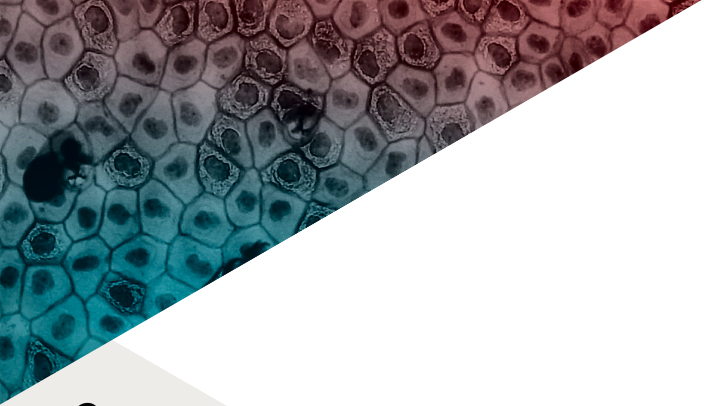
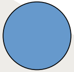


GROWTH-FRAGMENTATION AND QUASI-STATIONARY METHODS

Denis Villemonais **Alex Watson**

17 September 2021

A MODEL OF GROWTH-FRAGMENTATION



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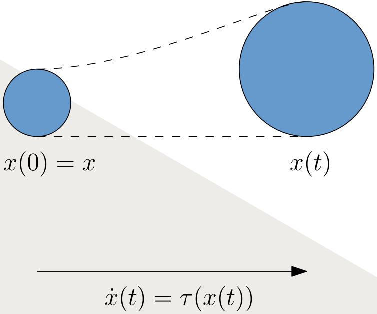

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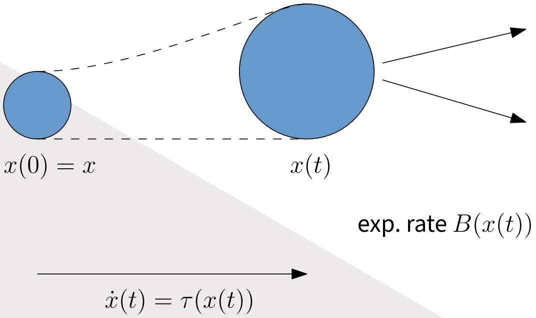

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exp. rate $B(x(t))$

$\dot{x}(t) = \tau(x(t))$

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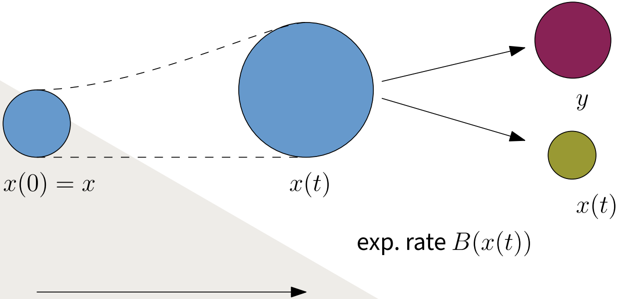

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y

$x(t) - y$

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chosen by $\kappa(x(t), \mathrm{d}y)$

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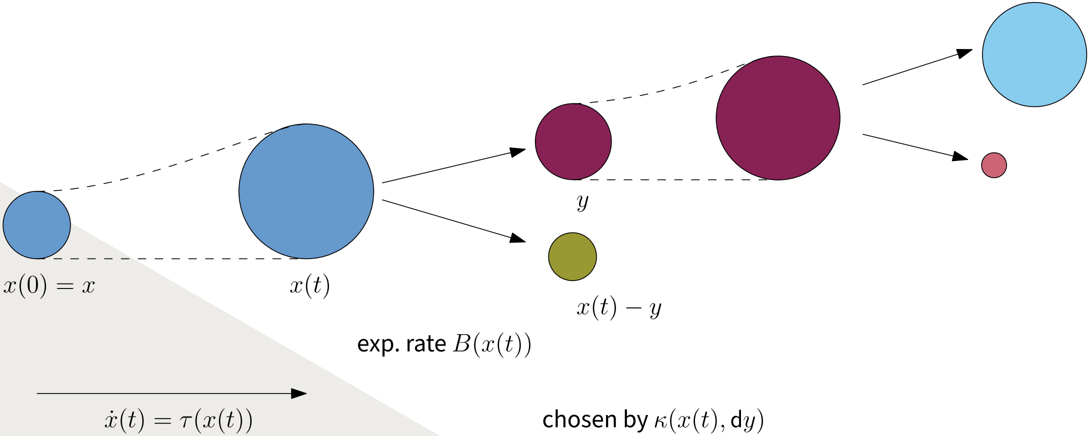

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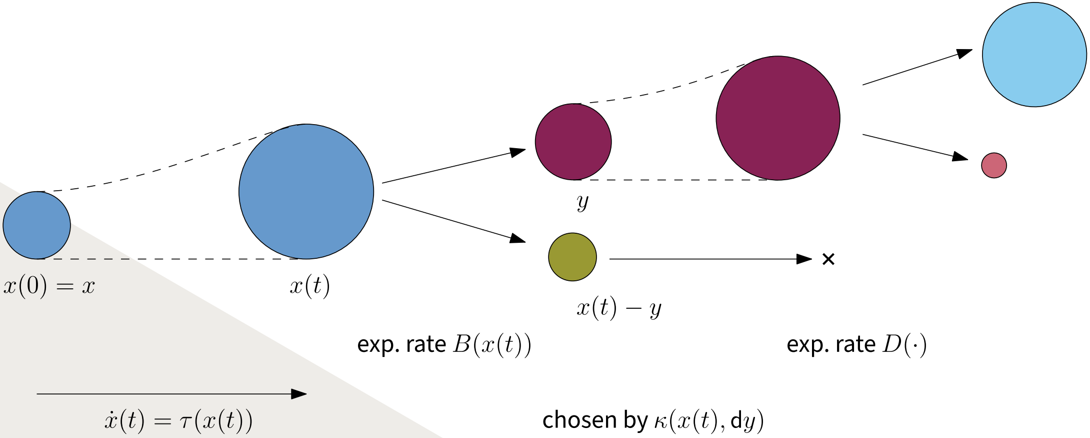

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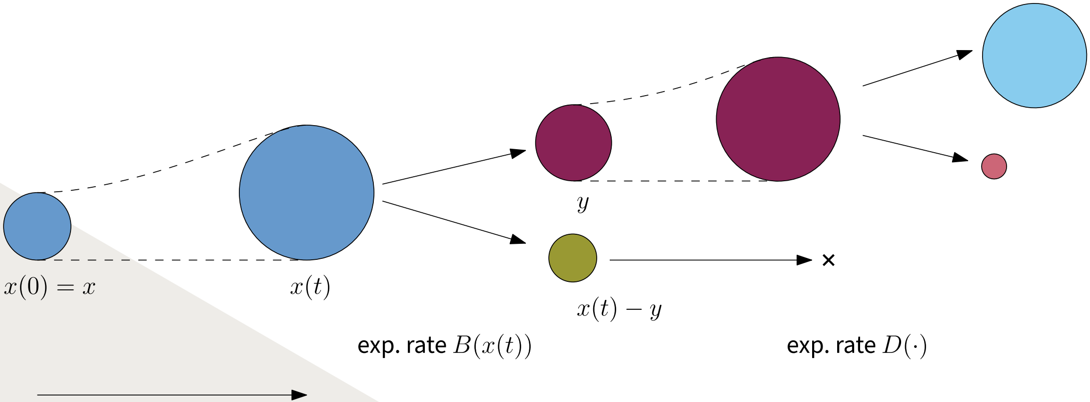

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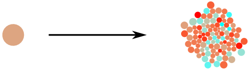
- List sizes at time t: $\mathbf{Z}(t) = (Z_u(t) \colon u \in U)$

Equilibrium behaviour

t = 0.00
1 cell

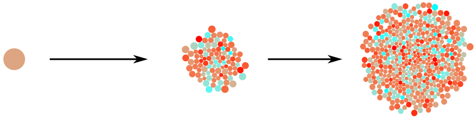
Equilibrium behaviour



t = 0.00
1 cell

t = 4.71
100 cells

EQUILIBRIUM BEHAVIOUR



t = 0.00
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t = 6.32
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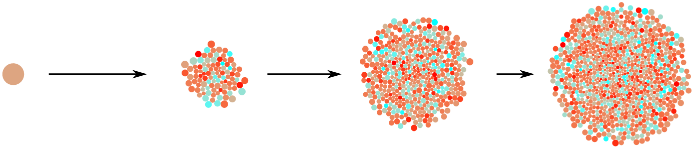

t = 0.00
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t = 7.02
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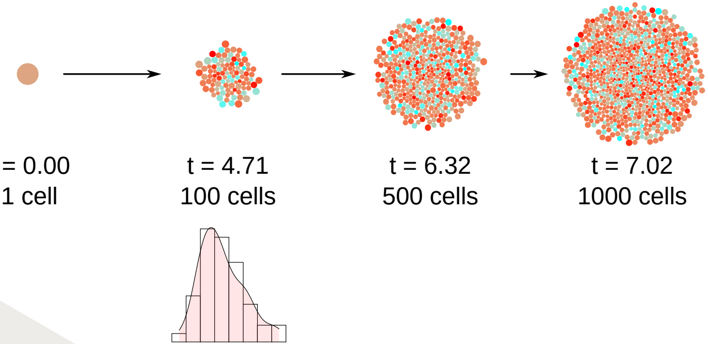

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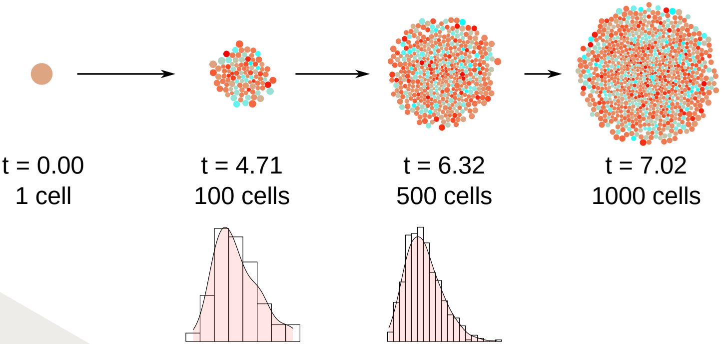

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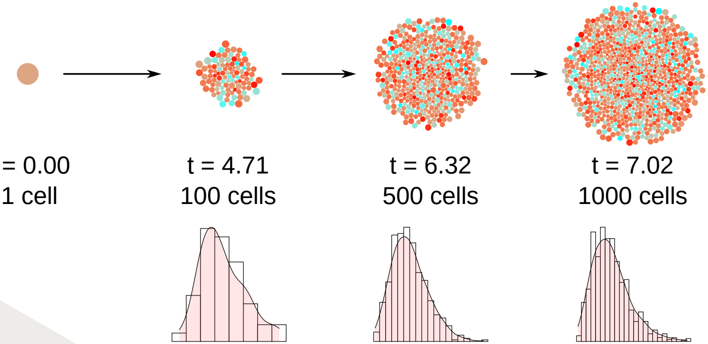

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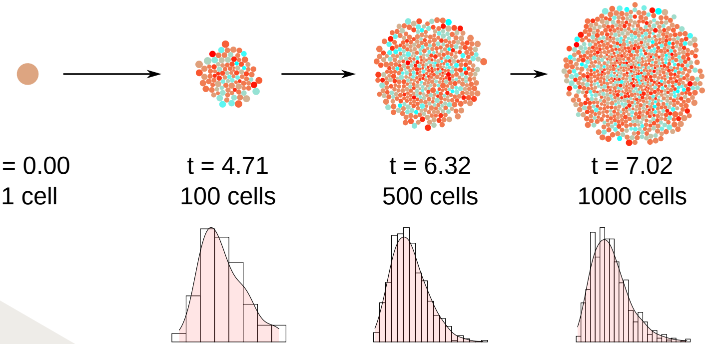

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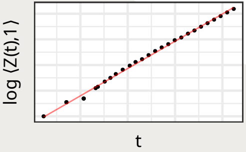

$$\mathbf{Z}(t) \sim Ce^{\lambda t}\nu$$

MEAN MEASURES

Look at $T_t f(x) = \mathbb{E}_x \left[\sum_u f(Z_u(t)) \right]$ (**formally**)

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...where $k(x, \mathrm{d}y) = 2B(x)\frac{\kappa(x,\mathrm{d}y)+\kappa(x,x-\mathrm{d}y)}{2}$, and $K(x) = B(x) + D(x)$.

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Questions

- Existence and uniqueness of such T_t? (For which coefficients; for which f?)
- Long term behaviour: $T_t f(x) \sim e^{\lambda t} h(x) \int f(y) \nu(\mathrm{d}y)$? Rate?

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- Spectral: find $\mathcal{A}h = \lambda h$, $v\mathcal{A} = \lambda v$ and use entropy methods

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- ‘Harris-type theorem for non-conservative semigroups’: Lyapunov function approach, Bansaye et al. (2019+)

OUR APPROACH

- Try to link to a **killed** Markov process

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- Try to link to a killed Markov process
- Study the quasi-stationary distribution (QSD) (‘stationary after conditioning on survival’)
- Find conditions for existence of the process and its QSD, and link back to desired semigroup T

Existence and uniqueness

FINDING A KILLED MARKOV PROCESS SPINE

- Fix $\alpha, \beta \in \mathbb{R}$ and let

$$V(x) = \exp\left(-\mathbb{1}_{\{x\leq 1\}}\alpha \int_x^1 \frac{\mathrm{d}y}{\tau(y)} + \mathbb{1}_{\{x>1\}}\beta \int_1^x \frac{\mathrm{d}y}{\tau(y)}\right)$$

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- $\mathcal{L}1 \leq 0$; it generates a killed Markov process

FINDING A KILLED MARKOV PROCESS SPINE

- $\mathcal{L}f(x) = \underset{\substack{\uparrow \\ \text{growth rate}}}{\tau}(x)f'(x) + \int_0^x \left[f(y) - f(x)\right] \underset{\substack{\uparrow \\ \text{jump rate}}}{k_V}(x, \mathrm{d}y) - \underset{\substack{\uparrow \\ \text{killing rate}}}{q}(x)f(x),$

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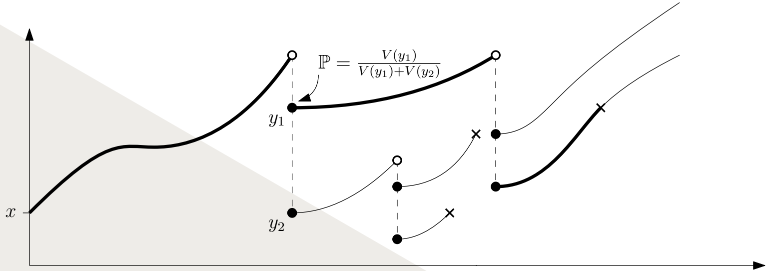

LEMMA

Assume, for all $M > 0$,

$$\sup_{x \in (0,M)} k_V(x, (0,x]) < \infty \qquad \text{and} \qquad \limsup_{x \to \infty} \Big[k_V(x, (0,x]) - K(x) \Big] < \infty.$$

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Then there is a Markov process X on $E = (0,\infty) \cup \{\partial\}$ with

$$Q_t f(x) := \mathbb{E}_x[f(X_t)] = f(x) + \int_0^t \mathbb{E}_x[\mathcal{L} f(X_s)] \, \mathrm{d}s$$

for $f : E \to \mathbb{R}$ such that $f|_{(0,\infty)}$ compactly supported and suitably differentiable.

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for $f : E \to \mathbb{R}$ such that $f|_{(0,\infty)}$ compactly supported and suitably differentiable.
Moreover, Q is the unique semigroup with these properties.

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  - Show **any** solution does not approach ∞ or 0 (supermartingale argument)
  - Compare solutions with solutions of martingale problem (a priori not necessarily the same!)

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Let

$$\mathcal{A}f(x) = \tau(x)f'(x) + \int_0^x f(y)k(x, \mathrm{d}y) - K(x)f(x)$$

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and

$$T_t f(x) = e^{bt} V(x) \mathbb{E}_x [f(X_t)/V(X_t)].$$

‘Unbias the spine motion and add the branching back in’.

Long-term behaviour

QUASI-STATIONARY DISTRIBUTIONS

- If X is a Markov process killed at T_∂, Champagnat and Villemonais (2018+) give criteria for
$$\mathbb{P}_x(X_t \in \mathrm{d}y \mid T_\partial > t) \to \nu^X(\mathrm{d}y),$$
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- ν^X is the **quasi-stationary distribution**.
- X is **killed** at random rate, our T has **branching** at random rate...

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with $\lambda_2 < \lambda_1$ and L compact, (plus boundary behaviour).

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In addition to our assumption about k_V, assume

$$\int_0^\infty \mathbb{1}_{\{k(y,(0,x])>0\}}\,\mathrm{d}y > 0, \quad \text{for } x > 0,$$

that there is a measure μ and a nonempty interval I with

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with $\lambda_2 < \lambda_1$ and L compact, (plus boundary behaviour).
Then...

THEOREM

…there exist $\lambda \in \mathbb{R}$, ν a measure, h a function and $\gamma > 0$, such that

$$\left\| e^{-\lambda t} T_t f(x) - h(x) \int f \, \mathrm{d}\nu \right\|_{TV} \leq C e^{-\gamma t} \psi(x)$$

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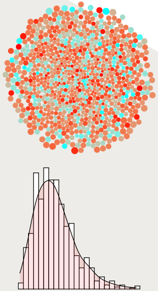

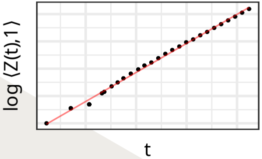

“$\mathbb{E}\mathbf{Z}(t) \sim e^{\lambda t} h(x) \nu$”

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  - cf. Laurençot and Walker (2021)

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- Spatially dependent fragmentation process – Callegaro and Roberts (2021+)

FURTHER READING

D. Villemonais and A. R. Watson
Asymptotic behaviour of growth-fragmentations via quasi-stationarity of the spine
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Thank you!