The hitting time of zero for stable processes, symmetric and asymmetric

A. R. Watson*

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Abstract. We will discuss a method to characterise explicitly the law of the first time at which a stable process reaches the point zero, using theories of self-similar Markov processes. When the process is symmetric, the Lamperti representation characterises the law of the hitting time of zero as equal to that of the exponential functional of a Lévy process. When the stable process is asymmetric, things are not quite so simple. However, using the newly-developed theory of real self-similar Markov processes, we demonstrate that the hitting time of zero is equal to the exponential functional of a Markov additive process. These laws have not been investigated very thoroughly in the past, but remarkably, we are able to set up and solve a two-dimensional functional equation for a vector-valued Mellin transform. Moreover, we can even write down the density of the hitting time. Finally, we will discuss an application to the stable process conditioned to avoid zero.

Based on [6], which is joint work with Alexey Kuznetsov (York, Canada), Juan-Carlos Pardo (CIMAT) and Andreas Kyprianou (Bath, UK).

About these notes. This document is generated from my `beamer` presentation based on notes left in the source. My hope is that the reader who wishes they had attended my talk, but found it necessary instead to file some expenses/write a grant proposal/play Minecraft, will be able to read this and obtain a similar experience in the comfort of their own home. It should read somewhere between an informal talk and a formal article, and as such there are likely to be some inaccuracies due to the informal language and the fact that I wrote most of it from memory.

This document is probably best read along with the accompanying slides, but 'flattened' (and somewhat mangled) versions are included for convenience,

*University of Zürich `alexander.watson@math.uzh.ch`

demarcated by horizonal rules.

Are you sitting comfortably? Then I'll begin.

To state the problem, we will begin with the stable process. This is simple a Lévy process – a process with stationary, independent increments – which also satisfies a certain property of scaling invariance. Stable processes in one dimension are characterised by two parameters, a *scaling parameter* $\alpha \in (1, 2]$; and a *positivity parameter* ρ. In general, the range of parameters is

$$\{\alpha \in (0,1], \rho \in [0,1]\} \cup \{\alpha \in (1,2), \rho \in [1-1/\alpha, 1/\alpha]\} \cup \{(\alpha,\rho) = (2,1/2)\}.$$

However, for reasons which will soon become apparent, we will focus on the case where $\alpha \in (1, 2)$; indeed, we will only deal with the parameter set

$$\{\alpha \in (1,2), \rho \in (1-1/\alpha, 1/\alpha)\}.$$

Stable process

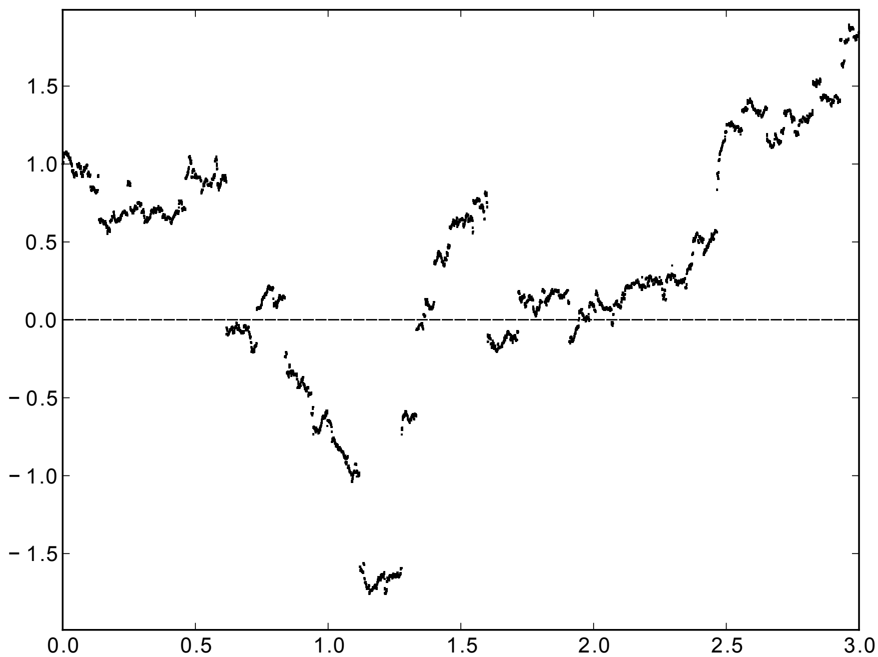
Begin with a stable process, X:

- Lévy process satisfying the *scaling property*

$$(cX_{tc^{-\alpha}})_{t\geq 0} \stackrel{d}{=} X, \qquad c > 0.$$

- Characterised by two parameters, α and ρ: $\rho = \mathrm{P}(X_t > 0)$.
- Our focus: $\alpha \in (1, 2)$.

Stable process: sample path, $(\alpha, \rho) = (1.3, 0)$



The big problem is to characterise the law of the first hitting time of zero. We will do it in two ways, firstly by computing the Mellin transform, and then by inverting that transform to give expressions for the density of the law.

Problem: statement

Let

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

The problem: *characterise* $\mathrm{P}_x(T_0 \in \cdot), \quad x \neq 0$

We will find:

- The *Mellin transform*, $\mathrm{E}_x[T_0^{s-1}]$
- The *density*, $\mathrm{P}_x(T_0 \in \mathrm{d}y)/\mathrm{d}y$

The problem has been studied previously. The more difficult spectrally one-sided case is given in [9], and the easier case may be deduced via the theory of scale functions. The symmetric case is considered in [10], where T_0 is characterised as the product of two laws; and [4], where an explicit expression for the Mellin transform is given.

These works rely heavily on potential theory. Our method is somewhat different, in that we make use of a path transformation. However, there is a way to obtain the Mellin transform that we get using potential theory—we will discuss this at the end.

Problem: history

- Spectrally one-sided case: Peskir (2008) and Simon (2011)
- Symmetric case: Yano, Yano, Yor (2009) and Cordero (2010)

We are going to do the symmetric case first: this relies on using the by now well-known Lamperti transform to relate X to another Lévy process.

Positive self-similar Markov processes

α-pssMp

- $[0,\infty)$-valued Markov process
- with initial measures P_x, $x > 0$
- *0 an absorbing state*
- satisfying the *scaling property*

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

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pssMps: examples

- Brownian motion killed on hitting zero
- Bessel process (of any dimension) killed on hitting zero
- Stable process...
  - ...killed on exiting $(0, \infty)$
  - ...conditioned to remain in $(0, \infty)$
  - ...conditioned to hit 0 continuously from above
  - ...censored outside of $(0, \infty)$

Finally, if X is a symmetric stable process, then $|X|$ is a pssMp, and if $\alpha > 1$, it hits zero.

The correspondence between such processes is given by the Lamperti transform.

Lamperti transform

α-pssMp X	**(killed) Lévy process ξ**
$X_t = \exp(\xi_{S(t)})$	$\xi_s = \log(X_{T(s)})$
$S(t)$ inverse to $\int_0^\cdot e^{\alpha \xi_u} \,\mathrm{d}u$	$T(s)$ inverse to $\int_0^\cdot (X_u)^{-\alpha} \,\mathrm{d}u$

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\} \qquad \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

$T_0 = T(\infty) = \int_0^\infty e^{\alpha \xi_u} \,\mathrm{d}u$

There are several examples of known Lamperti transforms, mostly associated with stable processes. In particular, the Lamperti-stable processes explored in Caballero and Chaumont [1], which are also members of the hypergeometric class of Lévy processes. However, this is a talk about the hitting time of zero, rather than Lamperti–Lévy processes, so we will focus just on the one example we want.

Apply to $|X|$

Write ξ for the Lamperti transform.

- Laplace exponent, $\mathbb{E}[e^{\lambda X_t}] = e^{t\psi(\lambda)}$:

  $$\psi(\lambda) = -2^{\alpha} \frac{\Gamma(\alpha/2 - \lambda/2)}{\Gamma(-\lambda/2)} \frac{\Gamma(1/2 + \lambda/2)}{\Gamma((1-\alpha)/2 + \lambda/2)},$$

  for $\operatorname{Re}\lambda \in (-1, 1/\alpha)$
- hypergeometric / extended hypergeometric class

When the Lévy process satisfies a certain Cramér condition, then the Mellin transform satisfies a functional equation. This equation, together with a certain asymptotic condition, is sufficient to characterise $\mathcal{M}$ completely: see the verification result in Kuznetsov and Pardo [5].

Exponential functional

$$T_0 = I := \int_0^\infty e^{\alpha\xi_u}\, \mathrm{d}u.$$

Let ξ be a Lévy process satisfying a *Cramér condition*, i.e. for some $\theta > 0$, $\psi(\theta) = 0$.

Write

$$\mathcal{M}(s) = \mathbb{E}[I^{s-1}].$$

Then

$$\mathcal{M}(s+1) = -\frac{s}{\psi(\alpha s)}\mathcal{M}(s),$$

and this, together with an asymptotic condition, is often enough to find $\mathcal{M}$.

We now present the main result for the symmetric case. In fact, a similar expression for the Mellin transform holds for a rather larger class of Lévy processes, the so-called *extended hypergeometric class*.

As we said, this has previously been derived using potential theory. Next, we will derive a Mellin transform for the hitting time of zero of an asymmetric Lévy process, and after we find a nice expression for this we will invert it to find a density; this result will apply equally to the symmetric case.

Main result, symmetric case

$$\mathrm{E}_1[T_0^{s-1}] = \mathbb{E}_0[I^{s-1}] = \mathcal{M}(s)$$

Proposition 1. *For* $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathcal{M}(s) = \sin(\pi/\alpha) \frac{\cos(\frac{\pi\alpha}{2}(s-1))}{\sin(\pi(s-1+\frac{1}{\alpha}))} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}$$

We will now turn to the asymmetric case. This follows the same pattern as the symmetric case, but is a bit more complicated and involves some quite recent developments.

Before we do, let's think about what is wrong with the previous approach. When X is asymmetric, $|X|$ still scales correctly, but it does not satisfy the Markov property, because one doesn't know the sign of X. Thus, the pssMp–Lévy process connection breaks down. The basic idea is that we should enlarge the state space to remedy this. Let's now discuss the necessary tools.

Our insight into the problem will come by viewing the stable process killed on hitting zero as a *real self-similar Markov process*. The theory behind these processes was developed by Chaumont et al. [2]. Note that, in the fourth bullet point of the slide, it is crucial that $c > 0$; this allows the process to have, essentially, two different behaviours depending on whether it is positive or negative.

We are going to make the assumption that X jumps both from positive to negative and from negative to positive in finite time. The processes where this does not occur have a similar structure but get 'stuck' in either a positive or negative state; it is simpler for us to omit them, but the majority of what we will discuss holds also for these processes.

Real self-similar Markov processes

α-rssMp

- $\mathbb{R}$-valued Markov process

- with initial measures P_x, $x \neq 0$
- *0 an absorbing state*
- satisfying the *scaling property*

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

Assume X jumps over zero in both directions

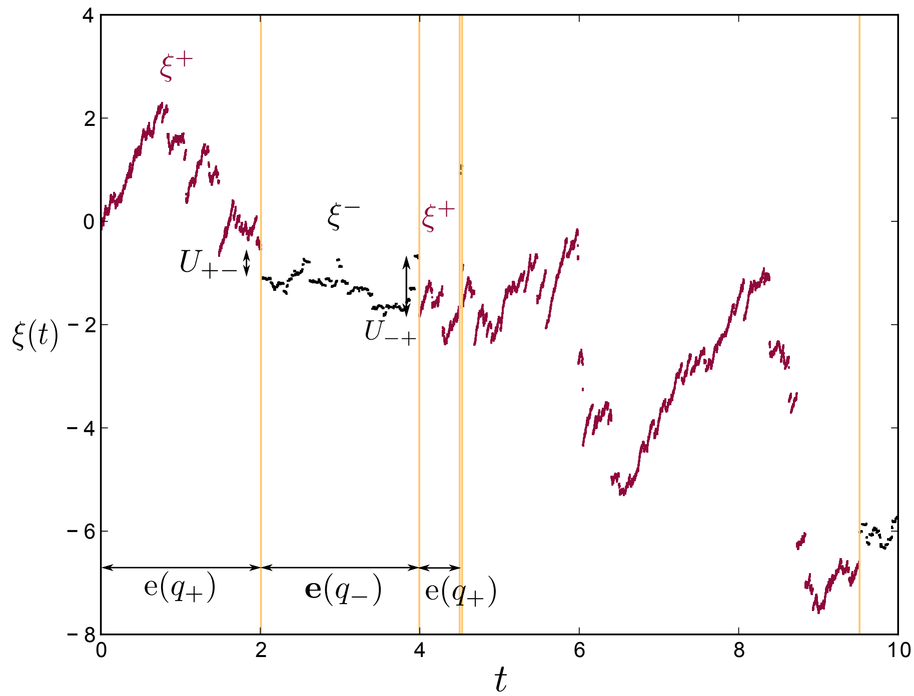
To describe the Lamperti–Kiu transform, we need to talk about Markov additive processes. For the sake of simplicity, we will just consider two-state MAPs; the full theory for finite-state MAPs is very similar, but the notation is more irritating. We will call the two states $+$ and $-$, for reasons which will be revealed in two slides' time.

A Markov additive process (ξ, J) runs as follows. Suppose that J begins in the state $+$. Then ξ follows a Lévy process with law ξ^+, and runs until a independent exponentially-distributed clock of rate q_+ rings; then, ξ makes a(n independent) jump distributed as U_{+-}, the process J switches to state $-$, and everything starts afresh, with ξ following a Lévy process ξ^- instead.

Two state $(+,-)$ Markov additive process

Lévy processes ξ^+,ξ^-; clock rates q_+,q_-; jumps U_{+-},U_{-+}

Markov additive process $(\xi(t), J(t))_{t\geq 0}$



Much as a Lévy process is characterised by its characteristic exponent, a MAP is characterised by a matrix, given here. The fact that it can be written fairly simply in terms of the components of the MAP will allow us to compute specific examples shortly.

MAP: characterisation

There exists a matrix $F(z) = \begin{pmatrix} F_{++}(z) & F_{+-}(z) \\ F_{-+}(z) & F_{--}(z) \end{pmatrix}$ such that

$$\left(e^{tF(z)}\right)_{ij} = \mathbb{E}\Big[e^{z\xi(t)}; J(t) = j \mid J(0) = i,\, \xi(0) = 0\Big].$$

F is determined by the components:

$$F(z) = \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+ G_{+-}(z) \\ q_- G_{-+}(z) & -q_- \end{pmatrix}$$

where $e^{\psi_i(z)} = \mathbb{E}[e^{z\xi^i(1)}]$ and $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$.

The Lamperti–Kiu transform, given by Chybiryakov [3] in the symmetric case and in general by Chaumont et al. [2], gives a bijection between rssMps and two-state Markov additive processes. The transform is essentially the same as the Lamperti transform (between positive self-similar Markov processes and Lévy processes) but it uses the Markov chain component of the Markov additive process to keep track of the sign of X.

We remark on the long-term behaviour of the processes X and ξ; in particular, when X hits zero continuously, the hitting time is given by an *exponential functional* of ξ.

Finally, the slide gives the results when starting the rssMp from ± 1. Since X satisfies a scaling property, it is not difficult to deduce from this the case where it starts from $x \neq 0$, but we omit the details.

Lamperti–Kiu transform (Chaumont, Pantí, Rivero)

α-rssMp X under $\mathrm{P}_{\pm 1}$	**Two-state MAP** (ξ, J)
$X_t = J(S(t)) \cdot \exp(\xi(S(t))$	$\xi(s) = \log(\lvert X_{T(s)} \rvert)$
	$J(s) = \operatorname{sgn}(X_{T(s)})$
$S(t)$ inverse to $\int_0^\cdot e^{\alpha\xi(u)}\,\mathrm{d}u$	$T(s)$ inverse to $\int_0^\cdot \lvert X_u \rvert^{-\alpha}\,\mathrm{d}u$
X never hits zero	$\xi \to \infty$ or ξ oscillates
X hits zero continuously	$\xi \to -\infty$

$T_0 = T(\infty) = \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u$

Before we get to the main topic of the talk, let's see an example – it is one of the few examples where we can compute the Lamperti–Kiu transform explicitly.

The components of this MAP were computed in [2]. We remark here that the diagonal entries in F are the Laplace exponents of hypergeometric, Lamperti-stable Lévy processes; and the off-diagonal entries correspond to 'log-Pareto' distributions.

Lamperti–Kiu transform: example

X: stable process killed on hitting zero (an α-rssMp)

Compute explicitly

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix}$$

(hypergeometric Lévy processes and 'log-Pareto' jump distributions)

Exponential functional

Recall:

$$T_0 = I(\alpha\xi) := \int_0^\infty e^{\alpha\xi(u)}\, \mathrm{d}u.$$

How to characterise the *exponential functional* of ξ?

Let

$$\mathcal{M}(s) = \begin{pmatrix} \mathcal{M}_+(s) \\ \mathcal{M}_-(s) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = +] \\ \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = -] \end{pmatrix},$$

the *Mellin transform* of $I(\alpha\xi)$.

The principal ingredient in finding $\mathcal{M}$ is the following vector-valued functional equation, which generalises the well-known scalar equation for the exponential functional of a Lévy process. It holds whenever the Markov additive process satisfies a *Cramér condition*, which we make precise here.

Whenever the matrix $F(z)$ is defined, it is known, by Perron–Frobenius theory, to have a simple real eigenvalue with maximal real part; denote this by $k(z)$. This leading eigenvalue, so called, may be viewed as an equivalent of the Laplace exponent of a Lévy process, and it has some similar properties; for instance, it is smooth and convex wherever it is defined.

We say that (ξ, J) satisfies the Cramér condition with Cramér number $\theta > 0$ if k is defined on $(0, \theta + \epsilon)$, for some $\epsilon > 0$, and $k(\theta) = 0$. This is a sufficient condition for $\mathcal{M}$ to be finite for all $\operatorname{Re} s \in (0, 1 + \theta)$ and for the functional equation in the slide to hold whenever $s \in (0, \theta)$.

In particular, this condition implies that $k'(0+) < 0$, so $\xi \to -\infty$.

Cramér condition & exponential functional

Whenever $F(z)$ is defined, it has a simple real eigenvalue $k(z)$ with maximal real part. We require:

Assumption 2 (Cramér condition)**.** *There exists some* $\theta > 0$*, such that* F *is defined on* $(0, \theta + \epsilon)$ *and* $k(\theta) = 0$*.*
Proposition 3. *Provided* (ξ, J) *satisfies the* θ*-Cramér condition,*

$$\mathcal{M}(s+1) = -s(F(\alpha s))^{-1}\mathcal{M}(s),$$

for $\operatorname{Re} s \in (0, 1 + \theta/\alpha)$*.*

We have a good intuition about the form of the Mellin transform due to the known cases mentioned, and when combined with the functional equation above, this is enough to deduce the correct solution. Verifying this is not easy, but the basic outline is much as in the case of Lévy processes, and consists of complex analytic arguments using the asymptotics of the proposed solution.

Solution: Mellin transform

Theorem 4. *For* $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$*,*

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$

Here we outline an alternative way to derive the Mellin transform of T_0 using potential theory. A rather similar approach has been taken by Letemplier and Simon [7].

The first identity is a well known fact from potential theory, but the expression on the right-hand side, which includes the λ-potential density of X, is rather difficult.

The naive approach to computing the law of T_0 would be to use series expressions for u^λ and attempt to invert these, but this fails. Instead, we integrate both sides of the equation, obtaining the Mellin transform of T_0 on the left and something which appears complicated on the right. All is well, however. Although no compact expression for u^λ is available, it may be written as the Fourier inversion integral of the expression $(\lambda + \Psi(\theta))^{-1}$. One thus obtains a double integral on the right-hand side. This can be evaluated by reversing the order of integration, and one thus obtains the Mellin transform.

In light of this we should perhaps offer some defence of our MAP-based approach. Although it seems rather more involved, the MAP framework allows one to work with rssMps other than stable processes. Some possible areas of interest where the MAP viewpoint seems to be useful are fluctuations and conditionings of rssMps, and the construction of entrance laws or recurrent extensions for rssMps started from the point 0.

Alternative derivation, cf. also Letemplier & Simon (2013)

It is known that

$$\mathrm{E}_1\left[e^{-\lambda T_0}\right] = \frac{u^\lambda(-1)}{u^\lambda(0)},$$

where

$$u^\lambda(x)\,\mathrm{d}x = \int_0^\infty e^{-\lambda t}\mathrm{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t.$$

Trick:

$$\mathrm{E}_1\left[T_0^{s-1}\right] = \frac{1}{\Gamma(1-s)}\int_0^\infty \frac{u^\lambda(-1)}{u^\lambda(0)}\lambda^{-s}\,\mathrm{d}\lambda$$

and

$$u^\lambda(x) = -\frac{1}{\pi}\,\mathrm{Im}\left[\int_0^\infty \frac{e^{tx}}{\lambda + t^\alpha e^{\mathrm{i}\pi\alpha\hat{\rho}}}\,\mathrm{d}t\right]$$

Computing the density is in principle rather a simple matter, involving simply the inversion integral for Mellin transforms and residue calculus. In practice, it turns out to be somewhat more involved, and the delicate bounds required to ensure convergence yield the following (partial) result; more on the 'dense set', and what to do outside this set, can be found in the article (or the appendix slides.)

Solution: density

Let $\mathrm{P}_1(T_0 \in \mathrm{d}t) = p(t)\,\mathrm{d}t$.
Theorem 5. *For a dense, full-measure set of $\alpha \in (1,2)$:*

$$\begin{aligned} p(t) &= \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat{\rho})}\sum_{k\geq 1}\frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))}\frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} \\ &\quad - \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat{\rho})}\sum_{k\geq 1}\frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)}\frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)}t^{-k-1+\frac{1}{\alpha}}. \end{aligned}$$

Here is a simple application of the expression for the density. The following is a special case of conditioning a Lévy process to avoid zero, which is discussed in Pantí [8]. In the general case, one has that the h-process is the Lévy process conditioned to have avoided zero up to an exponential time, as the rate goes to zero. In the stable case, we will do better on the next slide.

Application: conditioning to avoid zero. Cf. Pantí (2012)

Define

$$h(x) = \begin{cases} -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\hat{\rho})}{\pi}|x|^{\alpha-1}, & x > 0, \\ -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\rho)}{\pi}|x|^{\alpha-1}, & x < 0, \end{cases}$$

h is *invariant* for the stable process X killed on hitting zero. Let

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \frac{1}{h(x)}\mathrm{E}[h(X_t)\mathbb{1}_\Lambda;\ t < T_0], \qquad \Lambda \in \mathcal{F}_t,\ x \neq 0.$$

This is the *process conditioned to avoid zero*: for any stopping time T and $\Lambda \in \mathcal{F}_T$, and $x \neq 0$,

$$\lim_{q \downarrow 0} \mathrm{P}_x(\Lambda, T < \mathbf{e}_q | T_0 > \mathbf{e}_q) = \mathrm{P}_x^{\updownarrow}(\Lambda), \qquad \mathbf{e}_q \sim \mathrm{Exp}(q).$$

In the stable case, we can see the h-process as the stable process conditioned to avoid zero up to a fixed time s, as $s \to \infty$. The proof is an explicit calculation using the density of T_0 and ζ.

This is an interesting improvement on the general result. We may be able to offer another. Consider the h-function on the previous slide in, say, the spectrally negative case ($\rho = 1/\alpha$). Then h is identically zero on the negative half-line. It should be possible to use the more precise information in the density of T_0 to give an alternative (time-dependent) h-function to condition a spectrally one-sided Lévy process to avoid zero. This is still preliminary.

Application: conditioning to avoid zero

Write n for the excursion measure of X away from zero, and ζ for the lifetime of the excursion.

Proposition 6. *For $x \neq 0$,*

$$h(x) = \lim_{s\to\infty} \frac{\mathrm{P}_x(T_0 > s)}{n(\zeta > s)},$$

and for any stopping time T such that $\mathbb{E}_x[T] < \infty$, and $\Lambda \in \mathcal{F}_T$,

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \lim_{s\to\infty} \mathrm{P}_x(\Lambda | T_0 > T + s).$$

Extra material starts here.

We give some details on what was meant by a 'dense, full-measure set' on the previous slides. What can go wrong is that the series given in the previous slides may fail to converge for some α, essentially because the partial sums may periodically blow up. On the next slide we will see a way around this; for now, we note that the α where this may occur are given in the set $\mathcal{L}$; since this set has Hausdorff dimension zero, it also has zero Lebesgue measure.

The 'dense subset' ...

Let $\|x\| = \min_{n\in\mathbb{Z}}|x - n|$, and

$$\mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} \tfrac{1}{n} \ln\|nx\| = 0\}).$$

$\mathcal{L}$ is small – it has Hausdorff dimension zero.

The previous result holds for $\alpha \notin \mathbb{Q} \cup \mathcal{L}$.

Even when $\alpha \in \mathcal{L}$, expressions may be found for the density of T_0. All we need to do is avoid the set of N where the partial sums diverge. The set $\mathcal{K}(\alpha)$ here reasonably nice – it has density equal to one, in the sense that

$$\lim_{n\to\infty} \frac{\#\{\mathcal{K}(\alpha) \cap [1, n]\}}{n} = 1.$$

The expression for p doesn't look quite as nice, but it is just as useful as the series expressions.

. . . and what happens outside

Define
$$\mathcal{K}(\alpha) = \{N \in \mathbb{N} : \|(N - \tfrac{1}{2})\alpha\| > \exp(-\tfrac{\alpha-1}{2}(N-2)\ln(N-2))\}$$

For $\alpha \notin \mathbb{Q}$, it holds that
$$\begin{aligned} p(t) = \lim_{\substack{N\in\mathcal{K}(\alpha)\\ N\to\infty}} \Bigg[\frac{\sin(\frac{\pi}{\alpha})}{\pi\sin(\pi\hat{\rho})} \sum_{1\le k<\alpha(N-\frac{1}{2})-1} & \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \\ & \times \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} \\ - \frac{\sin^2(\frac{\pi}{\alpha})}{\pi\sin(\pi\hat{\rho})} \sum_{1\le k<N} & \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma(k-\frac{1}{\alpha})}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}} \Bigg]. \end{aligned}$$

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