(T\), which permits that successive values of \(H\) may be equal. For this reason, \((T, H)\) is also called the weak ladder process.

Since \(T_n\) is a stopping time for each \(n = 0, 1, 2, \ldots\) and \(S\) has i.i.d. increments, it follows that the increments of \((T, H)\) are independent and identically distributed with the same law as the pair \((N, S_N)\), where

\[N = \inf\{n > 0 : S_n \geq 0\}\]

is the first visit of \(S\) to \([0, \infty)\) strictly after time 0.

![Figure 9: Excursions of the random walk from its supremum.](image)

**Proof of Theorem 5.1** If \(T_n < \infty\), then we define the \((n + 1)\)-st excursion from the maximum to be the path segment \(\{S_m - H_n : m = T_n, \ldots, T_{n+1}\}\); where we understand this path segment to be infinite if \(T_{n+1} = \infty\). The path of the random walk may be broken into a random number \(\nu \in \{0, 1, 2, \ldots\}\) of (complete) excursions from the maximum, followed by an additional excursion which straddles the random time \(\Gamma\) (in the sense that, if \(k\) is the index of the left end point of the last complete excursion, then \(T_k \leq \Gamma\)). See Figure 9.
5. Wiener–Hopf factorisation

By the strong Markov property for random walks and lack of memory of the geometric distribution, the completed excursions must have the same law; namely, that of the random walk sampled on the time points \( \{1, 2, \ldots, N\} \) conditioned on the event that \( N \leq \Gamma \). Hence \( \nu \) is geometrically distributed with parameter \( 1 - P(N \leq \Gamma) \). Now observe that

\[
(G_\Gamma, S_{G_\Gamma}) \overset{d}{=} \sum_{i=1}^{\nu} (N^{(i)}, H^{(i)})
\]

where the pairs \( \{N^{(i)}, H^{(i)} : i = 1, 2, \ldots\} \) are independent having the same distribution as \((N, S_N)\) conditioned on \( \{N \leq \Gamma\} \). In other words, \((G_\Gamma, S_{G_\Gamma})\) is the component-wise sum of \( \nu \) independent copies of \((N, S_N)\) (where \((G_\Gamma, S_{G_\Gamma}) = (0, 0)\) if \( \nu = 0 \)). Infinite divisibility follows as a consequence of the fact that \((G_\Gamma, S_{G_\Gamma})\) is the sum of a geometric number of i.i.d. random variables; see Exercise 3. The independence of \((G_\Gamma, S_{G_\Gamma})\) and \((\Gamma - G_\Gamma, S_\Gamma - S_{G_\Gamma})\) follows from the decomposition described above. (We will go into more detail on this point later, for the general Lévy process case.)

Finally, Feller’s classic duality lemma (cf. Feller [Fel71, §XII.2]) for random walks, which is the analogue of Lemma 4.13, shows that \((\Gamma - G_\Gamma, S_\Gamma - S_{G_\Gamma})\) is equal in distribution to \((G_\Gamma^*, S_{G_\Gamma}^*)\).

\[\square\]

5.2. Local times of Markov processes

Note: in this section we will give precise definitions of local times and excursions of Markov processes, but for proofs we mostly defer to Bertoin [Ber96, Chapter IV]. The reader may also like to refer to Pardo [Par], which is based on [Ber96].

In this section, we will work in a fairly general framework to study the times at which a Markov process \( M \) takes the value 0. We will subsequently apply this when \( M \) is the Markov process \( \overline{X} - X \), where \( X \) is a Lévy process and \( \overline{X}_t = \sup\{0 \vee X_s : s \leq t\} \), the supremum process of \( X \).

If we try to follow the ideas presented for random walks, it would seem (following (5.1)) that it would be natural to attempt to define a local time

\[
L(t) = \int_0^t 1_{\{M_s = 0\}} \, ds.
\]

However, if \( M \) is, say, a standard Brownian motion \( B \), then \( L(t) \) would be zero for every \( t \geq 0 \) (to see this, take the expectation and use Fubini’s theorem) even though the zero set of a Brownian motion is quite large—the set \( \{t \in [0, 1] : B_t = 0\} \) has Hausdorff dimension 1/2\(^4\). So this definition will not work for many Markov processes.

Let us specify more precisely what we will require of \( M \). It must be ‘nice’ in the following sense. We require that \( M \) is càdlàg, adapted to a filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the natural conditions, and takes values in \( \mathbb{R}^d \). We suppose that there exist a family of probability measures \((P_x)_{x \in \mathbb{R}^d}\), which correspond to starting \( M \) from different points.

\(^4\)See [MP10, Theorem 4.24]. For a discussion of the local time of Brownian motion as a Hausdorff measure, see [MP10, §6.4]. This is certainly not the approach we will take in this section!
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We will write $P$ for $P_0$. We require that $M$ satisfy the strong Markov property, namely that for any finite $(\mathcal{F}_t)_{t \geq 0}$-stopping time $T$,

$$E(f(M_{T+t}, t \geq 0) \mid \mathcal{F}_T) = E_M(f(M_t, t \geq 0)),$$

for any real-valued Borel function $f$ on the space of càdlàg functions.

Certainly Feller processes and standard Markov processes are ‘nice’ in this sense, and the definition has several advantages; for instance, $M$ will satisfy the Blumenthal zero-one law, which states that every event in the $\sigma$-algebra $\mathcal{F}_0$ has probability zero or one.

Defining the stopping time $R = \inf\{t > 0 : M_t = 0\}$, the first return time to zero, we note that the probability $P(R = 0)$ is necessarily equal to zero or one. When $P(R = 0) = 0$, we say that 0 is irregular (for itself); otherwise, we say that 0 is regular (for itself). When 0 is regular, we define $S = \inf\{t \geq 0 : M_t \neq 0\}$, the first exit time from 0, and again we observe that $P(S = 0)$ is equal to zero or one. In the former case, we say that 0 is a holding point, and in the latter we say that 0 is instantaneous.

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For instance, 0 is irregular for itself if $M$ is a compound Poisson process plus non-zero drift, and a holding point if $M$ is a compound Poisson process. A simple example of a process where $M$ is instantaneous would be a Brownian motion.

Our interest in this section is in establishing a local time, analogous to that already discussed for reflected random walks, to measure the time spent by $M$ at the point 0. When 0 is irregular, these times are isolated; and when 0 is a holding point, $M$ spends positive time at zero before moving onto somewhere else. The analysis is only really difficult when 0 is instantaneous, and so we will focus on this case for the rest of the section.

5.2.1. Construction of the local time

From here up to §5.2.4 we assume that 0 is an instantaneous point.

We define $\mathcal{Z} = \{t \geq 0 : M_t = 0\}$, the zero set of $M$. To make the structure of $\mathcal{Z}$ clearer, consider the following argument. If $(t_n) \subset \mathcal{Z}$ and $t_n \downarrow t$ in $\mathbb{R}$, then the right-continuity of $M$ implies that $t \in \mathcal{Z}$ also. This means that every point of $\mathcal{Z} \setminus \mathcal{Z}$ is isolated on the right. Now, since $(\mathcal{Z})^c$ is an open subset of $\mathbb{R}$, it is a countable union of disjoint open intervals. It then follows that $\mathcal{Z}^c$ is a countable union of intervals of the type $(g, d)$ or $[g, d)$. One may also prove that $\mathcal{Z}$ is a nowhere dense set containing no isolated points.

---

5 This is a classical result which can be found in many textbooks, but here is a website with a collection of proofs: [http://math.stackexchange.com/questions/318299](http://math.stackexchange.com/questions/318299)
For $0 \leq g < d \leq \infty$ and, we say that $(g, d)$ is an excursion interval if it is maximal among the open intervals on which $M \neq 0$. This implies that $M_t \neq 0$ for all $t \in (g, d)$, and $g, d \in \mathbb{Z} \cup \{\infty\}$. Note that we do not say anything about the value of $M_g$ and $M_d$.

How should we go about building the local time? The first insight is that visits to zero correspond with excursions away from zero, and it is easier to count excursions than to count visits. Secondly, it’s clear that there are only a finite number of ‘large’ excursions (i.e., excursions with length greater than $a > 0$, for a given) which occur before time $t$; and these can simply be counted. However, as we reduce $a$ to zero, this number will increase to $+\infty$. We need some way to compensate it and get a limit, and it turns out that the following works, though we will not prove it.

Fix $c > 0$ such that, with positive probability, there exists an excursion with length greater than $c$. In fact, for any such $c$, the probability referred to is equal to one. We make the following argument to support this. Define

$$\Lambda_t = \{\text{all excursion intervals } (g, d) \text{ with } d < t \text{ have length } l \leq c\}.$$  

Pick $t > c$ such that $P(\Lambda_t) < 1$. Then, let $d_1$ be the first time that $M$ is at 0 after time $t$. We have, applying the Markov property at $d_1 < 2t$,

$$P(\Lambda_{3t}) \leq P(\Lambda_t \cap \{\text{all excursions intervals within } [d_1, 3t] \text{ have length } l \leq c\}) \leq (P(\Lambda_t))^2.$$  

Iterating, we obtain that for $n \geq 1$, $P(\Lambda_{3^{n}t}) \leq (P(\Lambda_t))^{2^n}$. Thus,

$$P(\text{all excursion intervals have length } l \leq c) = \lim_{n \to \infty} P(\Lambda_n) = 0,$$

which was our claim.

Denote the first excursion interval of length greater than $c$ by $(g_1(c), d_1(c))$, write $l_1(c) = d_1(c) - g_1(c)$ for its length, and let

$$\bar{\Pi}(a) = \begin{cases} 1/P(l_1(a) > c), & a \leq c, \\ P(l_1(c) > a), & a > c, \end{cases}$$

and

$$N_a(t) = \#\{\text{excursions of length greater than } a \text{ started strictly before } t\}.$$  

Although $\lim_{a \downarrow 0} N_a(t) = \infty$, one may show (see [Ber96], as usual) that the correct compensation for $N_a$ is precisely $\bar{\Pi}(a)$, and the limit

$$L(t) = \lim_{a \downarrow 0} \frac{N_a(t)}{\bar{\Pi}(a)}$$  

exists and is finite. $L$ is called a local time, and it is more or less unique, as stated in the following theorem.

**Theorem 5.2** (Characterisation of local time). There exists a local time $L$ such that
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(i) $L$ is continuous, increasing and adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$;

(ii) The support of the Stieltjes measure $dL$ is $\mathcal{E}$;

(iii) Let $T$ be a stopping time such that $M_T = 0$ a.s. on $\{T < \infty\}$. Then, the process $((M_{T+t}, L(T + t) - L(T)) : t \geq 0)$ is independent of $\mathcal{F}_T$ under $P(\cdot \mid T < \infty)$ and has the same law as $(M, L)$ under $P$.

Furthermore, if $A$ is another process satisfying the three points above, then $A = kL$ for some $k \geq 0$.

In view of this theorem, any non-zero stochastic process $L$ satisfying (i)–(iii) above is known as a local time of $M$ at zero.

Let us make a brief remark at this point. The excursion length $c$ chosen in the discussion above was arbitrary, and the construction of the local time depended on it. We can see from the above theorem that if we were to choose another excursion length $c'$ and construct a new local time $L'$ based on the value $c'$, there would exist some $k > 0$ such that $L' = kL$; that is, we would only change the normalisation of the local time.

Corollary 5.3. There exists a constant $d \geq 0$ such that

$$\int_0^t 1_{\{M_s = 0\}} \, ds = \int_0^t 1_{\{s \in \mathcal{F}\}} \, ds = dL(t), \quad t \geq 0.$$  

Proof. The sets $\mathcal{E}$ and $\mathcal{F}$ differ only at points where $M$ ‘jumps away’ from zero, and as the process has only countably many jumps (due to its càdlàg paths), the two integrals coincide. It is a simple matter to show that the integral satisfies the conditions of the previous theorem, and this proves the second equality.

We will shortly see that $d$ is the drift coefficient of a certain subordinator.

5.2.2. Inverse local time

We will now work to characterise the law of the inverse local time,

$$L^{-1}(t) = \inf \{s \geq 0 : L(s) > t\}, \quad t \geq 0.$$  

This is the right-inverse of $L$, in the sense that for all $t \geq 0$, $L(L^{-1}(t)) = t$; and the reader will note that it is right-continuous with left limits. It is also clear that $L^{-1}$ is increasing, and adapted to the time-changed filtration $(\mathcal{F}_{L^{-1}(t)})_{t \geq 0}$.

Proposition 5.4 (Properties of inverse local time). (i) For each $t \geq 0$, the random variables $L^{-1}(t)$ and $L^{-1}(t-)$ are stopping times.

(ii) Almost surely, for all $t \geq 0$,

$$L^{-1}(L(t)) = \inf \{L^{-1}(u) : L^{-1}(u) > t\} = \inf \{s > t : M_s = 0\}$$

and

$$L^{-1}(L(t-)) = \sup \{L^{-1}(u) : L^{-1}(u) < t\} = \sup \{s < t : M_s = 0\}.$$  

In particular, $L^{-1}(t) \in \mathcal{E}$ whenever $L^{-1}(t) < \infty$.  

Proof. Before embarking on the proof, note that because $L$ is increasing and continuous, for any $s, t \geq 0$, $L^{-1}(t) < s$ if and only if $L(s) > t$, and correspondingly, $L^{-1}(t) \geq s$ if and only if $L(s) \leq t$.

(i) Fix $t \geq 0$. For $s \geq 0$, observe $\{L^{-1}(t) < s\} = \{L(s) > t\} \in \mathcal{F}_s$, since $L$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$, and hence $L^{-1}(t)$ is a stopping time. Since $L^{-1}(t-)$ is the limit of a sequence of stopping times, it is also a stopping time.

(ii) We may immediately calculate $L^{-1}(L(t)) = \inf\{s : L(s) > L(t)\} = \inf\{L^{-1}(u) : L^{-1}(u) > t\}$. We define $D_t = \inf\{s > t : M_s = 0\}$, and consider two cases.

First, suppose that $D_t > t$. Then, $L$ only increases on $\mathcal{F}$, it is constant on $[t, D_t)$. Since $L$ is also continuous, we deduce that $L(D_t) = L(D_t-) = L(t)$. Applying $L^{-1}$, we obtain $D_t \leq L^{-1}(L(D_t)) = L^{-1}(L(t))$. Assume now that $D_t < \infty$, as otherwise we are done. By virtue of the definition of $D_t$, we have that for all $s > D_t$, $L(s) > L(D_t) = L(t)$. It then follows, using the definition of $L^{-1}$, that $D_t \geq L^{-1}(L(t))$, and we are done.

Now suppose instead that $D_t = t$. Then, $t \in \mathcal{F}$ and, as before, for every $s > t$, $L(s) > L(t)$. Thus, $t \geq L^{-1}(L(t))$. The reverse inequality follows immediately from the first identity in this result.

The identities for $L^{-1}(L(t)-)$ are very similar to prove.

Finally, take $t \geq 0$ such that $L^{-1}(t) < \infty$. Then, $L(\infty) < t$. Since $L$ is continuous and increasing, it is surjective as a map from $[0, \infty]$ to $[0, L(\infty)]$. Thus, there exists $s \geq 0$ such that $L(s) = t$, and then $L^{-1}(t) = L^{-1}(L(s)) = \inf\{u > s : M_u = 0\}$, and this is an element of $\mathcal{F}$ because (since it is an infimum) it is the limit of a decreasing sequence of elements of $\mathcal{F}$. \qed

Interestingly, $L^{-1}$ is a subordinator, and one can identify its characteristics in terms of the explicit construction we sketched above. Define

$$p := \bar{\Pi}(\infty) = P(l_1(c) = \infty) \geq 0.$$ 

If $p = 0$, then the first excursion of length greater than $c$ is finite with probability one; applying the Markov property successively at the right endpoints of excursions with $l > c$, one sees that every excursion is finite and hence that $\mathcal{F}$ is unbounded almost surely. On the other hand, if $p > 0$, a similar argument shows that $\mathcal{F}$ is bounded a.s.\textcolor{white}{.}

Let us state the result explicitly.

**Theorem 5.5.** The inverse local time $L^{-1}$ is a subordinator with drift $d$, killing rate $p$, and Lévy measure $\Pi$ given by

$$\Pi(s, t] = \bar{\Pi}(s) - \bar{\Pi}(t).$$

**Partial proof.** We will not derive the expression for $\Pi$.

It is plain that $L^{-1}$ is sent to $+\infty$, and remains there, at the time $L(\infty)$. To prove that $L^{-1}$ is a killed subordinator, we must show that $(L^{-1}(t), t < L(\infty))$ is equal in law to
(S_t, t < \tau) where S is a subordinator and \tau is exponentially distributed and independent of S.

Let’s first show that L(\infty) is exponentially distributed. We demonstrate the memory-less property. Fix s, t \geq 0; we make the following calculation using Theorem 5.2(iii).

\[ P(L(\infty) > s + t \mid L(\infty) > s) = P(L^{-1}(s + t) < \infty \mid L^{-1}(s) < \infty) \]
\[ = P(L^{-1}(s + t) - L^{-1}(s) < \infty \mid L^{-1}(s) < \infty) \]
\[ = P(L^{-1}(t) < \infty) = P(L(\infty) > t). \]

To show that L(\infty) is independent of the path of L^{-1}, it is sufficient to show that, for any t < t', the laws of (L^{-1}(s), s \leq t) under the measures P(\cdot \mid L^{-1}(t) < \infty) and P(\cdot \mid L^{-1}(t') < \infty) are equal. This can be shown by a similar calculation to the above.

We must still show the stationary independent increments property. Fix t \geq 0. By the previous proposition, L^{-1}(t) is a stopping time for \tau, and under the conditional measure P(\cdot \mid L^{-1}(t) < \infty), M_{L^{-1}(t)} = 0 a.s.. It then follows from the ‘additivity’ property Theorem 5.2(iii) that, still under the conditional probability measure, the shifted process (M, L) = ((M_{s+L^{-1}(t)}, L(s + L^{-1}(t)) - t), s \geq 0) has the same law as (M, L) and is independent of \mathcal{F}_{L^{-1}(t)}. It is then simple to calculate

\[ \tilde{L}^{-1}(s) = \inf\{u \geq 0 : L(u + L^{-1}(t)) - t > s\} \]
\[ = \inf\{v \geq 0 : L(v) > s + t\} - L^{-1}(t) \]
\[ = L^{-1}(s + t) - L^{-1}(t), \]

and this gives precisely the stationary, independent increments property of L^{-1}.

We have thus shown that L^{-1} is a killed subordinator. One may show (but we have not developed the methods here; see [Ber96]) that L(\infty) is exponentially distributed with parameter p = \bar{\Pi}(\infty); when this is zero, L^{-1} is an unkill subordinator.

We will prove that the drift of L^{-1} is d. Conditionally on L^{-1}(t) being finite, we have

\[ L^{-1}(t) = \int_0^{L^{-1}(t)} ds = \int_0^{L^{-1}(t)} 1_{\{s \in \mathcal{F}\}} ds + \sum_{s \leq t} \Delta L^{-1}(s). \]

This description hinges on the fact that the jumps of L^{-1} correspond precisely to the length of excursion intervals of M. We now apply Corollary 5.3 giving

\[ L^{-1}(t) = dt + \sum_{s \leq t} \Delta L^{-1}(s). \]

\[ 5.2.3. \text{Excursion measure} \]

Fix a > 0 such that there exists an excursion with length greater than a (or, equivalently, such that \bar{\Pi}(a) > 0), and denote by \mathcal{E}_a the space of excursions of M away from zero with lifetime \zeta > a:

\[ \mathcal{E}_a = \{\omega \in \tilde{\Omega} : \zeta > a, \omega(t) \neq 0 \text{ for } 0 < t < \zeta\}; \]
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where $\bar{\Omega}$ is the space of càdlàg functions on $[0, \zeta)$ for any $\zeta \leq \infty$ endowed with the Skorokhod topology. Let $n_a$ be the measure on $\mathcal{E}_a$ given by the law of $(M_{g_1(a)+t}, 0 \leq t < \zeta_{-1})$ under $P$.

A crucial consistency condition is given by the following lemma, which we are unfortunately not in a position to prove.

**Lemma 5.6.** For any $a > 0$ such that $\Pi(a) > 0$, and any choice of $0 < b < a$ and event $\Lambda \subset \mathcal{E}_a$, $\Pi(a)n_a(\Lambda) = \Pi(b)n_b(\Lambda)$.

This allows us to define a $\sigma$-finite measure $n$ on the excursion space $\mathcal{E} = \cup_{a > 0} \mathcal{E}_a$ by $n(\Lambda) = \Pi(a)n_a(\Lambda)$, $\Lambda \subset \mathcal{E}_a$.

This measure $n$ is known as the excursion measure of $M$ (away from 0). In particular, $n(\zeta > a) = \Pi(a)$, and $n(\Lambda | \zeta > a) = n_a(\Lambda)$ for $\Lambda \subset \mathcal{E}_a$, so that the measure $n_a$ may be seen as the law of a generic excursion conditioned on having lifetime greater than $a$. Note that we will generally write $\epsilon = \epsilon(t), t \in [0, \zeta)$ for the generic excursion under $n$.

It is important to note that the excursion measure satisfies the Markov property, in a sense which we will now describe. Fix $a > 0$. We see that $g_1(a) + a = \inf \{ t \geq a : M_s \neq 0 \text{ for } s \in [t-a, t] \}$ is a $(\mathcal{F}_t)_{t \geq 0}$-stopping time. Let us define, on excursion space, a filtration $\mathcal{G}_t = \sigma \{ \epsilon(s), s \leq t \}$. Notice that $n(f(\epsilon) | \zeta > a; \mathcal{G}_a) = n_a(g(\epsilon) | \mathcal{G}_a) = \mathbb{E}(f(M_{g_1(a)+t}, t \leq \zeta) | M_{g_1(a)+s}, s \leq a)$, for any real-valued Borel function $f$ on $\bar{\Omega}$. Using the Markov property of $M$ at the stopping time $g_1(a) + a$, we have $n(f(\epsilon(a + t), 0 \leq t < \zeta - a) | \zeta > a, \mathcal{G}_a) = \mathbb{E}_{\epsilon(a)}(f(M_t, 0 \leq t < R))$, for Borel $f$. (Recall that $R$ is the first return of $M$ to the point 0.) Thus, the law of the generic excursion under $n$ is given by an entrance law $\{ n(\epsilon(a) \in \cdot, \zeta > a) \}_{a > 0}$ together with the semigroup $\{ P_x(M_t \in \cdot, t < R) \}_{x \neq 0, t \geq 0}$.

We now see that $n$ can be viewed as the intensity of a Poisson random measure. Let $E \subset [0, \infty) \times \mathcal{E}$ be the set of local time points and paths of excursions:

$$E = \{(t, (M_s+L^{-1}(t-), 0 \leq s < L^{-1}(t) - L^{-1}(t-)): t \geq 0, \Delta L^{-1}(t) > 0\}.$$ We will identify $E$ with the support of a stopped Poisson random measure. By this, we mean the following. Let $N$ be a Poisson random measure on $[0, \infty) \times S$, for some Polish space $S$, with intensity $\text{Leb} \times \eta$. Fix a set $B \subset S$ and define the random time $T_B = \inf \{ t \geq 0 : N([0, t] \times B) > 0 \}$.

Then, by ‘$N$ stopped on entering $B$’, we mean the restriction $N|_{[0,T_B] \times S}$, which puts zero measure on any set which lies strictly ‘to the right of $T_B$’. We will write $N^B$ for this measure. The key result is the following:
Theorem 5.7. Let
\[ \tilde{K}(A) = \#(E \cap A), \]
for Borel sets A. Then, \((\tilde{K}, L(\infty))\) is equal in law to \((K^{\infty}, T_{\infty}), \) where \(K\) is a Poisson random measure on \([0, \infty) \times \mathcal{E}\) with characteristic measure \(\text{Leb} \times \nu.\)

The measure \(\tilde{K}\) is adapted to the filtration \(\mathcal{F}_{L^{-1}(t)}_{t \geq 0},\) in the sense that for any Borel \(B \subset \mathcal{E},\) \(\tilde{K}([0, t] \times B) \in \mathcal{F}_{L^{-1}(t)}\).

The random measure \(\tilde{K}\) never enters \(\mathcal{E}_\infty\) if and only if 0 is recurrent.

We omit the proof; it is rather similar to that of Theorem 5.5 but more involved.

A common way to view the situation we have just described is to define the excursion process \(e,\) which is essentially the process \(M\) viewed on the local time scale, so that excursions from zero appear as points. The process \(e\) takes values in \(\mathcal{E} \cup \{\Upsilon\},\) where \(\Upsilon\) is appended as an isolated point, and is given by:
\[
e_t = \begin{cases} \{(M_{s+L^{-1}(t)-}, 0 \leq s < L^{-1}(t) - L^{-1}(t-)), \quad \Delta L^{-1}(t) > 0, \quad t \geq 0. \\ \Upsilon, \quad \text{otherwise,} \end{cases}
\]
The precise relationship between \(e, E\) and the stopped Poisson random measure is left to the reader.

This theorem contains some rather useful information. For example, one can deduce the following facts about the local time. The Lévy measure of \(L^{-1}\) is specified by the arrival of excursions with certain lengths, and in fact \(\Pi(dx) = n(\zeta \in dx).\) The killing rate \(p\) of \(L^{-1}\) is given by \(n(\mathcal{E}_\infty),\) due to Exercise 10.

Perhaps the most useful consequence of this theorem is that it allows one to apply the compensation formula for Poisson random measures to the excursions of \(M.\) For each excursion interval \((g, d),\) denote by \(e_g\) the excursion starting at \(g,\) i.e. \(e_g = (M_{g+t}, 0 \leq t < d - g).\) Suppose that \(F: [0, \infty) \times \Omega \times \mathcal{E} \to [0, \infty)\) is a measurable function such that for each \(\epsilon \in \mathcal{E},\) the process \(t \mapsto F_t(\epsilon)\) is left-continuous and adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}.

Corollary 5.8 (Compensation formula for excursion theory). For \(F\) as above,
\[
\mathbb{E}\left[ \sum_g F_g(e_g) \right] = \mathbb{E}\left[ \int_0^\infty dL(s) \int_\mathcal{E} F_s(\epsilon) n(d\epsilon) \right],
\]
where the sum on the left-hand side is over all \(g\) which are left endpoints of excursion intervals.

Proof. We rewrite the left-hand side as:
\[
\mathbb{E}\left[ \sum_g F_g(e_g) \right] = \mathbb{E}\left[ \int F_{L^{-1}(t-)}(\epsilon) 1_{\{t \leq L(\infty)\}} \tilde{K}(dt, d\epsilon) \right]
\]
where \(\tilde{K}\) is as in the previous theorem. Noting that \(t \leq L(\infty)\) if and only if \(L^{-1}(t-) < \infty,\) we observe that the integrand is left-continuous and adapted to \((\mathcal{F}_{L^{-1}(t)})_{t \geq 0}.\) This then permits us to apply the compensation formula of §4.4 and the result follows after changing variables (see Corollary A.6).
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As usual, one may also extend the compensation formula to predictable $F$. For some examples using the compensation formula, see [Ber96, pp. 119–121].

5.2.4. Holding points and irregular points

Up to now, we have assumed that 0 is a regular, instantaneous point. We now offer some brief remarks about the other two cases.

If 0 is regular and a holding point, then it spends a.s. positive time at 0 before leaving it. We label the successive exit and return times to and from 0 by $S_1 < R_1 < S_2 < R_2 < \cdots$, and may simply set

$$dL(t) = \int_0^t 1_{\{M_s = 0\}} \, ds$$

for any choice of $d > 0$ in order to obtain a process satisfying the conditions in Theorem 5.2. From this we obtain that the inverse local time $L^{-1}$ is still a (killed) subordinator with drift $d$, whose jumps occur at the local times $L(S_i)$, for $i \geq 1$. One then considers the set of excursions, which again is given by a Poisson random measure stopped at the first infinite excursion; however, the intensity measure $n$ is finite, and is some multiple of the law of the first excursion, $(M_{t+S_1}, 0 \leq t < R_1 - S_1)$.

If 0 is irregular, then we consider the first return times: $R_0 = 0$, $R_{n+1} = \inf\{t > R_n : M_t = 0\}$. The natural thing to do, analogously with the random walk case, would be to define the local time to be $n(t) = \max\{i : R_i < t\}$, but this produces an integer-valued process, and if we tried to look at the inverse local time its domain would be $\mathbb{N}$. This is rather annoying, so we introduce an artificial time-scale for the local time; in particular, let $(\tau_i)_{i=0}^\infty$ be a sequence of i.i.d. exponentially-distributed random variables, and define

$$L(t) = \sum_{i=0}^{n(t)} \tau_i, \quad t \geq 0.$$  

Then $L$ does indeed increase on the set $\{t \geq 0 : M_t = 0\}$. The process is not adapted to $(\mathcal{F}_t)_{t \geq 0}$, but we can cheat by setting $\mathcal{F}_t' = \sigma(\mathcal{F}_t \cup \{L(s), s \leq t\})$, and then $L$ is adapted to $(\mathcal{F}_t')_{t \geq 0}$! $L$ is also discontinuous, but this is not a problem. (The requirement of continuity was really meant to ensure uniqueness of the local time.) Letting $L^{-1}$ be the right-inverse of the local time, as usual, we obtain that $L^{-1}$ is a killed subordinator. Once again, we can without difficulty define the set of excursions, which is a stopped Poisson process, and this time the intensity measure $n$ is a multiple of the law of $(M_t, 0 \leq t < R_1)$.

5.3. Wiener–Hopf factorisation for Lévy processes

In this section we are concerned with describing how Lévy processes reach new maxima and minima. We have already seen one result in this vein, [Proposition 4.12] for subordinators. The Wiener–Hopf factorisation will enable us to prove similar results for general Lévy processes, as well as many other theorems and identities. It can be seen as a decomposition of the path of a Lévy process into excursions from its maximum, and
we will rely a great deal on the construction of the local time and excursion measure given in the previous section.

Throughout this section, we are going to exclude the case of a compound Poisson process. However, we will state the main results of the Wiener–Hopf factorisation in a way that is valid also for the compound Poisson case.

Let us start by introducing notation. We define

\[
\bar{X}_t = \sup \{ 0 \vee X_s : s \leq t \}, \quad \underline{X}_t = \inf \{ 0 \wedge X_s : s \leq t \},
\]

which are called, respectively, the supremum and infimum process of \( X \). The process \( \bar{X} - X \) [resp. \( X - \underline{X} \)] is known as the Lévy process reflected in its supremum [resp. reflected in its infimum]. Notice that, under \( \mathbb{P}_x \), the initial value \( (\bar{X} - X)_0 \) of the reflected process is \( x \). For \( q > 0 \), let \( E_q \) be independent of the Lévy process \( X \), and have an exponential distribution with rate \( q \). Furthermore, let

\[
\bar{G}_t := \sup \{ s < t : X_s \vee X_{s-} = \bar{X}_t \}
\]

and

\[
G_t := \inf \{ s < t : X_s \wedge X_{s-} = \underline{X}_t \}.
\]

The random times \( \bar{G}_t \) and \( G_t \) refer to the last and first times at which \( X \) attains a final supremum and infimum. When \( X \) is not a compound Poisson process, it turns out that the time of attaining the supremum or infimum is unique, and then there are actually some nicer expressions for \( \bar{G}_t \) and \( G_t \) which we will use more often. We derive these now.

Let us first define regularity of sets. Namely, we say that \( 0 \) is regular for a set \( B \) if \( \mathbb{P}(\tau_B = 0) = 1 \), where \( \tau_B \) is the first passage time of \( B \), as in §4.3; and that \( 0 \) is irregular for \( B \) otherwise. (Since \( X \) is a nice Markov process, the Blumenthal zero-one law implies that, if \( 0 \) is irregular for \( B \), then it actually holds that \( \mathbb{P}(\tau_B > 0) = 1 \).)

We are now in a position to state and prove the following lemma.

**Lemma 5.9** (uniqueness of maxima). *Suppose that \( X \) is not a compound Poisson process. Almost surely, for any \( 0 \leq s < t \), if there exists \( u \in (s, t) \) such that \( \bar{X}_u = \underline{X}_u \), then \( \bar{X}_s < \bar{X}_t \).*

**Proof.** It suffices to prove the claim for \( s, t \) in some countable dense subset \( A \subset [0, \infty) \). We then get the claim for all \( s, t \) by taking \( (s_n) \subset A \) a decreasing sequence with limit \( s \) and \( (t_n) \subset A \) an increasing sequence with limit \( t \).

Suppose first that \( 0 \) is regular for \( (0, \infty) \). Consider \( s, t \) fixed, and let

\[
T = \inf \{ u > s : X_u = \bar{X}_u \}.
\]

On the event \( \{ T < t \} \), we apply the Markov property at \( T \), and deduce that \( \bar{X}_{T+u} > \bar{X}_T \) for all \( u > 0 \); in particular, we have that \( \bar{X}_s \leq \bar{X}_T < \bar{X}_t \).

 Suppose now instead that \( 0 \) is irregular for \( (0, \infty) \). Since \( X \) is not a compound Poisson process, it follows that \( 0 \) must be regular for \( (-\infty, 0) \). Applying the previous argument again we see that the claim holds for \( -X \). However, by the duality lemma, if we time-reverse \( X \) at \( t \), we obtain a process with the law of \( -X \); this proves the claim. \( \square \)
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**Corollary 5.10.** Suppose that $X$ is not a compound Poisson process. Then

\[
\bar{G}_t = \sup\{s < t : X_s = \bar{X}_s\}
\]

and

\[
G_t = \sup\{s < t : X_s = X_s\}.
\]

**Proof.** It follows immediately from the preceding lemma that, for any fixed $t \geq 0$,

\[
P(\exists s < u < t : \bar{X}_s = X_s = \bar{X}_u = \bar{X}_t) = 0,
\]

and thus we see that $\bar{G}_t$ is the unique moment at which $X$ attains its final supremum. Of course, these arguments hold equally for the infima and $G_t$.

The expressions from the corollary above are the ones we will usually use for $\bar{G}_t$ and $G_t$, and this simplifies matters a lot. It is largely for this reason that we omit compound Poisson processes from our study of the Wiener–Hopf factorisation.

We may as well state our main theorem right now; it will take some work to prove, however.

**Theorem 5.11** (Wiener–Hopf factorisation, first part). Let $X$ be a Lévy process which is not a compound Poisson process. Then, the pairs

\[(\bar{G}_{e_q}, \bar{X}_{e_q}) \quad \text{and} \quad (e_q - \bar{G}_{e_q}, \bar{X}_{e_q} - X_{e_q})\]

are independent and infinitely divisible, and the second pair is equal in distribution to $(G_{e_q}, -X_{e_q})$.

**Proof postponed to page 50.**

**Remark 5.12** (Proof of equality in distribution). The statement about equality in distribution is actually a straightforward application of duality, and our proof works also for compound Poisson processes. We proved in Lemma 4.14 that the equality in distribution of the second part of each pair holds at constant times $t$; the proof that $(t - \bar{G}_t, \bar{X}_t - X_t) \overset{d}{=} (G_t, -X_t)$ is identical, using the ‘general’ definitions of $\bar{G}_t$ and $G_t$. Then, since $e_q$ is independent of $X$, one has

\[
E[f(e_q - \bar{G}_{e_q}, \bar{X}_{e_q} - X_{e_q})] = \int_0^\infty e^{-qt}E[f(t - \bar{G}_t, \bar{X}_t - X_t)] dt
\]

\[
= \int_0^\infty e^{-qt}E[f(G_t, -X_t)] dt = E[f(G_{e_q}, -X_{e_q})],
\]

for Borel $f$, which proves the equality in distribution.

As alluded to, our main tool in proving Theorem 5.11 will be the local time and excursion theory we have already developed for Markov processes. Our first result is the following.

**Lemma 5.13.** The reflected process $\bar{X} - X$ is a strong Markov process in the filtration of the Lévy process $X$. 

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Proof. See Exercise 17 or [Ber96, Proposition VI.1].

We thus know that $\bar{X} - X$ is a nice Markov process in the sense of §5.2. We will denote the local time of $\bar{X} - X$ at zero by $L$, and refer to it as the local time at the supremum. As we have already alluded to, our technique will be to study the extrema of $X$ using the local time and excursions from the supremum. To this end, it is worth reflecting briefly on when $0$ is regular and instantaneous, when it is regular and a holding point, and when it is irregular (for itself, for $\bar{X} - X$).

One can distinguish the following three cases, which may help to clarify which sort of local time we are talking about. For a longer discussion with proofs, see [Kyp14, §6.1].

1. $X$ is of unbounded variation. Then, $0$ is regular for both $[0, \infty)$ and $(-\infty, 0)$. The point $0$ is an instantaneous point for the process $\bar{X} - X$. The local time $L$ of $\bar{X} - X$ is given by the construction in the previous section.

2. $X$ is of bounded variation and $0$ is regular for $[0, \infty)$. Then, $0$ is either an instantaneous point or a holding point for $\bar{X} - X$, and the local time is given by

$$dL(t) = \int_0^t 1_{\{X_s = \bar{X}_s\}} ds,$$

for some choice of $d > 0$. (See also Exercise 18.)

3. $X$ is of bounded variation and $0$ is irregular for $[0, \infty)$. Then, $0$ is irregular (for itself) for the process $\bar{X} - X$. The local time $L$ is discontinuous and given via the construction in §5.2.4.

We can give simple examples:

1. A Brownian motion (or any Lévy process with Brownian part).

2. A compound Poisson process with positive drift (in which case zero is a holding point), or a symmetric strictly $\alpha$-stable process with $\alpha \in (0, 1)$ (in which case zero is an instantaneous point).

3. A compound Poisson process with negative drift.

We will now embark on the proof of the theorem. We will rely on the theory of local times and excursions. In particular, recall from the previous section that the inverse local time at the maximum, $L^{-1}$, is a (possibly killed) subordinator with killing rate $p \geq 0$. Recall furthermore that the excursions of $\bar{X} - X$ form a (stopped) Poisson point process with intensity measure $\text{Leb} \times n$.

We will additionally define

$$H(t) = X_{L^{-1}(t)}, \quad t \geq 0,$$

(with the understanding that $H(t) = \partial$ if $L^{-1}(t) = \partial$). The bivariate process $(L^{-1}, H)$ is known as the ladder process or the ascending ladder process of $X$, and we have the following important fact.
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**Lemma 5.14.** The process \((L^{-1}, H)\) is a bivariate subordinator, which is unkillled if \(p = 0\) and killed at rate \(p\) otherwise.

We omit the proof as it is almost identical to that of Theorem 5.5.

We will denote by \(\kappa\) the Laplace exponent of the ladder process; that is,

\[
\mathbb{E}[e^{-\alpha L^{-1}(t) - \beta H(t)}] = e^{-t \kappa(\alpha, \beta)}.
\]

Although we do not have any way to express this bivariate exponent, it is worth recalling here that we do know the following expression for the marginal:

\[
\kappa(\alpha, 0) = \alpha d + \int_0^\infty (1 - e^{-\alpha x}) n(\zeta \in dx) = p + \alpha d + \int_0^\infty (1 - e^{-\alpha \zeta(\epsilon)}) \mathbb{1}_{\{\zeta(\epsilon) < \infty\}} n(\zeta(\epsilon) \in dx)
\]

where, as above, \(d \geq 0\) is the constant in Corollary 5.3 and \(n\) is the excursion measure of \(\bar{X} - X\); in the second line we have used the fact that \(p = n(\mathcal{E}_\infty)\) to separate the killing term from the Lévy measure.

As may be seen from the previous section, the rate \(p\) depends on the normalisation of the local time; in fact, since the killing time of \(L^{-1}\) is equal to \(L(\infty)\), we see that if we replace \(L\) by \(\tilde{L} = aL\), for some \(a > 0\), we will replace the rate \(p\) by \(\tilde{p} = p/a\). (Alternatively, one may observe as above that \(p = n(\mathcal{E}_\infty)\), the rate of arrival of an infinite-length excursion; the normalisation of \(n\) depends on the local time.)

The main result we need is the following decomposition, which allows us to split the path of \(\bar{X} - X\) into two independent parts at a random time which is not a stopping time.

**Lemma 5.15.**

(i) If 0 is irregular for \(\bar{X} - X\), then the processes

\[
(X_t, 0 \leq t \leq \bar{G}_{e_q}) \quad \text{and} \quad (X_{\bar{G}_{e_q} + t} - X_{\bar{G}_{e_q}}, 0 \leq t < e_q - \bar{G}_{e_q})
\]

are independent.

(ii) If 0 is regular for \(\bar{X} - X\), then the processes

\[
(X_t, 0 \leq t < \bar{G}_{e_q}) \quad \text{and} \quad (X_{\bar{G}_{e_q} + t} - X_{\bar{G}_{e_q} -}, 0 \leq t < e_q - \bar{G}_{e_q})
\]

are independent.

**Proof.** (i) Label as \(T_0, T_1, \ldots\) the times at which \(\bar{X} - X\) takes the value 0. One observes that, for each \(n\) and any \(s \geq 0\), the events \(\{\bar{G}_s = T_n\}\) and \(\{T_n < s \leq T_{n+1}\}\) are equal; this allows us to prove the statement in a discrete way. Fix \(n\), and denote by \(\tilde{X}_t = X_{T_n + t} - X_{T_n}\); since \(T_n\) is a stopping time, \(\tilde{X}\) is independent of \(\mathcal{F}_{T_n}\) and equal
in law to $X$. Fixing any bounded measurable functionals $F$ and $K$, we perform the following calculation:

$$
\mathbb{E}\left[F(X_t, 0 \leq t \leq \tilde{G}_{e_q}) K(X_{\tilde{G}_{e_q}+t} - X_{\tilde{G}_{e_q}}, 0 \leq e_q - \tilde{G}_{e_q}); T_n < e_q \leq T_{n+1}\right]
$$

$$
= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} e^{-qs} F(X_t, 0 \leq t \leq T_n)K(\tilde{X}_t, 0 \leq s - T_n) \, ds\right]
$$

$$
= \mathbb{E}\left[e^{-qT_n} F(X_t, 0 \leq t \leq T_n)\right]\mathbb{E}\left[\int_{0}^{\tilde{T}_n} e^{-qs} K(\tilde{X}_t, 0 \leq s) \, ds\right]
$$

Summing over $n$, we obtain

$$
\mathbb{E}\left[F(X_t, 0 \leq t \leq \tilde{G}_{e_q}) K(X_{\tilde{G}_{e_q}+t} - X_{\tilde{G}_{e_q}}, 0 \leq e_q - \tilde{G}_{e_q})\right]
$$

$$
= \mathbb{E}\left[K(X_t, 0 \leq t < e_q); e_q < T_1\right] \sum_{n \geq 0} \mathbb{E}\left[e^{-qT_n} F(X_t, 0 \leq t \leq T_n)\right]
$$

and this is sufficient to show that the two path sections are independent.

(ii) We will discretise the problem. Fix $\varepsilon > 0$, and let $N = \lfloor L(e_q)/\varepsilon \rfloor$, the integer part.

The random variable $L(e_q)$ has an exponential distribution: one way to see this is to replace $X$ in the foregoing discussion with $X^q$, the Lévy process killed at rate $q$, and to consider any excursion of $X^q$ in which the process is sent to $0$ to be an infinite excursion; then $L(e_q)$ is the first entry of the excursion process into $\mathcal{E}_\infty$, and hence is exponentially distributed. Since $L(e_q)$ is not lattice-valued, it follows that $\varepsilon N < L(e_q) < \varepsilon (N + 1)$, and hence that $L^{-1}(\varepsilon N) < e_q \leq L^{-1}(\varepsilon (N + 1))$. Then $\{N = n\}$ is equal to $\{L^{-1}(\varepsilon n) < e_q \leq L^{-1}(\varepsilon (n + 1))\}$, and we have

$$
\mathbb{E}\left[F(X_t, 0 \leq t \leq L^{-1}(\varepsilon N)) K(X_{L^{-1}(\varepsilon N)+t} - X_{L^{-1}(\varepsilon N)}, 0 \leq t \leq e_q - L^{-1}(\varepsilon N))\right]
$$

$$
= \sum_{n \geq 0} \mathbb{E}\left[F(X_t, 0 \leq t \leq L^{-1}(\varepsilon n)) K(X_{L^{-1}(\varepsilon n)+t} - X_{L^{-1}(\varepsilon n)}, 0 \leq t \leq e_q - L^{-1}(\varepsilon n)); L^{-1}(\varepsilon n) < e_q \leq L^{-1}(\varepsilon (n + 1))\right].
$$

For every $n$, $L^{-1}(\varepsilon n)$ is a stopping time, so we can do the same thing as in case (i) in order to split the process at $L^{-1}(\varepsilon N)$, giving

$$
\mathbb{E}\left[F(X_t, 0 \leq t \leq L^{-1}(\varepsilon N)) K(X_{L^{-1}(\varepsilon N)+t} - X_{L^{-1}(\varepsilon N)}, 0 \leq t \leq e_q - L^{-1}(\varepsilon N))\right]
$$

$$
= \mathbb{E}\left[F(X_t, 0 \leq t \leq L^{-1}(\varepsilon N))\right]\mathbb{E}\left[K(X_{L^{-1}(\varepsilon N)+t} - X_{L^{-1}(\varepsilon N)}, 0 \leq t \leq e_q - L^{-1}(\varepsilon N))\right].
$$

(5.3)

We then observe that $L^{-1}(\varepsilon N) \uparrow \tilde{G}_{e_q}$ as $\varepsilon \downarrow 0$, and using dominated convergence in (5.3) proves the theorem. \qed

It is worth remarking that a proof of this lemma using excursion theory is also possible; this forms Exercise 23. We should also remark that when $X$ drifts to $-\infty$ and $q = 0$, that is, $e_q = \infty$, this lemma is essentially given by Exercise 19.
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This lemma is already enough to prove the independence in Theorem 5.11, we need the following lemma to deduce infinite divisibility. Recall that multidimensional infinite divisibility and the multidimensional Lévy–Khintchine formula were discussed in Exercise 7.

**Lemma 5.16.** Let $Z$ be a $d$-dimensional Lévy process with characteristic exponent $\Psi : \mathbb{R}^d \to \mathbb{C}$, and $e_q$ an independent exponentially distributed random variable with parameter $q > 0$. Then the $(d + 1)$-dimensional random variable $(e_q, Z_{e_q})$ has an infinitely divisible distribution, and then

$$\mathbb{E}[e^{i\theta e_q + i\langle \theta, Z_{e_q} \rangle}] = \exp \left\{ \int_0^\infty \int_{\mathbb{R}^d} (e^{i\theta t + i\langle \theta, z \rangle} - 1) t^{-1} e^{-qt} \mathbb{P}(Z_t \in dz) \, dt \right\}$$

$$= \frac{q}{q - i\theta - \Psi(\theta)}.$$

**Proof.** The proof is fairly direct. To show that the pair is infinitely divisible, take $n \in \mathbb{N}$ and use the infinite divisibility of $e_q$ to write $e_q \overset{d}{=} \tau_1 + \cdots + \tau_n$ for i.i.d. $\tau_i$. We then simply write $(Z_{e_q}, e_q) \overset{d}{=} ((Z_{\tau_1} + \cdots + Z_{\tau_n} - Z_{\tau_1 + \cdots + \tau_n}) + \cdots + Z_{\tau_1}, \tau_1 + \cdots + \tau_n)$, and use the stationary independent increments of $Z$. Observe from the Lévy–Khintchine formula that $\Re \Psi(\theta) \leq 0$ for all $\theta \in \mathbb{R}^d$ [which was proved in one dimension in Exercise 16] and this justifies the second line in the following computation. The third line is an application of the Frullani integral from Lemma 2.1 and Exercise 5.

$$\mathbb{E}[e^{i\theta e_q + i\langle \theta, Z_{e_q} \rangle}] = \int_0^\infty q e^{-qt} e^{i\langle \theta + \Psi(\theta) \rangle t} \, dt$$

$$= q(q - i\theta - \Psi(\theta))^{-1}$$

$$= \exp \left\{ \int_0^\infty (e^{i\theta t + \Psi(\theta) t} - 1) t^{-1} e^{-qt} \, dt \right\},$$

and since $e^{\Psi(\theta)t} = \int_{\mathbb{R}^d} e^{i\langle \theta, z \rangle} \mathbb{P}(Z_t \in dz)$, this proves the lemma. \qed

We are now in a position to offer a proof of Theorem 5.11.

**Proof of Theorem 5.11** Independence. The irregular case is easy. Since the ladder times of $X$ form a discrete set, $X_{G_{e_q}} = \tilde{X}_{e_q}$. The independence then follows from the splitting lemma [Lemma 5.15].

The regular case needs a little more work. We need to prove that $X_{G_{e_q}} = \tilde{X}_{e_q}$, and then [Lemma 5.15] will give the result. Firstly, if $\tilde{G}_{e_q} = e_q$, there is nothing more to do since $X$ will be continuous at $e_q$ (by Proposition 4.6). So we will assume that $\tilde{G}_{e_q} < e_q$.

We point out first that $\tilde{X}_{e_q} = \tilde{X}_{G_{e_q}}$. There are now three cases. If $X$ is continuous at $\tilde{G}_{e_q}$, then $X_{G_{e_q}} = \tilde{X}_{G_{e_q}} = \tilde{X}_{e_q}$. If $X$ makes a downward jump at $\tilde{G}_{e_q}$, then $X_{\tilde{G}_{e_q}} = \tilde{X}_{\tilde{G}_{e_q}} = \tilde{X}_{\tilde{G}_{e_q}} > X_{G_{e_q}}$. Finally, it is not possible that $X$ makes an upward jump at $\tilde{G}_{e_q}$, as we now show. Let $\varepsilon > 0$, and define $T^\varepsilon_0 = 0$ and

$$T^\varepsilon_n = \inf \{ t > T^\varepsilon_{n-1} : \Delta X_t \geq \varepsilon \}, \quad n \geq 1, \quad T^\varepsilon_n < \infty,$$
the times at which $X$ jumps upwards by at least $\varepsilon$. Since $T_n^\varepsilon$ is a stopping time for every $n$, and 0 is regular for $[0, \infty)$, the process $(X_{T_n^\varepsilon + t} - X_{T_n^\varepsilon}, \ t \geq T_n^\varepsilon)$, hits $[0, \infty)$ immediately. Thus, if $X_{T_n^\varepsilon} = \bar{X}_{T_n^\varepsilon}$, the reflected process $\bar{X} - X$ will also hit 0 immediately after $T_n^\varepsilon$. It follows that at time $G_{eq}$, $X$ cannot jump further than $\varepsilon$ upwards, and $\varepsilon$ was arbitrary; we are done.

**Infinite divisibility.** We prove that the first pair is infinitely divisible; the second then follows automatically by Remark 5.12 (together with the fact that $X$ is not compound Poisson, so that $G_t$ is just $G_t$ calculated for the dual Lévy process $-X$.) We will give the proof in the regular case only. In the irregular case, the proof is very similar, and indeed more direct; see Lemma A.7.

The proof proceeds directly by calculating the Laplace transform. The calculation is a typical application of the compensation formula in excursion theory. Recall, from the `independence` part, that $\bar{X}_{eq} = X_{G_{eq} -} = \bar{X}_{G_{eq} -}$. Thus, for $\alpha, \beta > 0$, we compute

$$
\mathbb{E}[e^{-\alpha G_{eq} - \beta \bar{X}_{eq}}] = \mathbb{E}[e^{-\alpha \bar{G}_{eq} - \beta \bar{X}_{eq}}] = \mathbb{E}\left[\int_0^\infty q e^{-qt} e^{-\alpha \bar{G}_t - \beta \bar{X}_t} dt\right]
$$

$$
= \mathbb{E}\left[\int_0^\infty q e^{-qt} 1_{\{X_t = \bar{X}_t\}} e^{-\alpha \bar{G}_t - \beta \bar{X}_t} dt\right] + \mathbb{E}\left[\sum_{(g,d)} \int_0^d q e^{-qt} e^{-\alpha \bar{G}_t - \beta \bar{X}_t} dt\right]
$$

$$
= \mathbb{E}\left[\int_0^\infty q e^{-qt} 1_{\{X_t = \bar{X}_t\}} e^{-\alpha t - \beta \bar{X}_t} dt\right] + \mathbb{E}\left[\sum_{(g,d)} e^{-\alpha g - \beta \bar{X}_g} e^{-\eta g} \int_0^d q e^{-qt} dt\right],
$$

where in the final two lines, the sum is over excursion intervals $(g, d)$, and we have used, in the last line, the fact that $\bar{X}$ has only countably many jumps.

We rewrite the first term using Corollary 5.3 and changing variables (see Corollary A.6); here, $\zeta = L(\infty)$ represents the lifetime of $(L^{-1}, H)$.

$$
qd \mathbb{E}\left[\int_0^\infty e^{-qt} e^{-\alpha t - \beta \bar{X}_t} dL(t)\right] = qd \mathbb{E}\left[\int_0^{\zeta} e^{-(\alpha + q)L^{-1}(t) - \beta H(t)} dt\right]
$$

$$
= \frac{qd}{\kappa(\alpha + q, \beta)},
$$

with the last equality following by Exercise 14.

We express the second term using the compensation formula Corollary 5.8 with $F_s(\varepsilon) = e^{-(\alpha + q)s - \beta X_s} \int_0^{\zeta(\varepsilon)} q e^{-qt} dt$. (Below, $\zeta$ unfortunately represents both the lifetime of $(L^{-1}, H)$ and the length of the generic excursion under $n$. We use, in the first equality, the fact that $H$ has only countably many jumps.)

$$
\mathbb{E}\left[\int_0^\infty e^{-(\alpha + q)t - \beta \bar{X}_t} dL(t)\right] n(1 - e^{-\eta \bar{\zeta}}) = \mathbb{E}\left[\int_0^{\bar{\zeta}} e^{-(\alpha + q)L^{-1}(t) - \beta H(t)} dt\right] n(1 - e^{-\eta \bar{\zeta}})
$$

$$
= \frac{n(1 - e^{-\eta \bar{\zeta}})}{\kappa(\alpha + q, \beta)},
$$

again using Exercise 14.
Comparing with (5.2), we have that
\[
E[e^{-\alpha G_{eq} - \beta X_{eq}}] = \frac{\kappa(q, 0)}{\kappa(\alpha + q, \beta)},
\]
and we just need to show that the right-hand side is in fact the Laplace transform of an infinitely divisible random variable.

Let \( \tilde{\kappa}(\alpha, \beta) = \kappa(\alpha + q, \beta) - \kappa(q, 0) \). Denote by \( d \) and \( \Lambda \) the drift and Lévy measure of the ladder process \((L^{-1}, H)\). Using the Lévy–Khintchine formula, one sees that \( \tilde{\kappa} \) is the Laplace exponent of a subordinator with drift \( d \), Lévy measure \( e^{-\gamma x} \Lambda(dx, dy) \) and no killing. Let us call this subordinator \( \tilde{Z} \).

Finally, letting \( \tau \) be an exponential random variable with rate \( \kappa(q, 0) \), we obtain that, for \( \alpha, \beta > 0 \),
\[
E[e^{-t\tilde{\kappa}(\alpha, \beta)}] = \kappa(q, 0) \int_0^\infty e^{-\kappa(q, 0)t} e^{\tilde{\kappa}(\alpha, \beta)t} dt = \frac{\kappa(q, 0)}{\kappa(\alpha + q, \beta)}
\]
Hence, \((\tilde{G}_{eq}, \tilde{X}_{eq})\) has the same distribution as \( \tilde{Z}_\tau \), and by Lemma 5.16 this distribution is infinitely divisible.

We have thus proved a decomposition of the pair \((e_q, X_{eq})\) into two infinitely divisible random variables. We can say some more about these distributions; some of this information is extracted directly from the proof above.

**Theorem 5.17** (Wiener–Hopf factorisation, second part). Suppose that \( X \) is not a compound Poisson process. For every \( \gamma, \beta > 0 \),
\[
E[e^{-\gamma G_{eq} - \beta X_{eq}}] = \exp\left\{- \int_0^\infty \int_{[0,\infty)} (1 - e^{-\gamma t - \beta x})t^{-1}e^{-qt} \mathbb{P}(X_t \in dx) dt\right\},
\]
and
\[
E[e^{-\gamma (e_q - \tilde{G}_{eq}) - \beta (X_{eq} - \tilde{X}_{eq})}] = \exp\left\{- \int_0^\infty \int_{(-\infty,0)} (1 - e^{-\gamma t + \beta x})t^{-1}e^{-qt} \mathbb{P}(X_t \in dx) dt\right\}.
\]

**Proof.** Consider the pairs \((e_q, X_{eq})\), \((\tilde{G}_{eq}, \tilde{X}_{eq})\) and \((e_q - \tilde{G}_{eq}, X_{eq} - \tilde{X}_{eq})\). Denote the Lévy measures of these pairs by, respectively, \( \mu \), \( \mu^+ \) and \( \mu^- \). The support of \( \mu^+ \) is contained within \([0,\infty) \times [0,\infty)\) and that of \( \mu^- \) is contained within \([0,\infty) \times (-\infty,0]\). Since \( e_q \) can take arbitrarily small values with positive probability, the same holds for \((\tilde{G}_{eq}, \tilde{X}_{eq})\) and, using Remark 5.12, also for \((e_q - \tilde{G}_{eq}, X_{eq} - \tilde{X}_{eq})\); this implies that both of these pairs have zero drift. Moreover, neither may contain a Gaussian component since each is supported on a strict subset of \(\mathbb{R}^2\). Thus, the law of each pair is determined by its Lévy measure.

Now \((e_q, X_{eq}) = (\tilde{G}_{eq}, \tilde{X}_{eq}) + (e_q - \tilde{G}_{eq}, X_{eq} - \tilde{X}_{eq})\), and the two random variables on the right-hand side are independent by Theorem 5.11. It then follows that \( \mu = \mu^+ + \mu^- \), and we also know from Lemma 5.16 that
\[
\mu(dt, dx) = t^{-1}e^{-qt} \mathbb{P}(X_t \in dx) dt, \quad t > 0, \ x \in \mathbb{R}.
\]
Since $X$ is not a compound Poisson process, $\mathbb{P}(X_t = 0) = 0$ for a.e. $t > 0$ (see [Ber96, Proposition I.15] or [Sat99, Theorem 27.4].) Hence, using only the information about the support of these measures, we have

$$
\mu^+(dt, dx) = t^{-1}e^{-qt} \mathbb{P}(X_t \in dx) dt, \quad t > 0, x > 0,
$$

and

$$
\mu^-(dt, dx) = t^{-1}e^{-qt} \mathbb{P}(X_t \in dx) dt, \quad t > 0, x < 0.
$$

This proves the theorem. 

Let us briefly discuss the conjunction of Theorems 5.11 and 5.17. We have essentially discovered that the pair $(q, X e^q)$ can be factorised into two independent infinitely divisible distributions, and (due to the argument in the proof of Theorem 5.17) this factorisation is unique. This amounts to the following:

$$
\frac{q}{q - i\vartheta - \Psi(\theta)} = \mathbb{E}\left[e^{i\vartheta e^q + i\theta X e^q}\right] = \mathbb{E}\left[e^{i\vartheta \hat{G}_e + i\theta \hat{X}_e}\right] \mathbb{E}\left[e^{i\theta(e^q - \hat{G}_e) + i\theta(X_e - \hat{X}_e)}\right] = \Phi^+_q(\vartheta, \theta)\Phi^-_q(\vartheta, \theta),
$$

where $\Phi^+_q$ and $\Phi^-_q$ are the characteristic functions(!) of the random variables $(\hat{G}_e, \hat{X}_e)$ and $(G_e, X e^q)$ respectively. These $\Phi^+_q$ are sometimes called the Wiener–Hopf factors of $X$.

It is also useful to have the above results reworked in terms of the ladder processes. Recall that $\kappa$ is the Laplace exponent of the bivariate subordinator $(L^{-1}, H)$, which is known as the ascending ladder process. Denote by $(\hat{L}^{-1}, \hat{H})$ the equivalent subordinator computed for the dual process $\hat{X} = -X$, and write $\hat{\kappa}$ for its Laplace exponent. This essentially amounts to replacing the supremum and last instant at the supremum, $\bar{X}$ and $\bar{G}$, with the infimum and last instant at the infimum, $\underline{X}$ and $\underline{G}$, throughout. We call $(\hat{L}^{-1}, \hat{H})$ the descending ladder process of $X$.

**Theorem 5.18** (Wiener–Hopf factorisation, third part: ladder processes). Let $X$ be a Lévy process which is not a compound Poisson process.

(i) The random variables in Theorem 5.17 are related to the ladder processes by

$$
\mathbb{E}\left[e^{-\gamma \hat{G}_e - \beta \hat{X}_e}\right] = \frac{\kappa(q, 0)}{\kappa(\gamma + q, \beta)},
$$

and

$$
\mathbb{E}\left[e^{-\gamma(e^q - \hat{G}_e) - \beta(\hat{X}_e - X_e)}\right] = \mathbb{E}\left[e^{-\gamma \hat{G}_e + \beta \hat{X}_e}\right] = \frac{\hat{\kappa}(q, 0)}{\hat{\kappa}(\gamma + q, \beta)}.
$$

(ii) The Laplace exponent $\kappa$ may be identified as

$$
\kappa(\alpha, \beta) = k \exp \left\{ \int_0^\infty dt \int_{(0, \infty)} (e^{-t} - e^{-\alpha t - \beta x}) t^{-1} \mathbb{P}(X_t \in dx) \right\}, \quad \alpha, \beta \geq 0,
$$

for some $k > 0$ which depends on the normalisation of local time.
(iii) Let $\Psi$ be the characteristic exponent of $X$. Then, for some $c > 0$,
\[-c\Psi(\theta) = \kappa(0, -i\theta)\hat{\kappa}(0, i\theta), \quad \theta \in \mathbb{R}.
\]
This last equality is sometimes known as the (spatial) Wiener–Hopf factorisation of $\Psi$.

Proof.  
(i) The relationship between $(\bar{G}_{e_q}, \bar{X}_{e_q})$ and the ladder process is given in the course of the proof of Theorem 5.11, in (5.5). The second identity in the theorem is given by the first together with duality (Remark 5.12.)

(ii) For $\alpha > 1$ and $\beta \geq 0$, the result follows by taking $q = 1$ in part (i), using Theorem 5.17 with $\gamma = \alpha - 1$, and setting $k = \kappa(1, 0)$.

Fix $\beta \geq 0$. In order to extend the result to $\alpha > 0$, the first step is to notice that the right-hand side of (5.6), with $q = 1$, is analytic for $\gamma \in \{z \in \mathbb{C} : \text{Re } z > -1\}$. The proof of this is essentially the same as the proof of Proposition 4.11, and it would be distracting to give it here; see Corollary A.4.

Once we have observed this, it follows that the right-hand side of (5.8) is actually analytic for $\alpha \in \mathbb{C}_r^\circ = \{z \in \mathbb{C} : \text{Re } z > 0\}$. We already know, thanks to Proposition 4.11 that the left-hand side of (5.8) is analytic for $\alpha$ in $\mathbb{C}_r^\circ$, so by the remark about uniqueness in §4.6 we have proved the identity for $\alpha > 0$, $\beta \geq 0$.

Finally, for $\alpha, \beta = 0$ the identity follows by monotone convergence; note that integral may be equal to $-\infty$, in which case $\kappa(0, 0) = 0$.

(iii) From (ii) we deduce
\[
\kappa(\alpha, 0) = k \exp \left\{ \int_0^\infty (e^{-t} - e^{-\alpha t})t^{-1} \mathbb{P}(X_t \geq 0) \, dt \right\}, \quad \alpha > 0,
\]
and the equivalent expression for $\hat{\kappa}$, so that (recalling $\mathbb{P}(X_t = 0) = 0$ for almost every $t$, since $X$ is not a compound Poisson process)
\[
\kappa(\alpha, 0)\hat{\kappa}(\alpha, 0) = k\hat{k} \exp \left\{ \int_0^\infty (e^{-t} - e^{-\alpha t})t^{-1} \, dt \right\}, \quad \alpha > 0.
\]
Using the Frullani integral from Lemma 2.1, we see immediately that
\[
\kappa(\alpha, 0)\hat{\kappa}(\alpha, 0) = k\hat{k} \alpha, \quad \alpha > 0. \tag{5.9}
\]
Now, taking (5.7) with $\vartheta = 0$, and evaluating it using part (i) and (5.9) we obtain
\[
\frac{1}{q - \Psi(\theta)} = \frac{k\hat{k}}{\kappa(q, -i\theta)\hat{\kappa}(q, i\theta)}, \quad \theta \in \mathbb{R}, \tag{5.10}
\]
and letting $q \downarrow 0$ gives the result with $c = k\hat{k}$. \qed
Remark 5.19. Equation [5.10] may also be seen as a Wiener–Hopf factorisation. On the left-hand side, observe that \( q - \Psi(\theta) = -\Psi_q(\theta) \), where \( \Psi_q \) is the characteristic exponent of the process \( X \) killed at rate \( q \). On the right-hand side, we have the Laplace exponents of the ladder processes (in space and time).

Theorem 5.18(iii) or [5.10] can occasionally be used to find the ladder processes of \( X \), if one has a good guess for what \( \kappa \) and \( \hat{\kappa} \) should be, because they are unique; let us explain briefly why.

The idea is to run the proof backwards. Suppose that \( -\Psi_q(\theta) = v(q, -i\theta)\hat{v}(q, i\theta) \), for \( v, \hat{v} \) the Laplace exponents of some subordinators. Using the argument at the end of the proof of Theorem 5.11 we can construct an infinitely divisible random variable \( Y \) on \([0, \infty)\) with characteristic function \( v(q, 0)/v(q, -i\theta) \) and no Gaussian part or drift; and another infinitely divisible random variable \( Z \) on \((-\infty, 0] \) with characteristic function \( \hat{v}(q, 0)/\hat{v}(q, i\theta) \) and no Gaussian part or drift. But now have \( \mathbb{E}[e^{i\theta Y}] = \mathbb{E}[e^{i\theta Y}] \mathbb{E}[e^{i\theta Z}] \), i.e., we have factored \( X_{eq} \) into two infinitely divisible distributions with disjoint support who law is determined by their Lévy measures. By the argument in the proof of Theorem 5.17, such a factorisation is unique, and we already know that \( \mathbb{E}[e^{i\theta X_{eq}}] = \mathbb{E}[e^{i\theta X_{eq}}] \mathbb{E}[e^{i\theta X_{eq}}] \). It follows that \( Y \overset{d}{=} \bar{X}_{eq} \) and \( Z \overset{d}{=} X_{eq} \), and hence \( v = \kappa \) and \( \hat{v} = \hat{\kappa} \).

An example in which we deduce \( \kappa \) and \( \hat{\kappa} \) from Theorem 5.18(iii) is given in Exercise 25.

Example 5.20 (Brownian motion). It is rather rare that one can compute \( \Phi_q^\pm \) explicitly, but one case where this is possible is when \( X \) is a standard Brownian motion. Then \( \Psi(\theta) = -\theta^2/2 \), and we may write

\[
\frac{q}{q - i\theta + \theta^2/2} = \frac{\sqrt{2q}}{\sqrt{2q - 2i\theta}} \frac{\sqrt{2q}}{-i\theta}\frac{\sqrt{2q}}{-i\theta}.
\]

It is not obvious that these factors are characteristic functions of infinitely divisible distributions, but one may identify them as such using Theorem 5.18(i) and the ‘guess’

\[
\kappa(\alpha, \beta) = \hat{\kappa}(\alpha, \beta) = \sqrt{2\alpha + \beta}.
\]

This is certainly the Laplace exponent of a bivariate subordinator: the ladder process is a \( 1/2 \) stable subordinator (recall Exercise 15) together with an independent unit drift subordinator. By the reasoning in the proof of Theorem 5.11, which says that \( \kappa(q, 0)/\kappa(\alpha + q, \beta) \) is the Laplace exponent of an infinitely divisible distribution, together with the uniqueness of the Wiener–Hopf factorisation, \( \Phi_q^\pm \) must then be the factors identified.

In particular, we see that \( G_{eq} \sim \Gamma(q, 1/2) \), and \( \bar{X}_{eq} \sim \operatorname{Exp}(\sqrt{2q}) \). The same is true for the reflected variables, which is not a surprise since Brownian motion is symmetric.

Example 5.21 (Spectrally negative Lévy processes). Let \( X \) be a spectrally negative Lévy process. Such processes were introduced in Exercise 12: we say that a Lévy process \( X \) is spectrally negative if it has no positive jumps (and \( -X \) is not a subordinator.) We characterise spectrally negative Lévy processes via a function \( \phi: [0, \infty) \rightarrow \mathbb{R} \) given by \( \mathbb{E}[e^{\lambda X_1}] = e^{\phi(\lambda)} \).
5. Wiener–Hopf factorisation

In Exercise 21, we saw that $\bar{X}$ is a local time for $X$ at the supremum. By definition, we have that

$$L^{-1}(t) := \inf\{s > 0 : \bar{X}_s > t\} = \inf\{s > 0 : X_s > t\} =: \tau_t^+,$$

that is, the inverse local time of $X$ at its supremum is just the first passage time process of $X$. We studied this process (under a simplifying assumption) in Exercise 13. It turns out that the Laplace exponent of $L^{-1}$ is given by

$$\Upsilon(\alpha) := \sup\{\lambda \geq 0 : \phi(\lambda) = \alpha\}, \quad \alpha \geq 0.$$

Furthermore, since $X$ has no positive jumps, it is immediate that $H(t) = X_{L^{-1}(t)} = t$ for $t \geq 0$, and hence

$$\kappa(\alpha, \beta) = \Upsilon(\alpha) + \beta, \quad \alpha, \beta \geq 0.$$

It follows from Theorem 5.18(i) that

$$\Phi^+_q(\vartheta, \theta) = \frac{\Upsilon(q)}{\Upsilon(q - i\vartheta) - i\theta},$$

and, plugging this into (5.7) we see that

$$\Phi^-_q(\vartheta, \theta) = \frac{q}{\Upsilon(q)} \frac{\Upsilon(q - i\vartheta) - i\theta}{p - i\vartheta - \Psi(\theta)}.$$

Again applying Theorem 5.18(i) we obtain

$$\hat{k}(\alpha, \beta) = \frac{\alpha - \Psi(-i\beta)}{\Upsilon(\alpha) - \beta} = \frac{\alpha - \phi(\beta)}{\Upsilon(\alpha) - \beta}, \quad \alpha, \beta \geq 0.$$

This may not seem particularly helpful, but it is one of the few cases where one can obtain the law of the bivariate ladder processes $(L^{-1}, H)$ and $(\hat{L}^{-1}, \hat{H})$ even in semi-explicit form. For a systematic exposition of fluctuation theory for spectrally negative Lévy processes, which takes the above identities as a starting point, see Kyprianou [Kyp14, §§8–10] and Kuznetsov, Kyprianou and Rivero [KKR13].

Example 5.22 (Stable processes). Let $X$ be a strictly $\alpha$-stable process. We will use the expression for $\kappa$ in Theorem 5.18 to compute the marginal distributions of the ladder time process $L^{-1}$ and the ladder height process $H$ of $X$. Define

$$\rho = \mathbb{P}(X_t \geq 0) = \mathbb{P}(t^{1/\alpha}X_1 \geq 0),$$

where the second equality is due to the ‘scaling property’ ($\alpha$-ss) and shows that $\rho$ does not depend on $t$. $\rho$ is called the positivity parameter of $X$. We will exclude the cases $\rho = 0$ and $\rho = 1$, since in these cases either $-X$ or $X$ is a subordinator.
5. Wiener–Hopf factorisation

In order to find the distribution of the ladder time process $L^{-1}$, we make the following computation using the Frullani integral of Lemma 2.1:

$$\kappa(\lambda, 0) = k \exp \left\{ \int_0^{\infty} \int_{[0, \infty)} (e^{-t} - e^{-\lambda t}) t^{-1} P(X_t \in dx) \, dt \right\}$$

$$= k \exp \left\{ \int_0^{\infty} (e^{-t} - e^{-\lambda t}) \rho t^{-1} \, dt \right\}$$

$$= k\lambda^\rho.$$ 

Hence, $L^{-1}$ is a strictly $\rho$-stable subordinator. Recall that in Exercise 21, we showed that in the case of a spectrally negative stable process, $L^{-1}$ was a $1/\alpha$-stable subordinator. Thus, we have indirectly shown that $\rho = 1/\alpha$ when $X$ is spectrally negative.

In the same vein, we calculate the distribution of the ladder height process $H$. Specifically,

$$\kappa(0, \lambda) = k \exp \left\{ \int_0^{\infty} \int_{[0, \infty)} (e^{-t} - e^{-\lambda x}) t^{-1} P(X_t \in dx) \, dt \right\}$$

$$= k \exp \left\{ \int_0^{\infty} t^{-1} \mathbb{E} \left[ e^{-t} - e^{-\lambda X_t}; X_t \geq 0 \right] \, dt \right\}$$

$$= k \exp \left\{ \int_0^{\infty} s^{-1} \mathbb{E} \left[ e^{-s\lambda^{-\alpha}} - e^{-\lambda X_s}; \lambda X_s \geq 0 \right] \, ds \right\}$$ [setting $t = \lambda^{-\alpha} s$]

$$= k \exp \left\{ \int_0^{\infty} s^{-1} \mathbb{E} \left[ e^{-s\lambda^{-\alpha}} - e^{-X_s}; X_s \geq 0 \right] \, ds \right\}$$ [by scaling]

$$= k \exp \left\{ \int_0^{\infty} s^{-1} \mathbb{E} \left[ e^{-s} - e^{-X_s}; X_s \geq 0 \right] \, ds \right\} \exp \left\{ \int_0^{\infty} (e^{-s\lambda^{-\alpha}} - e^{-s}) \rho s^{-1} \, ds \right\}$$

$$= \kappa(0, 1) \lambda^\alpha$$ [by the Frullani integral].

Hence, $H$ is a strictly $\alpha\rho$-stable subordinator.

We remark that the joint distribution of $(L^{-1}, H)$ is significantly harder to determine. For certain choices of parameters, results were established in the 70s and 80s, and from 2008 onward a series of papers, making use of complex analysis and special functions, give explicit results in the general case. See the references in [Kyp14, p. 185] for details.

5.4. Some applications

Having proved the Wiener–Hopf factorisation for Lévy processes, and seen a few explicit and semi-explicit examples, we will bring the course to a close by looking at some sample applications. Of course, we are only going to scratch the surface, and the interested reader can find many more interesting facts and identities by consulting, for example, [Ber96, §§VI.3–5] or [Kyp14, §7]. The factorisation also forms the foundation of a great deal of current research on Lévy processes.

Since the coming results rely on our analysis in the previous section, we will assume throughout that $X$ is not a compound Poisson process. However, all the results in this section remain valid in that case.

**Theorem 5.23.** Let $X$ be a Lévy process which is not identically zero.
5. Wiener–Hopf factorisation

(i) If \( \int_1^\infty t^{-1} \mathbb{P}(X_t \geq 0) \, dt < \infty \), then \( \lim_{t \to \infty} X_t = -\infty \) almost surely, and we say that \( X \) drifts to \(-\infty\).

(ii) If \( \int_1^\infty t^{-1} \mathbb{P}(X_t \leq 0) \, dt < \infty \), then \( \lim_{t \to \infty} X_t = +\infty \) almost surely, and we say that \( X \) drifts to \(+\infty\).

(iii) If both integrals above are equal to \( \infty \), then \( \lim \sup_{t \to \infty} X_t = -\lim \inf_{t \to \infty} X_t = \infty \), and we say that \( X \) oscillates.

Since \( \int_1^\infty t^{-1} \, dt = \infty \), every Lévy process satisfies either (i), (ii) or (iii).

Proof. (i) We begin with (5.6) with \( \beta = 0 \) and take \( q \downarrow 0 \). By monotone convergence, we have that

\[
\mathbb{E}\left[e^{-\gamma G_\infty} ; G_\infty < \infty \right] = \exp\left\{-\int_0^\infty (1 - e^{-\gamma t}) t^{-1} \mathbb{P}(X_t \geq 0) \, dt \right\}, \quad \gamma > 0,
\]

where \( G_\infty = \sup\{t \geq 0 : X_t = X_1\} \). For small values of \( \gamma > 0 \), we have \( 0 \leq (1 - e^{-\gamma t}) \leq 1 \wedge t \), and hence

\[
\int_0^\infty (1 - e^{-\gamma t}) t^{-1} \mathbb{P}(X_t \geq 0) \, dt \leq \int_1^\infty t^{-1} \mathbb{P}(X_t \geq 0) \, dt + \int_0^1 \mathbb{P}(X_t \geq 0) \, dt < \infty.
\]

Using dominated convergence as \( \gamma \downarrow 0 \), we obtain that \( \mathbb{P}(G_\infty < \infty) = 1 \). In particular, this implies that \( \bar{X}_\infty < \infty \) a.s..

Now, the integral test implies that \( \int_1^\infty t^{-1} \mathbb{P}(X_t < 0) \, dt = \infty \). Considering (5.6) applied to the dual process \(-X\), and, as before, taking \( q \downarrow 0 \) using monotone convergence, we obtain that

\[
\mathbb{E}\left[e^{-\gamma G_\infty} ; G_\infty < \infty \right] = \exp\left\{-\int_0^\infty (1 - e^{-\gamma t}) t^{-1} \mathbb{P}(X_t \leq 0) \, dt \right\}, \quad \gamma > 0,
\]

However,

\[
\int_0^\infty (1 - e^{-\gamma t}) t^{-1} \mathbb{P}(X_t \leq 0) \, dt \geq (1 - e^{-\gamma}) \int_1^\infty t^{-1} \mathbb{P}(X_t < 0) \, dt = \infty,
\]

and so \( 0 \leq \mathbb{E}[e^{-\gamma G_\infty} ; G_\infty < \infty] \leq e^{-\infty} = 0 \) for every \( \gamma > 0 \). From this it follows that \( \mathbb{P}(G_\infty < \infty) = 0 \). That is to say, \( \bar{X}_\infty = -\infty \) a.s..

We have shown that \( \bar{X}_\infty < \infty \) and \( \underline{X}_\infty = -\infty \). It follows that the downward first passage time

\[
\tau_x^- = \inf\{t > 0 : X_t \leq -x\}
\]

is finite a.s. for every \( x > 0 \). On the other hand, since \( \bar{X}_\infty < \infty \), we have that

\[
\mathbb{P}(\tau_x^+ < \infty) \downarrow 0, \quad \text{as } x \uparrow \infty.
\]

Fix \( \varepsilon > 0 \). By the preceding limit, there exists \( x_\varepsilon > 0 \) such that

\[
\mathbb{P}(\tau_x^+ < \infty) < \varepsilon, \quad \text{for all } x \geq 2x_\varepsilon.
\]
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Then, by applying the Markov property at $\tau_x^-$, we have

$$
P(X_t > -x/2 \text{ for some } t > \tau_x^-) \leq \varepsilon, \quad x \geq x_\varepsilon
$$

and we have that, for all $x \geq x_\varepsilon$,

$$
1 - \varepsilon \leq P(X_t \leq -x/2 \text{ for all } t > \tau_x^-) \leq P(\exists u > 0 : \forall t > u : X_s \leq -x/2) \leq P(\limsup_{t \to \infty} X_s \leq -x/2) = P(\limsup_{t \to \infty} X_t \leq -x/2)
$$

By taking $x \to \infty$ and $\varepsilon \to 0$ we get that $\limsup_{t \to \infty} X_t = -\infty$ almost surely, and this completes the proof of (i).

(ii) This follows from (i) applied to the dual process $-X$.

(iii) If both integral tests fail, then by the reasoning given in the proof of (i), $\bar{X}_\infty = \infty$ and $X_\infty = -\infty$ almost surely, and this is precisely what was claimed. \qed

We now offer quite an extensive generalisation of Proposition 4.12 to any Lévy process. Before we begin, recall the bivariate subordinator $(L^{-1}, H)$, the ascending ladder process; we will denote by $U$ the bivariate potential measure of this subordinator, that is,

$$
U(dt, dx) = \mathbb{E}\left[\int_0^\zeta 1_{(L^{-1}(s)\in dt,H(s)\in dx)} ds\right], \quad x, t \geq 0,
$$

where $\zeta$ is the lifetime of $L^{-1}$. This is also called the ascending renewal measure of $X$. Recall also (see Exercise 14) that

$$
\int_{[0,\infty) \times [0,\infty)} e^{-\alpha t - \beta x} U(dt, dx) = \frac{1}{\kappa(\alpha, \beta)}, \quad \alpha, \beta \geq 0.
$$

We have already said that the descending ladder process $(\hat{L}^{-1}, \hat{H})$ is the same quantity with $X$ replaced by $\hat{X} = -X$, and we define $\hat{U}$, the descending renewal measure, similarly. Of course, we also have

$$
\int_{[0,\infty) \times [0,\infty)} e^{-\alpha t - \beta x} \hat{U}(dt, dx) = \frac{1}{\hat{\kappa}(\alpha, \beta)}, \quad \alpha, \beta \geq 0.
$$

Recall that the local time of $X$ at its supremum is only defined up to a constant multiple, and changing the normalisation will affect $U$ by a constant multiple too (i.e., replacing $L$ with $cL$ will replace $U$ by $cU$). For the coming theorem, we need to fix the normalisation of local time, and we do it in such a way that $k\hat{k} = 1$, where $k > 0$ is the constant appearing in (5.8) and $\hat{k}$ is the equivalent for $(\hat{L}^{-1}, \hat{H})$. This has the effect that (5.9) reads

$$
\kappa(\alpha, 0)\hat{\kappa}(\alpha, 0) = \alpha, \quad (5.11)
$$

and this fact will prove useful.
Theorem 5.24 (Quintuple law). In the normalisation of local time given above, for each $x > 0$, it holds that for $u > 0, v \geq y, y \in [0, x], s, t \geq 0$,
\[
\mathbb{P}(\bar{G}_{\tau_x^+} \in ds, \tau_x^+ - \tau_x^- \in dt, X_{\tau_x^+} - x \in du, x - X_{\tau_x^+} \in dv, x - \bar{X}_{\tau_x^+} \in dy)
= \mathcal{U}(ds, x - dy) \hat{\mathcal{U}}(dt, dv - y) \Pi(du + v),
\]
where $\Pi$ is the Lévy measure of $X$ and $\mathcal{U}, \hat{\mathcal{U}}$ are the renewal measures.

Before starting the proof, let us recall a result which may be familiar, and which broadly speaking is part of the so-called ‘continuity theorem’ for Laplace transforms. First we recall the meaning of the Laplace transform in higher dimensions: if $\mu$ is a measure on $[0, \infty)^d$, then its Laplace transform is the function $\phi$ given by
\[
\phi(\lambda) = \int_{[0, \infty)^d} e^{-\langle \lambda, x \rangle} \mu(dx), \quad \lambda \in (0, \infty)^d.
\]

With this definition, we have the following theorem:

Lemma 5.25. Suppose that $\mu_n$ are measures on $[0, \infty)^d$ with Laplace transforms $\phi_n$, and $\mu$ is a measure on $[0, \infty)^d$ with Laplace transform $\phi$. Suppose that $\phi_n(\lambda) \to \phi(\lambda)$ for all $\lambda \in (0, \infty)^d$. If $f: [0, \infty)^d \to [0, \infty)$ is a continuous function with compact support, then
\[
\lim_{n \to \infty} \int f(x) \mu_n(dx) = \int f(x) \mu(dx).
\]

Proof. The one-dimensional version is given in [Fel71, Theorem XIII.1.2a]. Alternatively, one may adapt the proof of [Kal02, Theorem 5.22] (the tightness argument there is unnecessary.)

We are ready.

Proof of Theorem 5.24. Let $m, k, f, g$ and $h$ be positive, continuous functions with compact support, such that $f(0) = 0$. We begin by using the compensation formula for the jumps of $X$ in order to rewrite the law; in the fourth equality we set $w(z) =$
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Figure 10: A symbolic depiction of the quantities involved in the quintuple law.

Figure 11: A symbolic depiction of the quantities on the right-hand side of the quintuple law. The spatial components can be interpreted as follows: we first reach the maximum \( x - y \), then enter an excursion away from the maximum with final value \( v - y \), and then exit this excursion via a jump of \( X \) with size \( u + v \).
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\[ g(z) \int_{(z,\infty)} f(u-z) \Pi(du) \]

\[
\int_{u>0,y\in[0,x],v\in[0,y],s\geq 0,t \geq 0} k(s)m(t)f(u)g(v)h(y) \\
\times \mathbb{P}(\bar{G}_{\tau^+_x} \in ds, \tau^+_x - \bar{G}_{\tau^+_x} \in dt, X_{\tau^+_x} - x \in du, x - X_{\tau^+_x} \in dv, x - \bar{X}_{\tau^+_x} \in dy) \\
= \mathbb{E}\left[ k(\bar{G}_{\tau^+_x})m(\tau^+_x - \bar{G}_{\tau^+_x})f(X_{\tau^+_x} - x)g(x - X_{\tau^+_x})h(x - \bar{X}_{\tau^+_x}) \right] \\
= \mathbb{E}\left[ \int_{(0,\infty) \times \mathbb{R}} k(\bar{G}_{t-})m(t - \bar{G}_{t-})f(X_{t-} + z - x)g(x - X_{t-})h(x - \bar{X}_{t-}) \\
\times 1_{\{X_{t-} \leq x\}} 1_{\{X_{t-} + z > x\}} N(dt, dz) \right] \quad \text{[since \( X_t = X_{t-} + \Delta X_t \)}
\]

\[
= \mathbb{E}\left[ \int_{0}^{\infty} dt \ k(\bar{G}_{t})m(t - \bar{G}_{t})g(x - X_{t})h(x - \bar{X}_{t}) 1_{\{X_{t} \leq x\}} \right] \\
\times \int_{(x-X_{t},\infty)} \Pi(d\phi) f(X_{t} + \phi - x) \quad \text{[compensation, countable jumps]}
\]

\[
= \mathbb{E}\left[ \int_{0}^{\tau^+_x} k(\bar{G}_{t})m(t - \bar{G}_{t})h(x - \bar{X}_{t})w(x - X_{t}) dt \right] \quad \text{[definition of \( w \)}} \quad (5.12)
\]

Now, let \( q > 0 \); we will work with the final expression above integrated against \( e^{-qt} \) and then let \( q \downarrow 0 \).

\[
\mathbb{E}\left[ \int_{0}^{\tau^+_x} k(\bar{G}_{t})m(t - \bar{G}_{t})h(x - \bar{X}_{t})w(x - X_{t})e^{-qt} dt \right] \\
= q^{-1} \mathbb{E}\left[ k(\bar{G}_{e_q})m(e_q - \bar{G}_{e_q})h(x - \bar{X}_{e_q})w(x - X_{e_q}; e_q < \tau^+_x) \right] \\
= q^{-1} \int_{(0,\infty) \times [0,x]} \mathbb{P}(\bar{G}_{e_q} \in ds, \bar{X}_{e_q} \in d\theta) k(s)h(x - \theta) \\
\times \int_{(0,\infty) \times [0,\infty)} \mathbb{P}(e_q - \bar{G}_{e_q} \in dt, \bar{X}_{e_q} - X_{e_q} \in d\phi) m(t)w(x - \theta + \phi) \\
[\text{since \( \{e_q < \tau^+_x\} = \{\bar{X}_{e_q} \leq x\} \) and by writing \( X_{e_q} = X_{e_q} - (\bar{X}_{e_q} - X_{e_q}) \)]
\]

\[
= q^{-1} \int_{(0,\infty) \times [0,x]} \mathbb{P}(\bar{G}_{e_q} \in ds, \bar{X}_{e_q} \in d\theta) k(s)h(x - \theta) \\
\times \int_{(0,\infty) \times (0,\infty)} \mathbb{P}(e_q \in dt, -\bar{X}_{e_q} \in d\phi) m(t)w(x - \theta + \phi) \quad \text{Th 5.11}
\]

Now, we know already from Theorem 5.18(i) that

\[
\frac{1}{\kappa(q, \theta)} \mathbb{E}[e^{-\alpha \bar{G}_{e_q} - \beta \bar{X}_{e_q}}] = \frac{1}{\kappa(\alpha + q, \beta)} ,
\]

and the limit of the right-hand side as \( q \downarrow 0 \) is just \( 1/\kappa(\alpha, \beta) \), which is the Laplace transform of the measure \( \mathcal{U} \). It follows from Lemma 5.25 that

\[
\lim_{q \downarrow 0} \frac{1}{\kappa(q, \theta)} \int_{(0,\infty) \times [0,x]} k(s)h(x - \theta) \mathbb{P}(\bar{G}_{e_q} \in ds, \bar{X}_{e_q} \in d\theta) \\
= \int_{(0,\infty) \times [0,x]} k(s)h(x - \theta) \mathcal{U}(ds, d\theta),
\]

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and we can say the same regarding the laws \( \mathbb{P}(G_{eq} \in dt, -X_{eq} \in d\phi) / \hat{k}(q, 0) \) and \( \hat{U} \).

We now rewrite \( q = \kappa(q, 0) \hat{k}(q, 0) \) by \( 5.11 \) and take the limit as \( q \downarrow 0 \) in the calculation above. This yields:

\[
\begin{align*}
(5.12) = & \mathbb{E} \left[ \int_0^{\tau_x^+} k(\bar{G}_t) m(t - \bar{G}_t) h(x - \bar{X}_t) w(x - X_t) \, dt \right] \\
= & \int_{[0,\infty) \times [0,x]} \mathcal{U}(ds, d\theta) k(s) h(x - \theta) \int_{[0,\infty) \times [0,\infty)} \hat{U}(dt, d\phi) w(x + \theta + \phi) \\
= & \int_{[0,\infty) \times [0,x]} \mathcal{U}(ds, d\theta) k(s) h(x - \theta) \\
& \times \int_{[0,\infty) \times [0,\infty)} \hat{U}(dt, d\phi) g(x - \theta + \phi) \int_{(x-\theta+\phi,\infty)} \Pi(d\eta) f(\eta - x + \theta - \phi)
\end{align*}
\]

We now substitute \( y = x - \theta, y + \phi = v \) and \( \eta = v + u \) and bring everything together, which gives:

\[
\begin{align*}
& \int_{u>0, y \in [0,x], v \geq y, s \geq 0, t \geq 0} k(s) m(t) f(u) g(v) h(y) \\
& \times \mathbb{P}(\bar{G}_{\bar{r}_x^+} \in ds, \tau_{x+}^+ - \bar{G}_{\bar{r}_x^+} \in dt, X_{\bar{r}_x^+} - x \in du, x - X_{\bar{r}_x^+} \in dv, x - \bar{X}_{\bar{r}_x^+} \in dy) \\
= & \int_{[0,\infty) \times [0,x]} \mathcal{U}(ds, d\theta) \int_{[0,\infty) \times [y,\infty)} \hat{U}(dt, dv - y) \\
& \times \int_{[0,\infty)} \Pi(du + v) k(s) m(t) f(u) g(v) h(y).
\end{align*}
\]

This proves the theorem.

You may have observed that we ‘cheated’ in the previous theorem, much as we did in §4.6 by excluding the possibility that \( \{X_{\bar{r}_x^+} = x\} \), which is called ‘creeping’. For a version of the quintuple law which includes this event, see [GM11].

As ever with results in fluctuation theory, it is one thing to prove a general theorem and quite another to give explicit results for particular \( \text{Lévy} \) processes. The general quintuple law is often inaccessible in practice, but the following corollary (which follows by integrating out \( s \) and \( t \)) is more frequently applicable.

**Corollary 5.26** (Triplet law). For \( u > 0, y \in [0,x], v \geq y, \)

\[
\mathbb{P}(X_{\bar{r}_x^+} - x \in du, x - X_{\bar{r}_x^+} \in dv, x - \bar{X}_{\bar{r}_x^+} \in dy) = U(x - dy) \hat{U}(dv - y) \Pi(du + v),
\]

where \( U \) (resp., \( \hat{U} \)) is the potential measure of \( H \) (resp., \( \hat{H} \)) and \( \Pi \) is the \( \text{Lévy} \) measure of \( X \).

To give one example, in the case of the stable process, we deduce (from Example 5.22 and Exercise 15) that

\[
\mathbb{P}(X_{\bar{r}_x^+} - x \in du, x - X_{\bar{r}_x^+} \in dv, x - \bar{X}_{\bar{r}_x^+} \in dy) = c \frac{(x - y)^{\alpha - 1} (v - y)^{\alpha(1 - \rho) - 1}}{(u + v)^{\alpha + 1}} du \, dy \, dv,
\]

for some constant \( c > 0 \).
A. Miscellaneous results

A.1. Analytic extension of characteristic exponents

We will give a criterion for the characteristic exponent of a Lévy process to be analytic; we give it for \( \mathbb{R}^d \)-valued Lévy processes, which we discussed in Exercise 7.

In the following result, the symbol \( \langle \cdot, \cdot \rangle \) indicates the Euclidean inner product, and \( |\cdot| \) the Euclidean norm. For \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), we define \( \text{Re} \, z := (\text{Re} \, z_1, \ldots, \text{Re} \, z_d) \) and \( \text{Im} \, z := (\text{Im} \, z_1, \ldots, \text{Im} \, z_d) \). If \( D \subset \mathbb{C}^d \), then we say that a function \( f : D \to \mathbb{C} \) is analytic if: for every \( p \in D^o \) and \( j \in \{1, \ldots, d\} \), there exists some ball \( B_{\varepsilon} \) around \( 0 \in \mathbb{C} \) such that the function \( f_j : B_{\varepsilon} \to \mathbb{C} \), given by \( f_j(p_1, \ldots, p_{j-1}, p_j + z, p_{j+1}, \ldots, p_d) \), is analytic; and, furthermore, \( f \) is continuous on all of \( D \). That is, we say that \( f \) is analytic if it is analytic in each variable separately.

Theorem A.1. Let \( X \) be a killed \( d \)-dimensional Lévy process with characteristic exponent \( \Psi : \mathbb{R}^d \to \mathbb{C} \), with Lévy–Khintchine representation

\[
\Psi(\theta) = i(a, \theta) - \frac{1}{2}\langle \theta, Q \theta \rangle + \int_{\mathbb{R}^d} (e^{i\langle \theta, x \rangle} - 1 - i\langle \theta, x \rangle 1_{\{|x| \leq 1\}}) \Pi(dx) - q, \tag{A.1}
\]

where \( q \geq 0 \), \( a \in \mathbb{R}^d \), \( \Pi \) is a measure on \( \mathbb{R}^d \) such that \( \int_{\mathbb{R}^d} 1 \wedge |x|^2 \Pi(dx) < \infty \), and \( Q \) is a symmetric, non-negative definite matrix.

Furthermore, define

\[
A = \left\{ c \in \mathbb{R}^d : \int_{|x| > 1} e^{\langle c, x \rangle} \Pi(dx) < \infty \right\}.
\]

Then,

(i) \( A \) is a convex set, and \( 0 \in A \).

(ii) \( c \in A \) if and only if \( \mathbb{E}[e^{\langle c, X_t \rangle}] < \infty \) for some \( t > 0 \), or equivalently, for all \( t \geq 0 \).

(iii) If \( \theta \in \mathbb{C}^d \) is such that \( -\text{Im} \, \theta \in A \), then the right-hand side of (A.1) is well-defined; denote it by \( \Psi(\theta) \). The function \( \Psi \) is analytic on the set \( \{ \theta \in \mathbb{C}^d : -\text{Im} \, \theta \in A \} \). For such \( \theta \), \( \mathbb{E}[|e^{\langle \theta, X_t \rangle}|] < \infty \), and

\[
\mathbb{E}[e^{\langle \theta, X_t \rangle}] = e^{t\Psi(\theta)}, \quad t \geq 0.
\]

Proof. See [Sat99, Theorem 25.17] and its proof.

Corollary A.2 (Proposition 4.11). If \( X \) is a one-dimensional possibly killed subordinator, then its Laplace exponent \( \kappa \) is given by

\[
\kappa(\lambda) = q + d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi(dx), \quad \lambda \in \mathbb{C}_r,
\]

where \( \mathbb{C}_r = \{ \lambda \in \mathbb{C} : \text{Re} \, \lambda \geq 0 \} \), and \( \kappa \) is analytic on \( \mathbb{C}_r \); furthermore, \( -\kappa(-i\theta) = \Psi(\theta) \) for \( \theta \in \mathbb{R} \).
A. Miscellaneous results

Proof. This follows using the theorem and the identification $\theta = i\lambda$.  

Corollary A.3. Let $X$ be a one-dimensional killed spectrally negative Lévy process and $\phi: [0, \infty) \to \mathbb{R}$ such that $\mathbb{E}[e^{\lambda X}] = e^{\phi(\lambda)}$. Then, $\phi$ has an analytic extension to $\mathbb{C}_r$, and

$$\phi(\lambda) = -q - a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1_{\{x \geq -1\}}) \Pi(dx), \quad \lambda \in \mathbb{C}_r.$$ 

Furthermore, $-\phi(-i\theta) = \Psi(\theta)$ for $\theta \in \mathbb{R}$. 

Corollary A.4. The equation (5.6), namely

$$\mathbb{E}[e^{-\gamma G_{eq} - \beta X_{eq}}] = \exp\left\{-\int_0^\infty dt \int_{(0,\infty)} (1 - e^{-\gamma t-\beta x}) t^{-1} e^{-q t} \mathbb{P}(X_t \in dx)\right\},$$

holds on $\{(\gamma, \beta) : \text{Re} \gamma > -q, \text{Re} \beta \geq 0\}$, and both sides are analytic (separately in $\gamma$ and $\beta$) on this domain.

Proof. The Lévy measure of the infinitely divisible random variable $(\bar{G}_{eq}, \bar{X}_{eq})$ is given by $e^{-qt-1} \mathbb{P}(X_t \in dx) dt 1_{\{t,x \geq 0\}}$; simply apply the criterion in the theorem.  

A.2. Change of variables formula 

We give a very general version of the change of variables formula for Stieltjes integrals, taken from [Sha88, §A4], and we then specialise it to the case of the local time of a Markov process.

Let $f$ be an increasing function with finite values defined on an interval $(a, b)$. The right (resp., left) continuous inverse of $f$ is defined to be $\rho$ (resp., $\lambda$) such that

$$\rho(s) = \inf\{t \in (a, b) : f(t) > s\} \wedge b, \quad s \in \mathbb{R},$$

$$\lambda(s) = \sup\{t \in (a, b) : f(t) < s\} \vee a, \quad s \in \mathbb{R}.$$ 

One can show that $\rho$ and $\lambda$ are increasing functions taking values in $[a, b]$, as well as that $\rho(s) = \lambda(s+) \quad \text{and} \quad \lambda(s) = \rho(s-)$. 

Proposition A.5. Let $r$ be a finite-valued right-continuous increasing function defined on $(a, b)$ and let $l$ be its left-continuous inverse function. Let $r(a) = r(a+)$ and $r(b) = r(b-)$. Then,

(i) For all $t \in (a, b)$ and $s \in \mathbb{R}$, $t < l(s)$ if and only if $r(t) < s$.

(ii) $r$ is the restriction to $(a, b)$ of the right-continuous inverse of $l$.

(iii) If $m$ is a right-continuous, increasing function of $[r(a), r(b)]$, then for every positive Borel function $g$ on $(a, b)$,

$$\int_{(a,b)} g(t) dm(r(t)) = \int_{[r(a), r(b)]} g(l(s)) 1_{\{a < l(s) < b\}} dm(s).$$
A. Miscellaneous results

Proof. See [Sha88, Theorem A4.3].

The specialisation of this to the local time $L$ of a Markov process and its right-continuous inverse local time $L^{-1}$ is given as follows.

**Corollary A.6.** For every positive Borel function $g$ on $(0, \infty)$,

$$
\int_0^\infty g(s) \, dL(s) = \int_0^\zeta g(L^{-1}(t)) \, dt,
$$

where $\zeta = L(\infty)$ is the lifetime of the inverse local time.

**Proof.** Take $a \downarrow 0, b \to \infty$, $m(t) = t$, $r = L$ and $l(t) = L^{-1}(t-)$. The previous proposition then yields

$$
\int_0^\infty g(s) \, dL(s) = \int_0^{L(\infty)} g(L^{-1}(t-)) \, dt,
$$

but $L^{-1}$ has only countably many jumps, so one may replace $L^{-1}(t-)$ by $L^{-1}(t)$ on the right-hand side. Finally, we see that $L^{-1}$ is sent to $+\infty$ at time $L(\infty)$, so this is its lifetime.

A.3. Excursion theory in the irregular case

We will prove (5.5) when 0 is irregular for $\tilde{X} - X$. Recall that in this case we may enumerate by $0 = T_0 < T_1 < \cdots$ the discrete times at which $\tilde{X} - X$ takes the value 0.

Recall that we constructed the local time $L$ out of a sequence of i.i.d. exponentially distributed random variables, independent of $X$, and we wrote $L(t) = \sum_{i=0}^{n(t)} \tau_i$, with $n(t) = \max \{ i : T_i < t \}$. Take the rate of the $\tau_i$ to be equal to 1. Then, $L^{-1}$ is a compound Poisson subordinator whose Lévy measure is given by $\mathbb{P}(T_1 \in \cdot)$. Hence,

$$
\kappa(\alpha, 0) = \int_{(0, \infty)} (1 - e^{-\alpha x}) \mathbb{P}(T_1 \in dx) = \mathbb{E}[1 - e^{-\alpha T_1}],
$$

which is consistent with (5.2). It also follows that for Borel $f$,

$$
\mathbb{E} \left[ \int_0^\infty f(L^{-1}(t)) \, dt \right] = \mathbb{E} \left[ \sum_{n=0}^\infty f(T_n) \tau_n \right] = \sum_{n=0}^\infty \mathbb{E}[f(T_n)].
$$

These facts will be enough to show (5.5).

**Lemma A.7.** When 0 is irregular for $\tilde{X} - X$,

$$
\mathbb{E}[e^{-\alpha G_{t\beta}}] = \frac{\kappa(q, 0)}{\kappa(\alpha + q, \beta)}, \quad \alpha, \beta > 0.
$$
Proof. Recall that, in the irregular case, $X_{\tilde{e}_{eq}} = \tilde{X}_{eq}$. Furthermore, $T_n < e_q \leq T_{n+1}$ if and only if $\tilde{G}_{eq} = T_n$. Therefore, we make the following calculation, drawing together everything we have discussed so far; in the fifth equality we apply the Markov property at time $T_n$, and in the sixth we use Exercise 14.

\[
\mathbb{E}[e^{-\alpha \tilde{G}_{eq} - \beta \tilde{X}_{eq}}] = \mathbb{E}[e^{-\alpha \tilde{G}_{eq} - \beta \tilde{X}_{eq}}] \\
= \sum_{n=0}^{\infty} \mathbb{E}[e^{-\alpha T_n - \beta X_{T_n}}; T_n < e_q \leq T_{n+1}] \\
= \sum_{n=0}^{\infty} \mathbb{E}\left[ \int_{T_n}^{T_{n+1}} q e^{-qt} e^{-\alpha T_n - \beta X_{T_n}} \, dt \right] \\
= \sum_{n=0}^{\infty} \mathbb{E}\left[ e^{-(\alpha q)T_n - \beta X_{T_n}} \int_{0}^{T_{n+1}-T_n} q e^{-qs} \, ds \right] \\
= \mathbb{E}\left[ \int_{0}^{\infty} e^{-(\alpha+q)L^{-1}(t) - \beta H(t)} \, dt \right] \mathbb{E}\left[ \int_{0}^{T_1} q e^{-qs} \, ds \right] \\
= \frac{\kappa(q,0)}{\kappa(\alpha + q, \beta)}.
\]

We are done. \hfill \Box

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References


References


