Spectral properties of a growth-fragmentation equation

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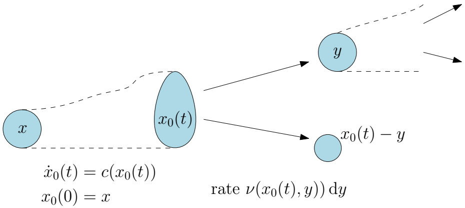
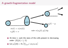
A growth-fragmentation model



- At time t, rank the sizes of the cells present in decreasing order: $(X_i(t), i \geq 1)$.
- Let $\mu_t^x(A) = \mathbb{E}_x\left[\sum_{i\geq 1} \mathbb{1}_{\{X_i(t)\in A\}}\right]$

A growth-fragmentation model



In this talk, we are interested in a model for growing and fragmenting cells. These could be biological, but they could equally be something more abstract – growth-fragmentation models have found applications in random maps and the study of random recursive trees, for example. The important thing is that the cells are characterised by a 'size' or a 'mass'.

We start with a single cell of size x; it grows according to the function $x_0(t)$, which solves an ODE, until it splits at rate $\nu(x_0(t)) = \int \nu(x_0(t), y)\,\mathrm{d}y$. Upon splitting, at time t, it creates two new cells of sizes y and $x_0(t) - y$, where y is distributed according to the probability measure $\frac{\nu(x_0(t),y)\,\mathrm{d}y}{\nu(x_0(t))}$.

The new cells evolve independently of the past and of each other.

The measure μ_t is called the *mean measure* of the process, and describes the mean number of cells with certain sizes at time t. We can describe its evolution explicitly.

Note that here $\mathbb{P}_x$ refers to starting this growth-fragmentation model from the initial condition in which there is a single cell of size x.

The growth-fragmentation equation

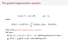
$$\partial_t \langle \mu_t^x, f\rangle = \langle \mu_t^x, \mathcal{A}f\rangle, \qquad \mu_0^x = \delta_x,$$

where

$$\mathcal{A}f(x) = c(x)f'(x) + \int_0^x f(y)k(x,y)\,\mathrm{d}y - K(x)f(x).$$

This is the growth-fragmentation equation.
We have:

- $k(x,y) = \nu(x,y) + \nu(x,x-y)$: splitting giving rise to size y
- $K(x) = \int_0^x \frac{y}{x} k(x,y)\,\mathrm{d}y$: total splitting rate



The evolution of the mean measures (μ_t) can be described by the *growth-fragmentation equation*, which holds (at least) for f once differentiable and compactly supported on $(0,\infty)$.

On this slide, and hereafter, $\langle \mu, f\rangle = \int f\,\mathrm{d}\mu$.

To be rigorous, our work requires some assumptions not on the slide:

- $\sup_{x>0} \frac{c(x)}{x} < \infty$
- $(0,\infty) \ni x \mapsto k(x,\cdot) \in L^1(0,\infty)$ is a bounded function

Note that the operator $\mathcal{A}$ is the generator of a semigroup, but not a contraction semigroup. So $\mathcal{A}$ is not the generator of a Markov process. But we can connect it to one, and this will be the first step in our method.

Finally, note that operators of form $\mathcal{A}$ are not in one-to-one correspondence with growth-fragmentation models: there are many different models which give rise to the same mean measures (μ_t), and hence the same operator $\mathcal{A}$; but if we restrict ourselves to binary models, then there is essentially a bijection.

The question

Asymptotic behaviour of (μ_t): expect

$$e^{-\lambda t}\langle \mu_t^x, f\rangle = \alpha(x)\langle m, f\rangle + o(e^{-\beta t}).$$

Determine λ (Malthus exponent), α and m (eigenelements of $\mathcal{A}$) and β (bound for spectral gap).

The question

Our goal for this talk is to find conditions for existence of these spectral properties of the growth-fragmentation equation, using probabilistic tools as far as possible.

The Malthus exponent is the leading eigenvalue of $\mathcal{A}$. Note that it could be negative.

A many-to-one lemma

Let X be a Markov process with generator

$$\mathcal{G}f(x) = c(x)f'(x) + \int_0^x [f(y) - f(x)]\bar{k}(x,y)\,\mathrm{d}y,$$

where $\bar{k}(x,y) = \frac{y}{x}k(x,y)$.

Lemma

Let

$$\mathcal{E}_t = e^{\int_0^t \frac{c(X(s))}{X(s)}\,\mathrm{d}s}.$$

Then,

$$\langle \mu_t^x, f\rangle = \mathbb{E}_{\delta_x}\Big[\sum_{i=1}^{\infty} f(X_i(t))\Big] = x\mathbb{E}_x\big[\mathcal{E}_t \frac{f(X(t))}{X(t)}\big].$$

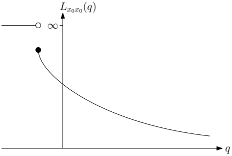
The process X is a piecewise deterministic Markov process: it follows the flow $\dot{x}_0(t) = c(x_0(t))$ until it jumps to a new position y at rate $\bar{k}(x_0(t), y)\,\mathrm{d}y$, and then this evolution begins again.

This lemma can be seen as a simple relation between semigroups, and it can be proved using the operators $\mathcal{A}$ and $\mathcal{G}$. However, something lies behind it, as indicated by the title of the slide. The process X is the size of a so-called *tagged cell* or *spine* in the growth-fragmentation model: one starts with the single initial cell and follows it until splitting, and at that time, one chooses a new cell to follow in proportion to the sizes of the children.

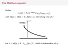
This idea will be familiar from branching process theory.

The Malthus exponent

Define
$$L_{xy}(q) = \mathbb{E}_x\left[e^{-qH(y)}\mathcal{E}_{H(y)}\mathbb{1}_{\{H(y)<\infty\}}\right],$$
with $H(y) = \inf\{t > 0 : X(t) = y\}$ the hitting time of y.



Let $\lambda = \inf\{q \in \mathbb{R} : L_{x_0x_0}(q) \le 1\}$, which is independent of x_0.



We can now define the *Malthus exponent*.

The function $q \mapsto L_{xy}(q)$ has the shape indicated: it approaches ∞ as $q \to -\infty$; it is right-continuous at the point at which it jumps below ∞ (if any); it is convex; and it approaches 0 as $q \to \infty$.

These properties imply the existence of some minimal λ such that $L_{x_0x_0}(\lambda) \le 1$, and in fact, it is not difficult to show that λ is independent of x_0. We call this λ the Malthus exponent.

We shall soon see that, if the 'good' case shown does hold, which implies that $L_{x_0x_0}(\lambda) = 1$, then λ does indeed determine the growth or decay of (μ_t), and we can answer our question.

The result

Theorem

Suppose *that, for some* $q \in \mathbb{R}$, $1 < L_{x_0 x_0}(q) < \infty$.
Let $\ell(x) = L_{x x_0}(\lambda)$, $\bar{\ell}(x) = x\ell(x)$ *and*

$$m(\mathrm{d}x) = \frac{\mathrm{d}x}{\bar{\ell}(x)c(x)|L'_{xx}(\lambda)|}, \qquad x > 0.$$

Then, *there exists* $\beta > 0$ *such that*

$$e^{-\lambda t}\langle \mu_t^x, f\rangle = \bar{\ell}(x)\langle m, f\rangle + o(e^{-\beta t}), \qquad t \to \infty.$$

Moreover, $\mathcal{A}\bar{\ell} = \lambda\bar{\ell}$ *and* $\mathcal{A}^* m = \lambda m$.

We are now able to present a complete result about the asymptotic behaviour of solutions to the growth-fragmentation equation.

It is notable that we are able to present this result by putting a condition on the Markov process X, and that the eigenelements are expressed in terms of X. This can be used to produce sufficient conditions for convergence which are easier to check than the one in this theorem, and this is done in the paper.

Note that the condition $L_{x_0 x_0}(q) \in (1, \infty)$ implies that $|L'_{xx}(\lambda)| < \infty$, so m is well-defined. In fact, the theorem is true (with $\beta \geq 0$) so long as m is well-defined, and this is a slightly weaker condition than the one given.

Finally, note that, to write down the equation $\mathcal{A}\bar{\ell} = \lambda\bar{\ell}$, we are assuming that $\bar{\ell}$ lies in the domain of $\mathcal{A}$, which we have not discussed. This is not obvious, and actually there is an unstated assumption here, but we leave discussion of this to the paper.

An idea of the proof

- Let $M_t = e^{-\lambda t}\mathcal{E}_t \frac{\ell(X(t))}{\ell(x)}$.
- M is a $\mathbb{P}_x$-martingale: for intuition as to why, observe that, by definition of λ, $\mathbb{E}_x[M_{R_n}] = 1$, where R_n is the n-th return time to x of the process X.
- Let $\frac{\mathrm{d}\mathbb{Q}_x}{\mathrm{d}\mathbb{P}_x}\Big|_{\mathcal{F}_t} = M_t$, and denote by Y the canonical measure under $\mathbb{Q}$., so $\langle \mu_t^x, f\rangle = x\mathbb{E}_x\big[\mathcal{E}_t \frac{f(X(t))}{X(t)}\big] = e^{\lambda t}\bar{\ell}(x)\mathbb{Q}_x\big[\frac{f(Y(t))}{\bar{\ell}(Y(t))}\big]$.
- Y is a recurrent Markov process which hits points, and therefore has *cycles*. Since $|L'_{x_0x_0}(\lambda)| < \infty$, it is positive recurrent, and in fact,

$$\bar{\ell}(y)m(\mathrm{d}y) = \frac{1}{\mathbb{Q}_x(H(x))}\mathbb{Q}_x\int_0^{H(x)} \mathbb{1}_{\{Y(s)\in \mathrm{d}y\}}\,\mathrm{d}s.$$

- Therefore, $\langle \mu_t^x, f\rangle = e^{\lambda t}\bar{\ell}(x)\mathbb{Q}_x\big[\frac{f(Y(t))}{\bar{\ell}(Y(t)}\big] \sim e^{\lambda t}\bar{\ell}(x)\langle m, f\rangle$.

An idea of the proof

The key insight here is that the Markov process X hits points. Each time X hits its starting value, it regenerates, and so the paths of X can be broken down into i.i.d. cycles between hitting times of the starting point. However, X need not be point recurrent, or even recurrent.

To fix this, and to remedy the discrepancy between the stochastic growth $\mathcal{E}_t$ and the deterministic growth $e^{\lambda t}$, we make a change of measure. (Here, we see where the equation for λ comes from: it is akin to averaging the stochastic versus deterministic growth over a cycle of X.)

After changing measure, we arrive at a recurrent Markov process Y, which has the same regeneration properties (cycle structure) as X. For such a process, the cycle potential $\mathbb{Q}_x\int_0^{H(x)} \mathbb{1}_{\{Y(s)\in \mathrm{d}y\}}\,\mathrm{d}s = \frac{\mathrm{d}y}{c(y)}$ is always invariant for the semigroup of Y, and we may show that the total mass of this measure is $|L'_{xx}(\lambda)|$. So, when the latter is finite, Y is positive recurrent, and the asymptotic behaviour of (μ_t) follows.

Finally, the result on the spectral gap ($\beta > 0$) follows from the fact that $\mathbb{Q}_x[e^{\varepsilon H(x)}] < \infty$ for some $\epsilon > 0$, and an application of Kendall's renewal theorem (adapted to a continuous time setting).

Simpler sufficient conditions

Assume $c(x) = ax$.

Proposition

If *$K(x) \sim \beta_0 x^{\gamma_0}$ for some $\gamma_0 > 0$ as $x \to 0$, $K(x) \to \beta_\infty$ as $x \to \infty$, and*

$$a < \beta_\infty \frac{1 - \sup_{x>0} \int_0^x \left(\frac{y}{x}\right)^s \frac{\bar{k}(x,y)}{K(x)}\,\mathrm{d}y}{s}, \qquad \text{for some } s > 0,$$

then *$\lambda = a$, $\bar{\ell}(x) = x$ and there exist $C, \beta > 0$ such that*

$$\mathrm{d}_{TV}\left(e^{-at}\frac{y}{x}\mu_t^x(\mathrm{d}y), ym(\mathrm{d}y)\right) \le Ce^{-\beta t}.$$

How does this result work? The condition on K at zero ensures that the system fragments slower than it grows when cells are close to zero. The growth and fragmentation are of the same order at infinity, and the (uglier) condition then ensures also that fragmentation happens marginally faster than growth at infinity.

These conditions conspire to give that the process X of the tagged cell is recurrent; this is shown using a Foster-Lyapunov argument. Then $\lambda = a$ since $H(x) < \infty$ a.s. for X, and moreover $X = Y$. Improving on this Foster-Lyapunov argument shows that $X = Y$ is indeed exponentially ergodic, which is essentially what the result says. (In this direction, recall that $ym(\mathrm{d}y)$ is the stationary distribution of Y.)

Further work

- A strong law of large numbers for the growth-fragmentation model
  - **Theorem** (Bertoin): If

    $$\limsup_{x\downarrow 0} \tfrac{c(x)}{x} < \lambda \text{ and } \limsup_{x\to\infty} \tfrac{c(x)}{x} < \lambda,$$

    then result of this talk holds.
  - **Conjecture**: If the same condition holds, then $e^{-\lambda t}\sum_{i\geq 1} f(X_i(t))\bar{\ell}(X_i(t)) \to W\langle m, f\rangle$, a.s.
- Alternative sufficient conditions for asymptotics of mean – via quasi-stationary distributions – requires a (local) Doeblin condition or Harnack inequality
- Incorporate position or shape of cells as types

Further work

We close with some indications of further work. One place where these results can certainly be extended is the strong law of large numbers. The results presented here are essentially a weak law (convergence in mean). We can expect that, in fact, to obtain convergence of weighted empirical measures:

$$e^{-\lambda t}\sum_{i\geq 1} f(X_i(t))\bar{\ell}(X_i(t)) \to W\langle m, f\rangle, \qquad \text{a.s.},$$

where W is a random variable. This is connected to the uniform integrability of certain 'Malthusian' martingales for $(X_i(t) : i \geq 1, t \geq 0)$.

It may be interesting to extend the model to incorporate particle positions; it is common biologically to consider an inhomogeneous growing medium, so that the growth and fragmentation rate depend on not just the size of the cell, but also the position. This can be done by incorporating types. If the type space is compact, then we may expect that analogous results will hold.

Further reading

J. Bertoin and A. R. Watson
A probabilistic approach to spectral analysis of growth-fragmentation equations
J. Funct. Anal., 274, no. 8, 2163–2204. 2018.

Further reading