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The Wiener-Hopf factorisation of Lévy processes

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- X has an infinitely divisible distribution if for any $n \geq 1$ there exist iid $X^{(k)}$ such that $X \overset{d}{=} X^{(1)} + \cdots + X^{(n)}$
- When restricting to infinitely divisible distributions, the answer is yes

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- Lévy–Khintchine formula: $\psi(z) = \mathrm{i}az - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \mathrm{i}zx\mathbb{1}_{[-1,1]}(x)\right)\Pi(\mathrm{d}x)$

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- a incorporates drift, σ the Gaussian coefficient, jumps of size $\mathrm{d}x$ occur at rate $\Pi(\mathrm{d}x)$

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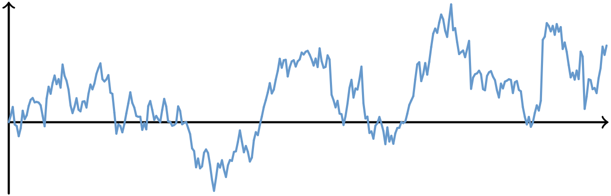
- Summands are independent, infinitely divisible and have disjoint support
- Therefore, such a factorisation is unique

Wiener-Hopf factorisation (version 2, path picture)



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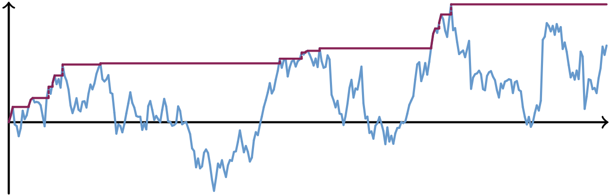

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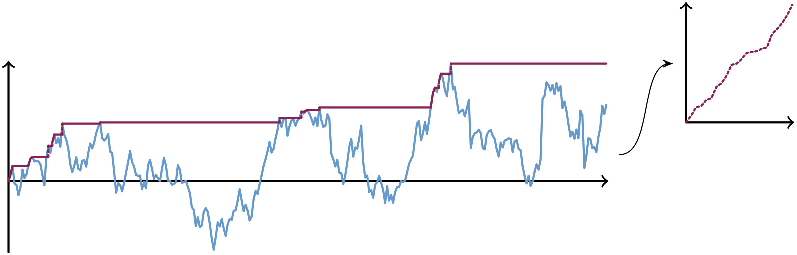

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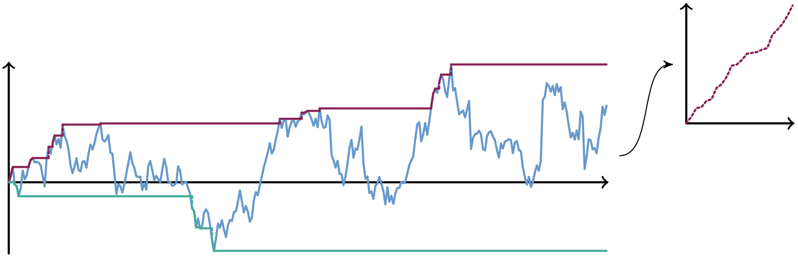

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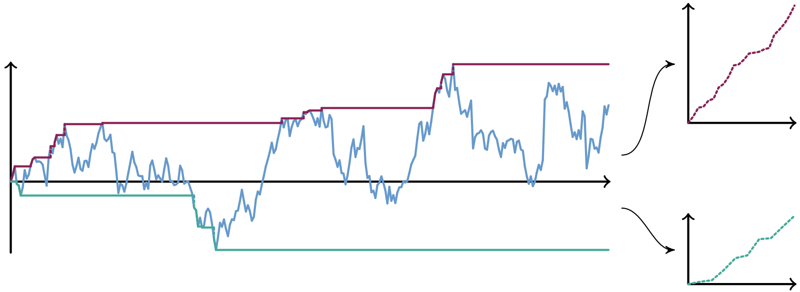

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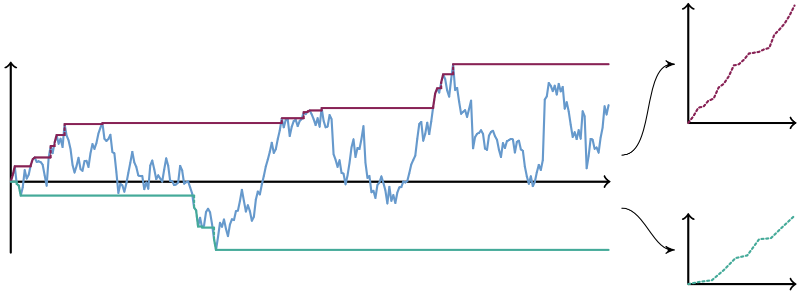

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$H^\pm$ are subordinators (increasing Lévy processes), possibly killed

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- ⚠ This does not mean $X_t = H_t^+ - H_t^-$!

The result: the Wiener-Hopf factorisation is unique

Theorem (DSTW, 2023+)

Let $\kappa'_\pm$ be the characteristic exponents of two subordinators, such that

$$-\psi(z) = \kappa_+(z)\kappa_-(-z) = \kappa'_+(z)\kappa'_-(-z), \quad z \in \mathbb{R}.$$

Then $\kappa_+(z) = c\kappa'_+(z)$ and $\kappa'_-(z) = c\kappa_-(z)$ for some $c > 0$.

We also proved an analogous result for random walks.

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$$q - \psi(z) = \kappa_{+}(q,z)\kappa_{-}(q,-z), \quad z \in \mathbb{R}.$$

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Theorem (Rogozin (1966) or earlier)

Fix $q > 0$ and let $\kappa_\pm(q, \cdot)$ and $\kappa'_\pm(q, \cdot)$ be characteristic exponents of subordinators, such that

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Then $\kappa_+(q, z) = c\kappa'_+(q, z)$ and $\kappa'_-(q, z) = c\kappa_-(q, z)$ for some $c > 0$ and all $z \in \mathbb{R}$.

Sketch proof

- $\kappa(q,z)$ can be extended to holomorphic functions on $\operatorname{Im} z \geq 0$ (where $\kappa \in \{\kappa_+, \kappa_-, \kappa'_+, \kappa'_-\}$)

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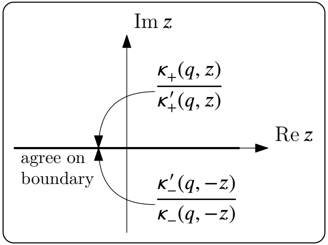

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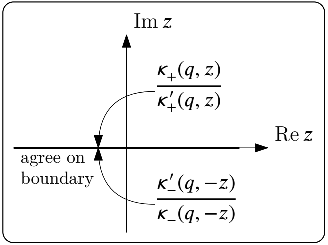

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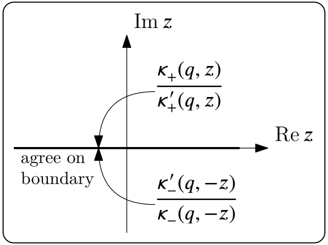

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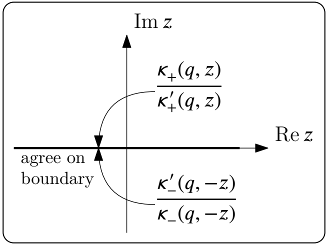

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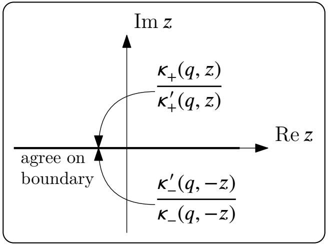
- Liouville's theorem: $F(z) = c$



Prior art: killed Lévy process, III

- When $q = 0$, may have $\liminf_{|z|\to\infty, z\in\mathbb{R}} |\kappa(z)| = 0$
- Then $\log \kappa(z)$ is unbounded and Liouville argument fails

Why is it important?

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- When is there a Lévy process X with CE ψ such that $-\psi(z) = \kappa_{+}(z)\kappa_{-}(-z)$?
- When such X exists, we call $H^{\pm}$ friends

Philanthropy

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Theorem (Vigon, 2002)

Any two philanthropists can be friends.

Construction of Lévy processes

If we have philanthropists with CEs $\kappa_\pm$, then

$$\psi(z) = -\kappa_+(z)\kappa_-(-z)$$

is the CE of a Lévy process.

Example (Kuznetsov and Pardo, 2013)

Let $\beta_\pm \geq 0, \gamma_\pm \in (0,1)$. Then

$$\kappa_\pm(z) = \frac{\Gamma(\beta_\pm + \gamma_\pm - \mathrm{i}z)}{\Gamma(\beta_\pm - \mathrm{i}z)}$$

gives rise to a hypergeometric Lévy process.

Fluctuations of constructed Lévy processes

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- If X is constructed via friendship, does H^+ have CE κ_+?

Proving uniqueness

Tempered distributions

- Rapidly decaying functions:
  $\mathscr{S} = \{\phi\colon \mathbb{R} \to \mathbb{C} \colon \forall \alpha, \beta \in \mathbb{N} \cup \{0\} \ \lim_{|x|\to\infty} |x^\alpha \phi^{(\beta)}(x)| < \infty\}$

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Examples

- Radon measures with slow growth near $\pm\infty$: $\langle \mu, \phi \rangle = \int \phi(x)\, \mu(\mathrm{d}x)$
- $\langle \delta, \phi \rangle = \phi(0)$
- $\langle D\delta, \phi \rangle = -\phi'(0)$

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- Differentiation: $\langle Dh, \phi \rangle = -\langle h, \phi' \rangle$
- Fourier transform:
  - $\mathscr{F}\phi(z) = \int e^{ixz}\phi(x)\,\mathrm{d}x$; then $\mathscr{F} \colon \mathscr{S} \to \mathscr{S}$
  - $\langle \mathscr{F}h, \phi \rangle = \langle h, \mathscr{F}\phi \rangle$

Lévy–Khintchine formula

For H^+ we have

$$\kappa_+(z) = -q_+ + \mathrm{i}d_+z + \int_{(0,\infty)} (e^{\mathrm{i}xz} - 1)\,\mu_+(\mathrm{d}x).$$

Let

$$\langle G_+, \phi\rangle = -q_+\langle \delta, \phi\rangle - d_+\langle D\delta, \phi\rangle + \int_{(0,\infty)} \big(\phi(x) - \phi(0)\big)\mu_+(\mathrm{d}x).$$

Then $G_+ \in \mathcal{S}'$ and $\mathcal{F}G_+(z) = \kappa_+(z)$.

Some useful representations

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Potentials

The potential of H^+: $U_+(\mathrm{d}x) = \int_0^\infty \mathbb{P}(H^+ \in \mathrm{d}x)\,\mathrm{d}t$.

Then $U_+ \in \mathscr{S}'$ and $\mathscr{F}U_+ = 1/\kappa_+$ (away from zero)

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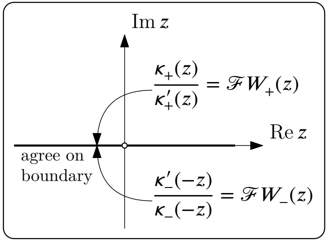

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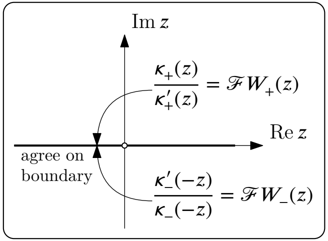

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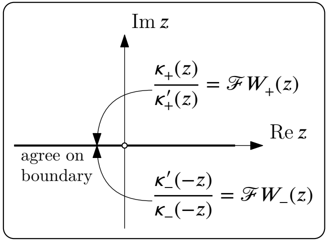
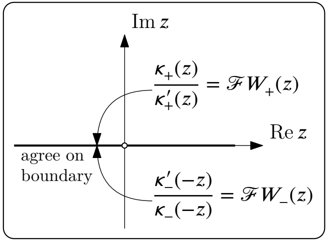


Sketch of the proof, II



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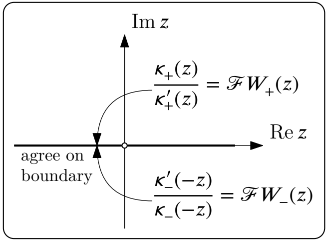

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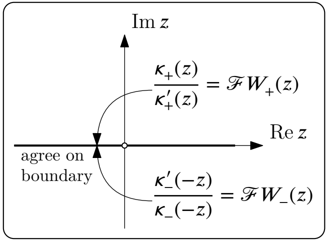

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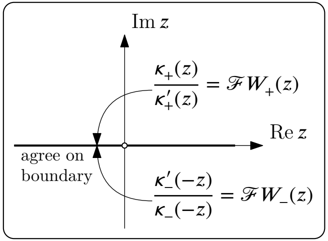

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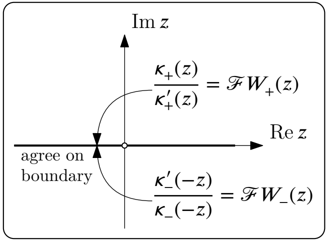

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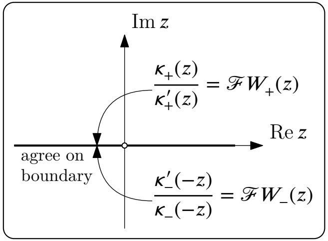

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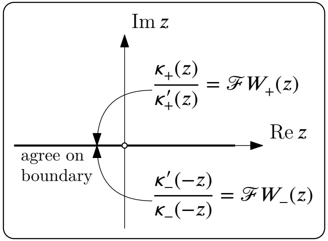

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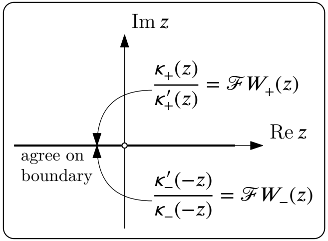

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- Asymptotics of F near zero and comparison with W_- (again) yield $F(z) = b_0$: we are done.

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Swept under the carpet

- Convolvability of distributions is tricky
- When X is lattice valued, ψ has zeroes on $\eta\mathbb{Z}$ for some $\eta > 0$: support arguments are trickier

The outlook

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  where U_+ and $\widehat{U_-}$ are potentials of H^+ and $-H^-$ (but if X is recurrent, U is very bad!)
- In Fourier space, this decomposition reads:

$$-\frac{1}{\psi(z)} = -\frac{1}{\kappa_+(z)} \times -\frac{1}{\kappa_-(-z)}$$

and we prove that it is unique (among potentials of subordinators)

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- Is there a representation of these, other than $\mathscr{F}U_+ = 1/\kappa_+$ etc.?

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- Does it hold in general?

Further reading

L. Döring, M. Savov, L. Trottner and A. R. Watson
The uniqueness of the Wiener–Hopf factorisation of Lévy processes and random walks
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Thank you!