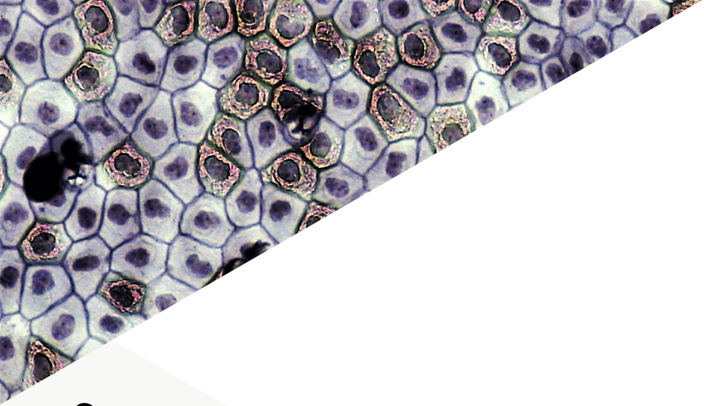


Strong laws for growth-fragmentation processes with bounded cell size

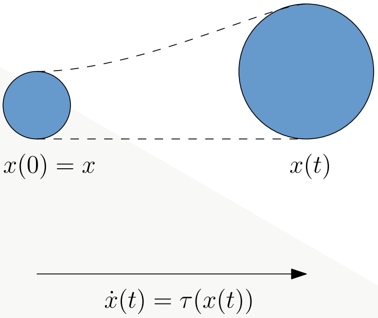
Emma Horton   Alex Watson

IECL seminar, 6 May 2021

A model of growth-fragmentation

$x(0) = x$

A MODEL OF GROWTH-FRAGMENTATION



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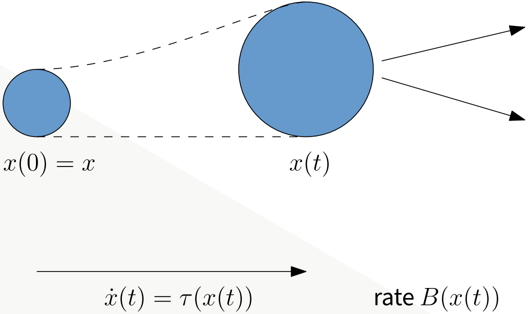

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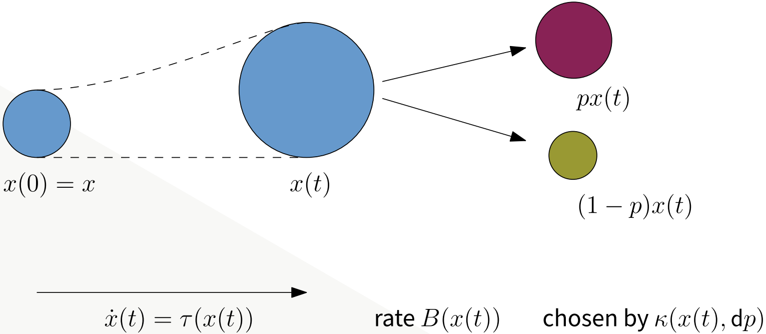

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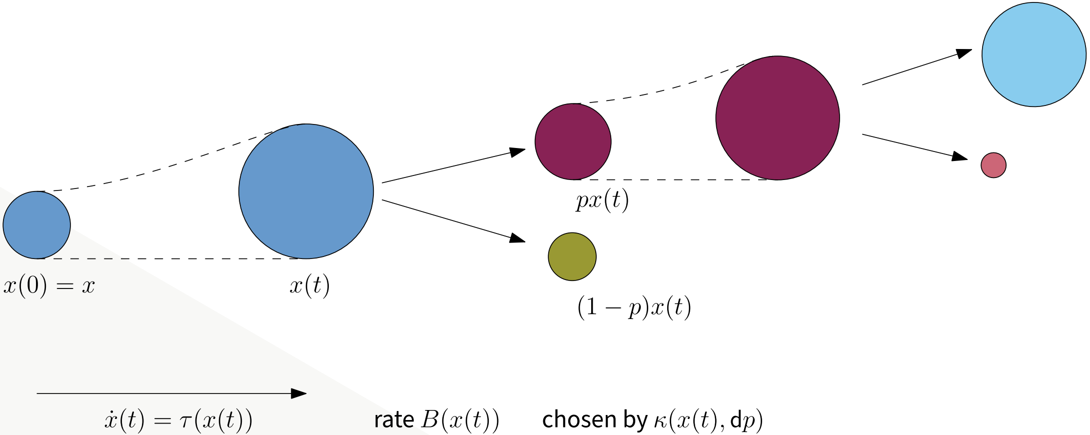

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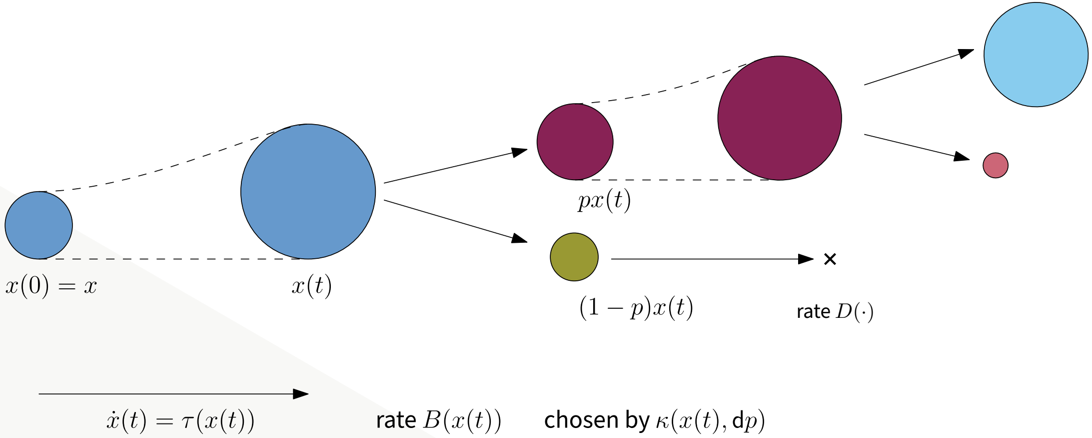

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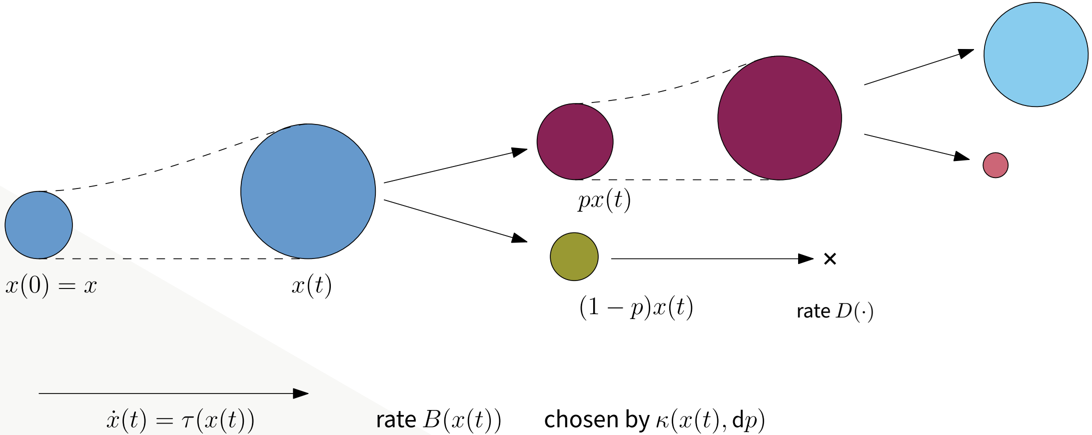

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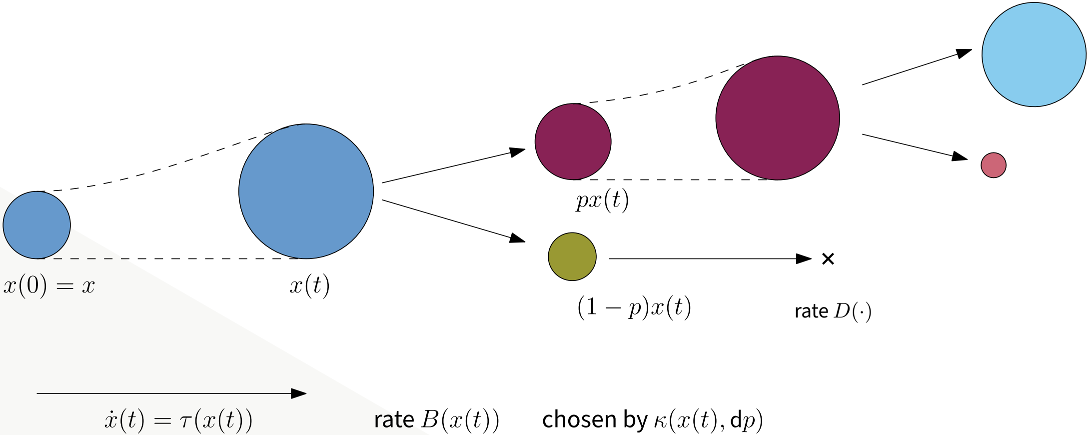

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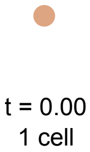
- $\mathbf{Z}(t) = \sum_{u \in U} \delta_{Z_u(t)}$

Equilibrium behaviour



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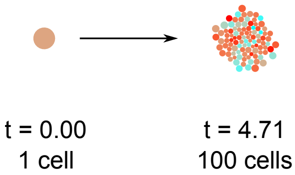

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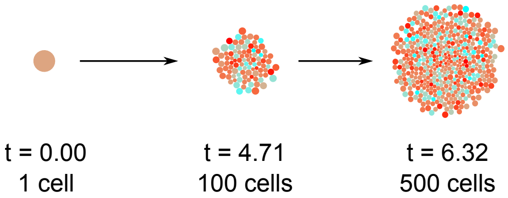

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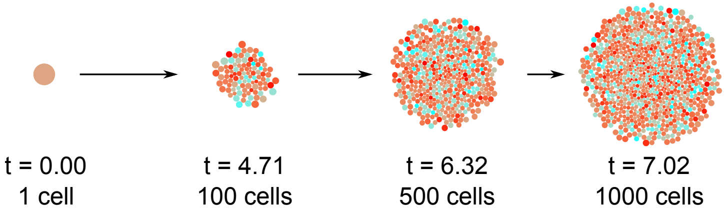

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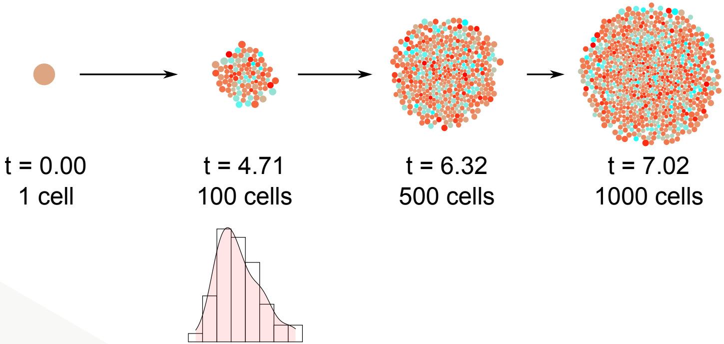

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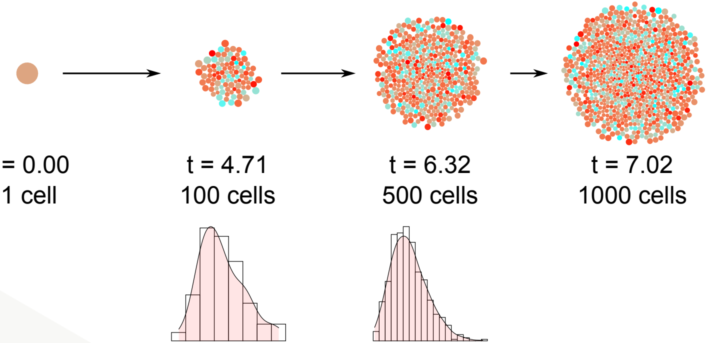

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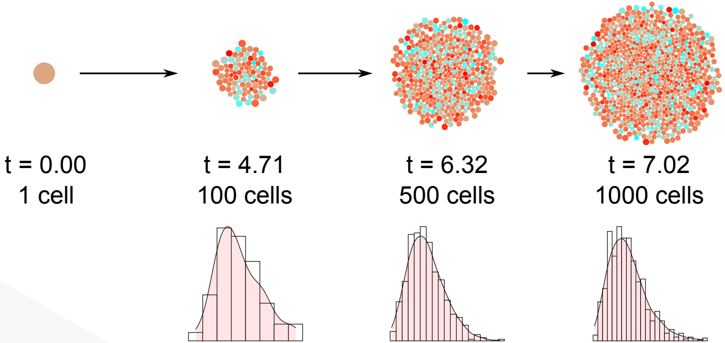

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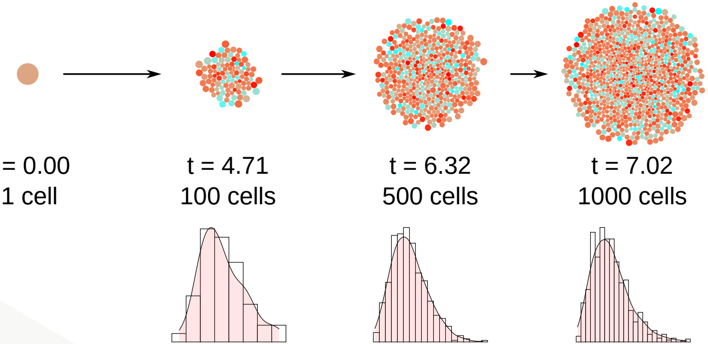

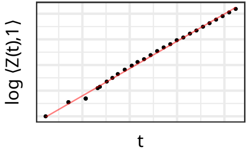

$$\langle \mathbf{Z}(t), f\rangle \sim e^{\lambda t}\langle \mathbf{Z}(0), h\rangle\langle \nu, f\rangle$$

Homogeneous model

- Exponential growth and size-independent rates: $\tau(x) = ax$, $B(x) = B$, $D(x) = \mathtt{k}$, $\kappa(x, \cdot) = \kappa(\cdot)$

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- No equilibrium behaviour
- Underlying Lévy process: $\mathbb{E}_x\langle \mathbf{Z}(t), f\rangle = e^{at}\mathbb{E}_x[e^{-\chi_t} f(e^{\chi_t})]$

Perturbations

- ‘Refracted process’:

$$\tau(x) = \begin{cases} ax, & 0 < x < c, \\ a'x, & x > c, \end{cases}$$

  where $a > a'$.
  Cavalli (2020, Acta Appl. Math.): equilibrium behaviour of mean measures

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- Underlying both: perturbed Lévy processes

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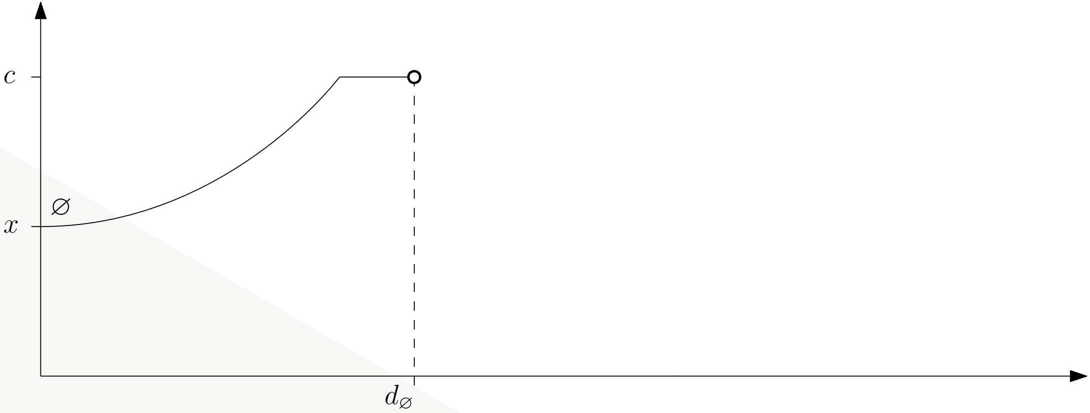
Growth-fragmentation process – schematic



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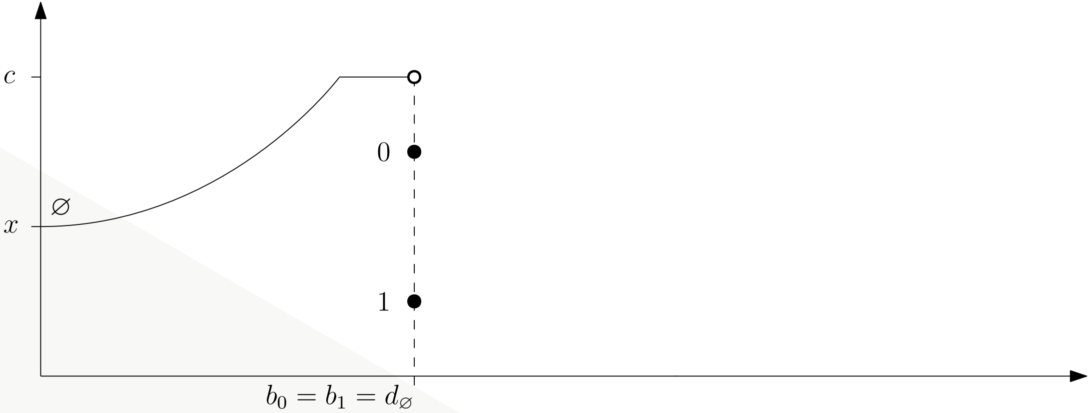

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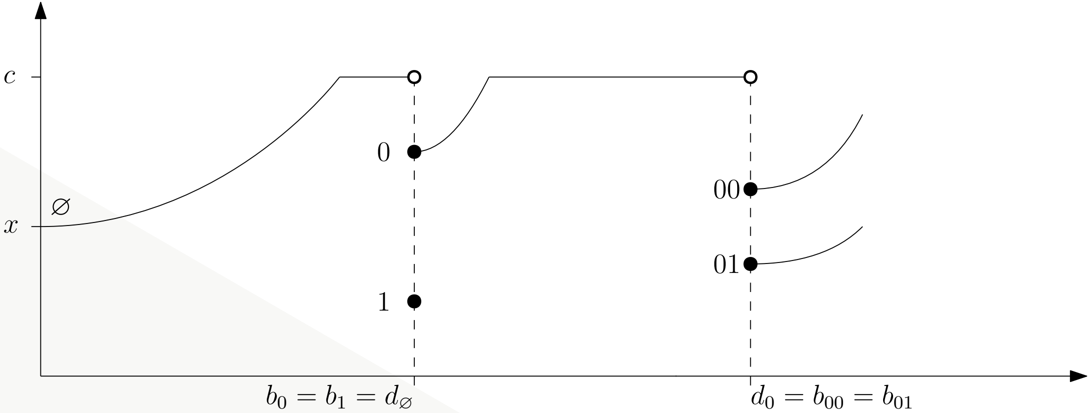

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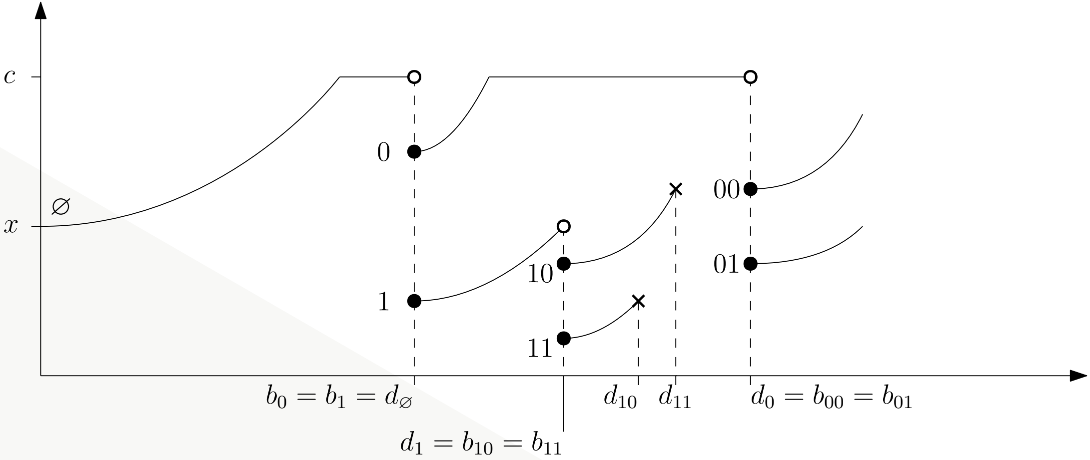

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...where $\tilde{\kappa}(\mathrm{d}p) = \frac{\kappa(\mathrm{d}p) + \kappa(1 - \mathrm{d}p)}{2}$.

Theorem 1(a)

If

$$a > 2B \int_0^1 (-\log p)\tilde{\kappa}(\mathrm{d}p),$$

then for f continuous and bounded,

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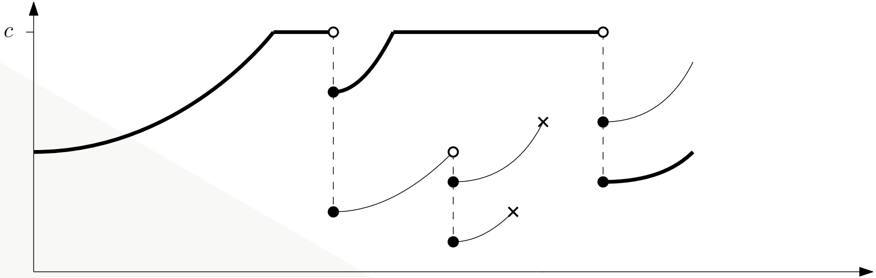
where $\langle \nu, h\rangle = 1$ and

$$\mathcal{A}h = \lambda h$$
$$\nu\mathcal{A} = \lambda\nu.$$

Proof ideas

Lemma (Many-to-one)

*Let η be a Lévy process with Lévy measure $2B\tilde{\kappa} \circ \log^{-1}$, drift a, and **reflection above** at $b = \log c$. Call η the **spine**. Then, $\langle \mu_t, f \rangle = e^{(B-\Bbbk)t}\mathbb{E}[f(e^{\eta_t}) \mid \eta_0 = \log x]$.*



Idea: conditional on living to t, follow offspring uniformly

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The invariant distribution m satisfies

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- $\langle \mu_t, f \rangle = e^{(B-\mathtt{k})t} \mathbb{E} f(e^{\eta_t}) \sim e^{(B-\mathtt{k})t} \langle m, f(e^{\cdot}) \rangle$.
- We get Theorem 1(a), with $\lambda = B - \mathtt{k}$, $h = 1$ and $\nu = m \circ \exp^{-1}$.

Theorem 1(b)

If

$$a > 2B \int_0^1 (-\log p)\tilde{\kappa}(\mathrm{d}p) \qquad \textbf{and} \qquad \int_0^1 p^{-(r+\epsilon)}\tilde{\kappa}(\mathrm{d}p) < \infty, \quad \text{some } r, \epsilon > 0,$$

then there exist $C, k > 0$ such that for f continuous,

$$\left|e^{-\lambda t}\langle \mu_t, f\rangle - h(x)\langle \nu, f\rangle\right| \leq \|f_r\|_\infty\big((c/x)^r + C\big)e^{-kt},$$

where $f_r(x) = (x/c)^r f(x)$.

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- Similar proof: exponential recurrence of reflected Lévy process

Theorem 2

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then, for continuous bounded f and $x \in (0, c]$,

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where M_∞ is a random variable with $\mathbb{E}_x M_\infty = h(x) = 1$.

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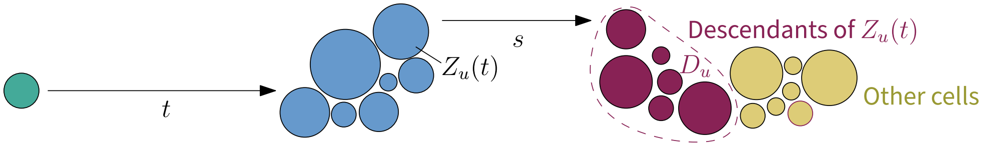
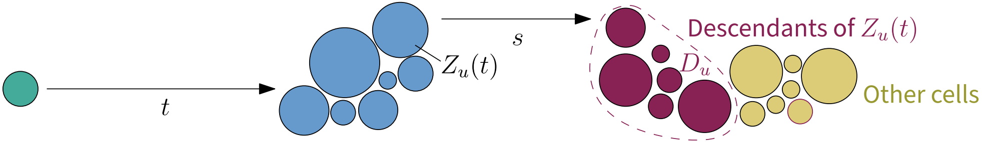
- M_∞ is its (a.s. or L^1) limit

Proof ideas



$$e^{-\lambda(t+s)}\mathbf{Z}(t+s)$$

Proof ideas



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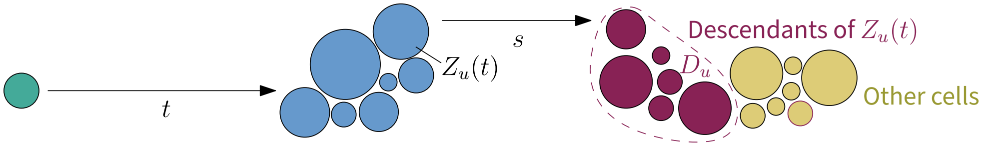

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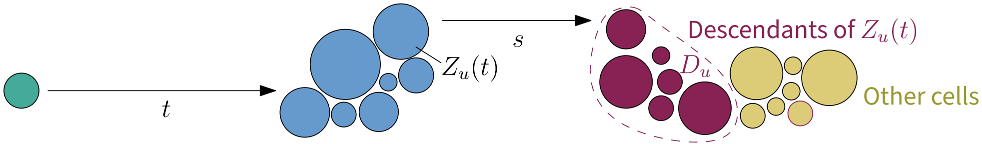

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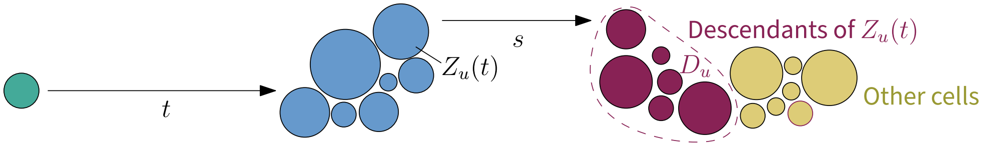

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$$+ M_t\nu$$

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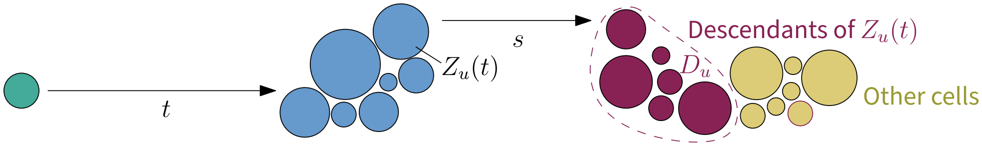

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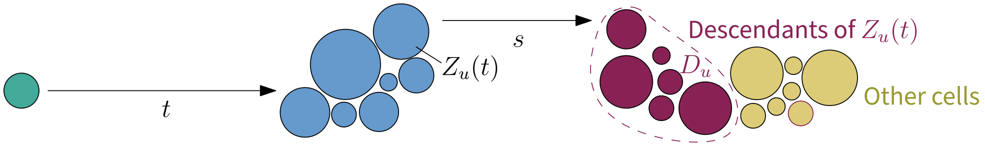
$$+ M_t\nu$$

Term 1 $\to 0$ by comparison with M. Term 3 $\to M_\infty\nu$.

Proof ideas



$$
\begin{aligned}
e^{-\lambda(t+s)}\mathbf{Z}(t+s) &= e^{-\lambda t}\sum_u e^{-\lambda s}\sum_{v\in D_u}\delta_{Z_v(t+s)} \\
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Term 1 $\to 0$ by comparison with M. Term 3 $\to M_\infty\nu$. **Term 2?**

Proof ideas

$$|\langle \text{Term } 2, f\rangle| = \left| e^{-\lambda t} \sum_u \left(\mathbb{E}_x \left[\sum_{v \in D_u} e^{-\lambda s} f(Z_v(t+s)) \middle| \mathcal{F}_t \right] - \langle \nu, f\rangle \right) \right|$$

Proof ideas

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- Use exponential rate asymptotics:

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...and the RHS is bounded in t.

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- Thus $\mathbb{E}_x R_{\delta n, \delta n}$ is summable, and Borel-Cantelli yields a.s. convergence along δn.

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...and the RHS is bounded in t.

- Thus $\mathbb{E}_x R_{\delta n, \delta n}$ is summable, and Borel-Cantelli yields a.s. convergence along δn.
- This is the core of the proof.

Proof ideas

- What if `k` > 0, i.e., cells may be killed?

PROOF IDEAS

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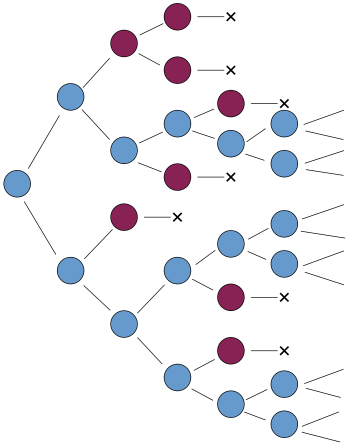
- Colour blue lines of descent which live forever, others red



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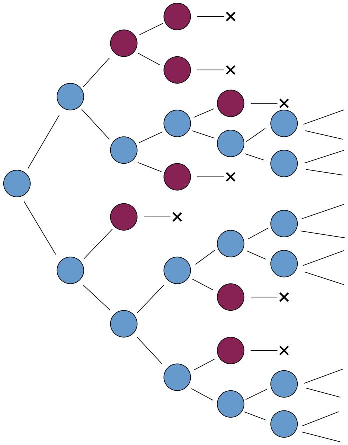

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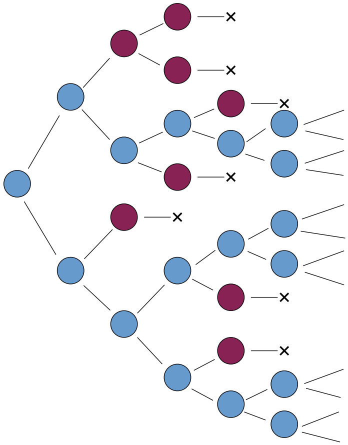
- Show that red tree has negligible contribution to limit



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- Require $B < \infty$, both for $\lambda < \infty$ to hold and for passage from $\mathbf{Z}(\delta n)$ to $\mathbf{Z}(t)$.
  - Could imagine more general Lévy processes appearing, but h will not be 1

Theorem 3: transient case

If

$$B > \mathtt{k}, \quad a < 2B \int_0^1 (-\log p)\tilde{\kappa}(\mathrm{d}p) \quad \text{(plus extra condition)}$$

there exist $\lambda_0 < \lambda$, $q_0 \in \mathbb{R}$ such that for f continuous with $f(x) = O(x^{q_0})$ as $x \to 0$,

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- Cell sizes decay to zero too fast to be seen by f
- Analysis involves nice reflected Lévy process with killing at reflection boundary

Further reading

E. Horton and A. R. Watson
Strong laws of large numbers for a growth-fragmentation process with bounded cell size
arXiv:2012.03273 [math.PR]

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Thank you!