Lévy processes conditioned to avoid an interval

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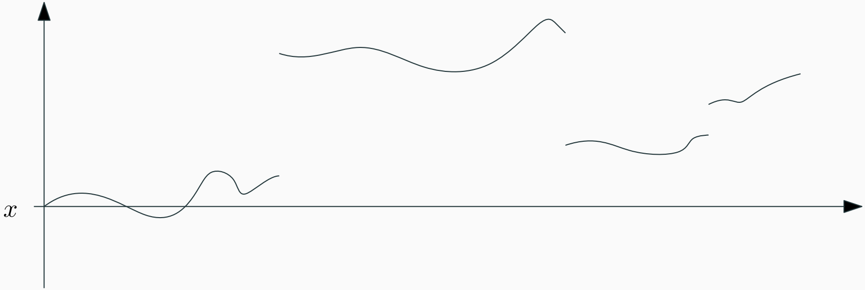
University of Manchester

Days of very low probability

- X: recurrent Markov process on $\mathbb{R}$



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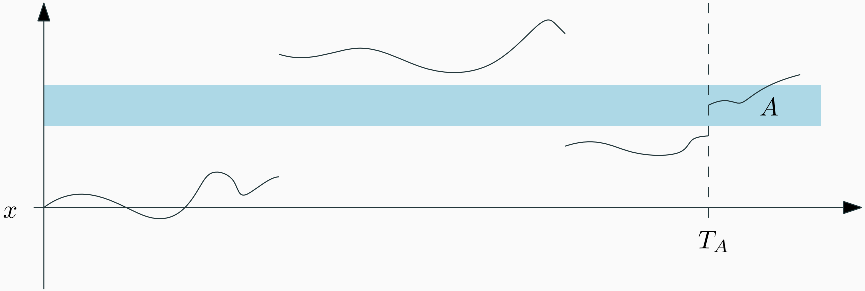
- $A \subset \mathbb{R}$, T_A the hitting time of A



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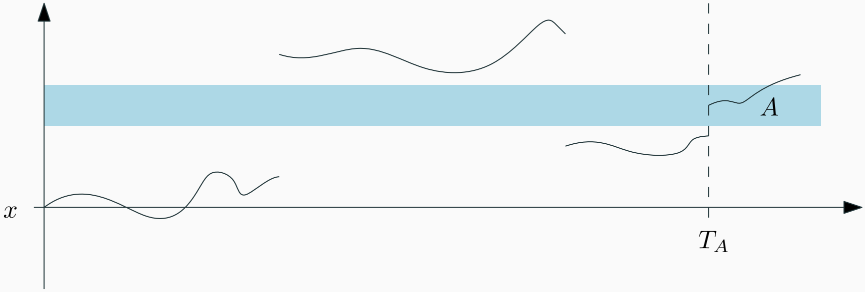
- $T_A < \infty$ almost surely: what is 'X conditioned to avoid A'?



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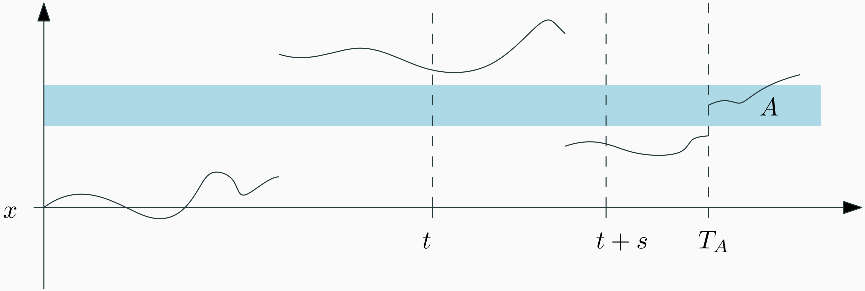

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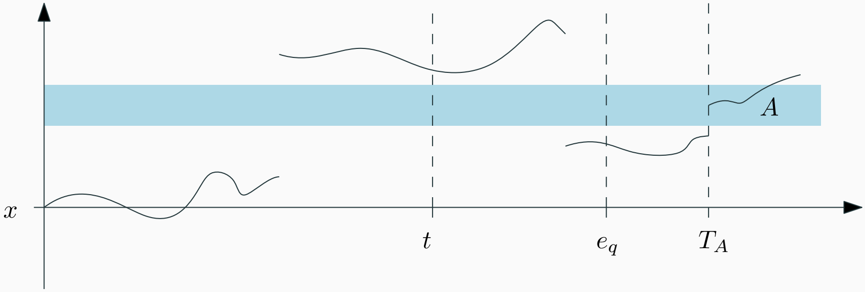
  - $\mathbb{P}^x_A(\Lambda) = \lim_{s\to\infty} \mathbb{P}^x(\Lambda \mid s + t < T_A)$
  - $\mathbb{P}^x_A(\Lambda) = \lim_{q\downarrow 0} \mathbb{P}^x(\Lambda; t < e_q \mid e_q < T_A)$,
    with $e_q \sim \mathrm{Exp}(q)$ independent of X



Transient Markov processes

If $h_A(x) = \mathbb{P}^x(T_A = \infty) > 0$ for $x \notin A$, then

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- $h_A(X_t)$ is a martingale for the process X killed on hitting A,

The measures $(\mathbb{P}^x_A)_{x \in \mathbb{R} \setminus A}$ are a Doob h-transform of $(\mathbb{P}^x)_{x \in \mathbb{R}}$.

Recurrent Markov processes

If $h^q(x) = \mathbb{P}^x(T_A > e_q)$ for $q > 0, x \notin A$, then

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- For recurrent processes, $h^q(x) \to 0$ as $q \downarrow 0$...
- ...but if $h^q(x) \sim f(q) h(x)$, we get back to an h-transform formula: asymptotic factorisation.

A non-exhaustive literature review

Lots of work in this area.

- Random walks...
- Lévy processes...
- Self-similar processes...

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conditioned to

- ...avoid a half-line (Bertoin, Chaumont, Doney)
- ...avoid a point (Kyprianou, Pantí, Rivero, Satitkanitul, W., Yano)
- ...remain in a cone (Denisov, Wachtel)
- ...remain in an interval (Lambert, Kyprianou, Rivero, Şengül)
- ...avoid an interval (next slide)

The question

A Lévy process X is a stochastic process with

- independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$ (for $s < t$)
- stationary increments: $X_t - X_s \stackrel{d}{=} X_{t-s}$.

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An analogous question was studied for arithmetic random walks by Kesten and Spitzer (1963), partially; random walks with finite variance by Vysotsky (2015) and for stable processes by Döring, Kyprianou, Weißmann (2018+).

Assumptions

Assume:

$$X \text{ has zero mean and finite variance,}$$
$$\text{and is not a compound Poisson process} \qquad (A)$$

$$X \text{ can jump upwards by more than } b - a \qquad (B)$$

$$X \text{ can jump downwards by more than } b - a \qquad (\hat{B})$$

Theorem
There exist functions h_+, h_- and constant C such that:

- *For $\Lambda \in \mathcal{F}_t$, $\lim_{q\downarrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]}, X_{e_q} > b)$ exists, and is a Doob h-transform of $\mathbb{P}^\cdot$ by h_+.*

An answer (and more)

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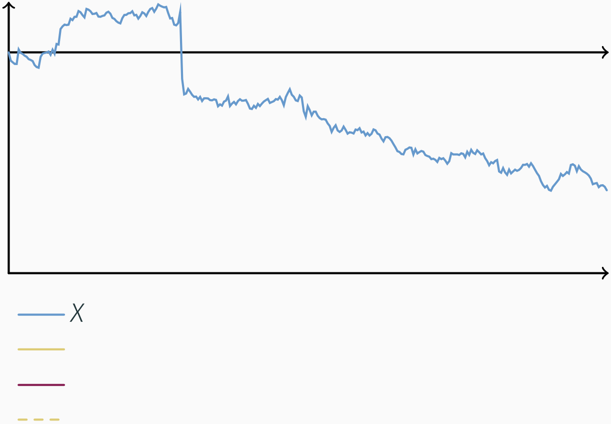
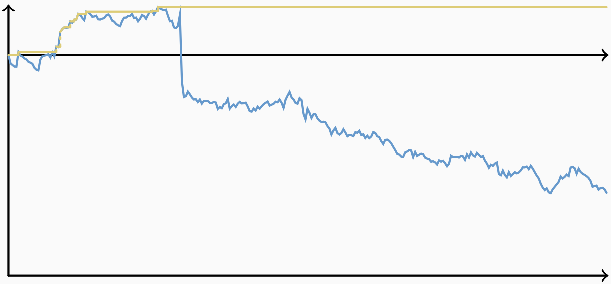
- *For $\Lambda \in \mathcal{F}_t$, $\lim_{q\downarrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]})$ exists, and is a Doob h-transform of $\mathbb{P}^\cdot$ by $h(x) := h_+(x) + Ch_-(x)$.*


X



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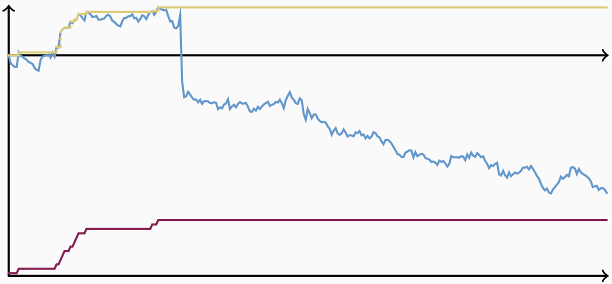
$\bar{X}_t = \sup_{s \le t} X_s$, the running supremum



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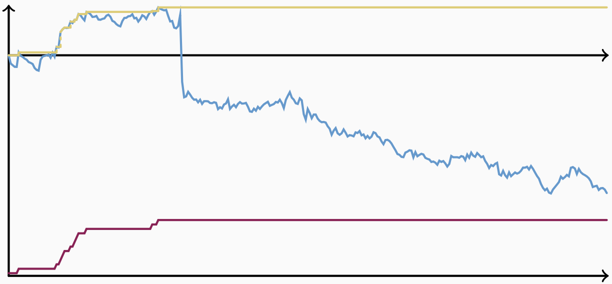
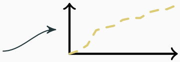
- L, the local time at the maximum





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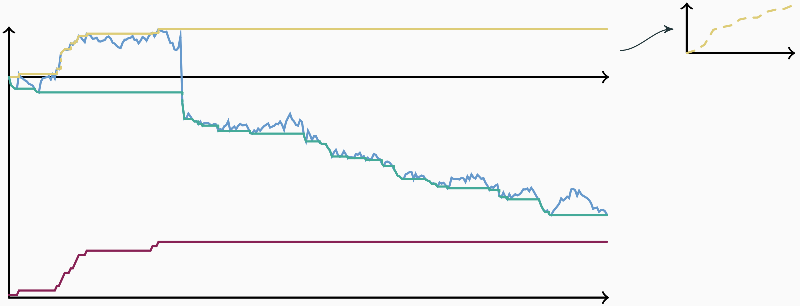
- $H_+(t) = X_{L_t^{-1}}$, the ladder height process: maxima 'stitched together'

The ladder height process is a Lévy process! Define its potential: $U_+(x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{H_+(t)\le x, t<L_\infty\}}\,\mathrm{d}t$.

Discussion of $h_\pm$: Wiener–Hopf factorisation 1/2



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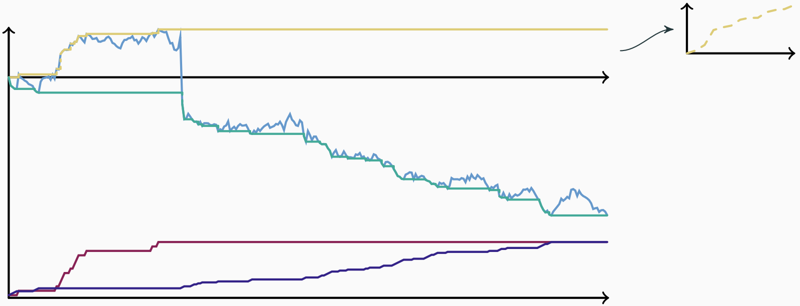

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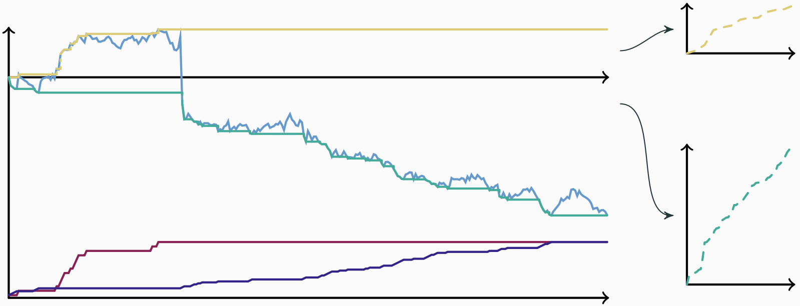

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The ladder height process is a Lévy process! Define its potential: $U_+(x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{H_+(t)\le x, t< L_\infty\}}\, dt$.

Likewise, the running infimum, the local time at the minimum, the downward ladder height and its potential U_-.

The definition of $h_\pm$ involves the overshoot measures of X.

- $\tau_0 = 0$,
- $\tau_k = \inf\{t > \tau_{k-1} : X_{t-} > b, X_t \leq b\}$
  $\wedge \inf\{t > \tau_{k-1} : X_{t-} < a, X_t \geq a\}$

Discussion of $h_\pm$

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- $\nu_k^x(\mathrm{d}y) = \mathbb{P}^x(X_{\tau_k} \in \mathrm{d}y, \tau_k \leq T_{[a,b]})$

Then:

$$h_+(x) = \begin{cases} \sum_{k=0}^{\infty} \int_b^{\infty} U_-(y-b)\,\nu_{2k}^x(\mathrm{d}y), & x > b, \\ \sum_{k=0}^{\infty} \int_b^{\infty} U_-(y-b)\nu_{2k+1}^x(\mathrm{d}y), & x < a. \end{cases}$$

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Key result:

Proposition

- $\mathbb{P}^x(e_q < T_{(-\infty,b]}) \sim \hat{\kappa}(q)U_-(x-b)$ *(well-known)*
- $\mathbb{P}^x(e_q < T_{[a,b]}, X_{e_q} > b) \sim \hat{\kappa}(q)h_+(x)$ *(new)*

Proof of $\mathbb{P}^x(e_q < T_{[a,b]}, X_{e_q} > b) \sim \hat{\kappa}(q) h_+(x)$

$$\mathbb{P}^x(e_q < T_{[a,b]}, X_{e_q} > b) = \sum_{k=0}^{\infty} \mathbb{P}^x(e_q \in [\tau_{2k}, \tau_{2k+1}), e_q < T_{[a,b]})$$

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&= \sum_{k=0}^{\infty} \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{2k+1} < T_{[a,b]}\}} \mathbb{P}^{X_{\tau_{2k}}}(e_q < T_{(-\infty,b]})\right]
\end{aligned}$$

Proof of $\mathbb{P}^x(e_q < T_{[a,b]}, X_{e_q} > b) \sim \hat{\kappa}(q)h_+(x)$ 2/2

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This is the asymptotic factorisation that we need for the h-transform to condition to avoid $[a, b]$ and end up above.

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Recall h is the supposed h-transform function for conditioning to avoid $[a, b]$.

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Asymptotic factorisation – leads to an h-transform for conditioning to avoid $[a, b]$

Other results

- Behaviour of the conditioned process – drifts to either $+\infty$ or $-\infty$ (but doesn't oscillate)

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Open problems

- Lévy process with infinite variance (stable known, but via ssMp)

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  - An idea: look at potential density of X (killed),

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- Lévy process conditioned to hit an interval continuously (just differentiate?)
- Analogues for self-similar processes (in an annulus?)

Further reading

L. Döring, A. R. Watson, P. Weißmann.
Lévy processes with finite variance conditioned to avoid an interval
arXiv:1807.08466 [math.PR]

An example

Let

$$X_t = \sqrt{2}B_t + \sum_{i=1}^{N_t} Y_i,$$

with B a standard Brownian motion, N a Poisson process with rate 1 and (Y_i) iid with pdf

$$f_Y(y) = \frac{1}{2}\eta e^{-\eta y}\mathbb{1}_{\{y>0\}} + \frac{1}{2}\eta e^{-\eta(-y)}\mathbb{1}_{\{y<0\}}.$$

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By explicitly finding the iterated overshoot distributions ν_k^x, we can find

$$h(x) = \begin{cases} \frac{\eta}{\beta}(x-b) + \left(\frac{\beta-\eta}{\beta^2} + \frac{2c}{\beta(1-c)}\right)(1-e^{-\beta(x-b)}), & x > b, \\ \frac{\eta}{\beta}(a-x) + \left(\frac{\beta-\eta}{\beta^2} + \frac{2c}{\beta(1-c)}\right)(1-e^{-\beta(a-x)}), & x < a, \end{cases}$$

with $\beta = \sqrt{\eta^2+1}$ and $c = \frac{\beta-\eta}{\beta+\eta}e^{-\eta(b-a)}$.

Behaviour of the conditioned process

Write $\mathbb{P}^{\cdot}_{+}$ for the law of the process conditioned to avoid $[a, b]$ and end up above b, and $\mathbb{P}^{\cdot}_{[a,b]}$ for that of the process just conditioned to avoid $[a, b]$.

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Trajectories under $\mathbb{P}^{\cdot}_{[a,b]}$ *do not oscillate.*

Transient processes

Drop assumptions about finite variance and zero mean.

If $X_t \to \infty$ a.s., and $\mathbb{E}[H_+(1)], \mathbb{E}[H_-(1)] < \infty$,

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- $h_-(x)$ can still be used to condition X to avoid $[a, b]$ and end up below it – but we end up with a killed Markov process. Analogous to increasing Lévy process conditioned to stay below a level (Kyprianou et al. 2017).