Models for fragmentation with growth

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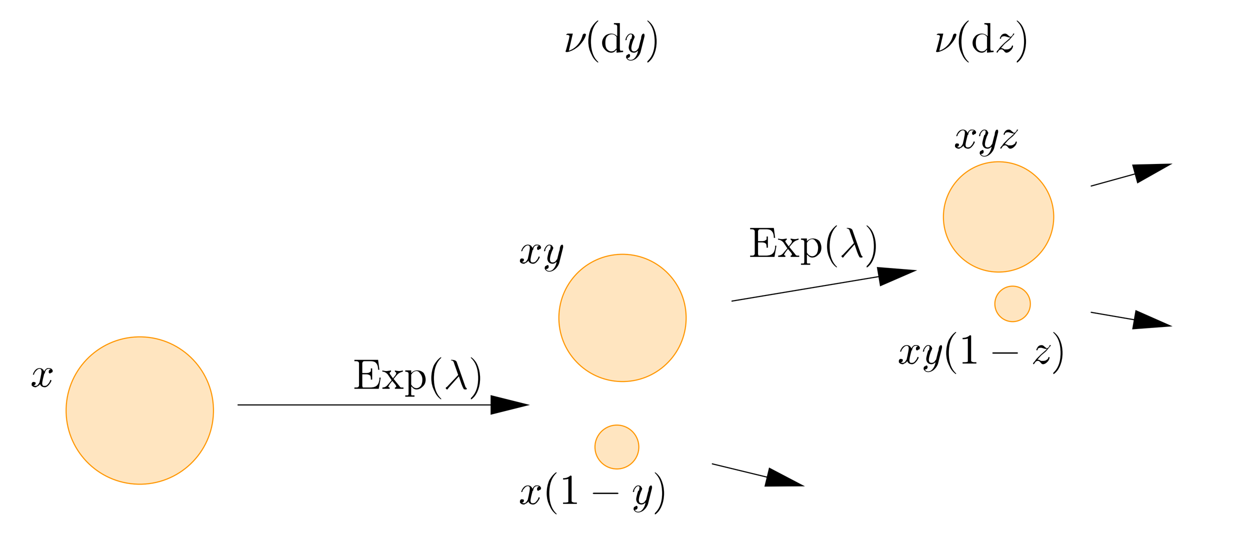
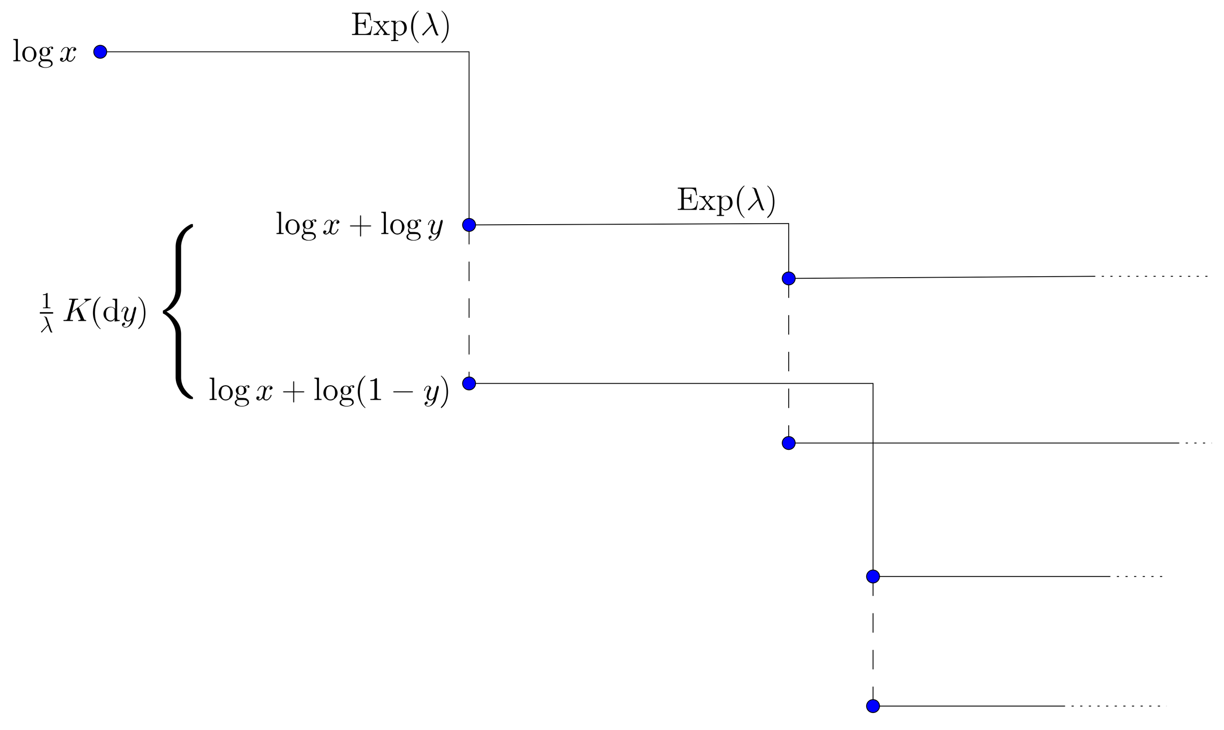
A simple model of fragmentation



We begin with a single particle of size x. It waits for an exponential time of rate λ, and then splits into two daughter particles with sizes xy and $x(1-y)$, where y is chosen from some probability distribution ν with support in $[\frac{1}{2}, 1]$. These act independently of each other and of the past, each waiting an exponential time and fragmenting according to the same rule.

Evolution of the particle sizes

To study this model, let $\mathcal{Z}(t) = \sum_{\text{particles } u} \delta_{\log(\text{size of } u)} \mathbb{1}_{\{u \text{ present at time } t\}}$. We plot the evolution of these particle sizes; let $K(\mathrm{d}y) = \lambda\nu(\mathrm{d}y)$.

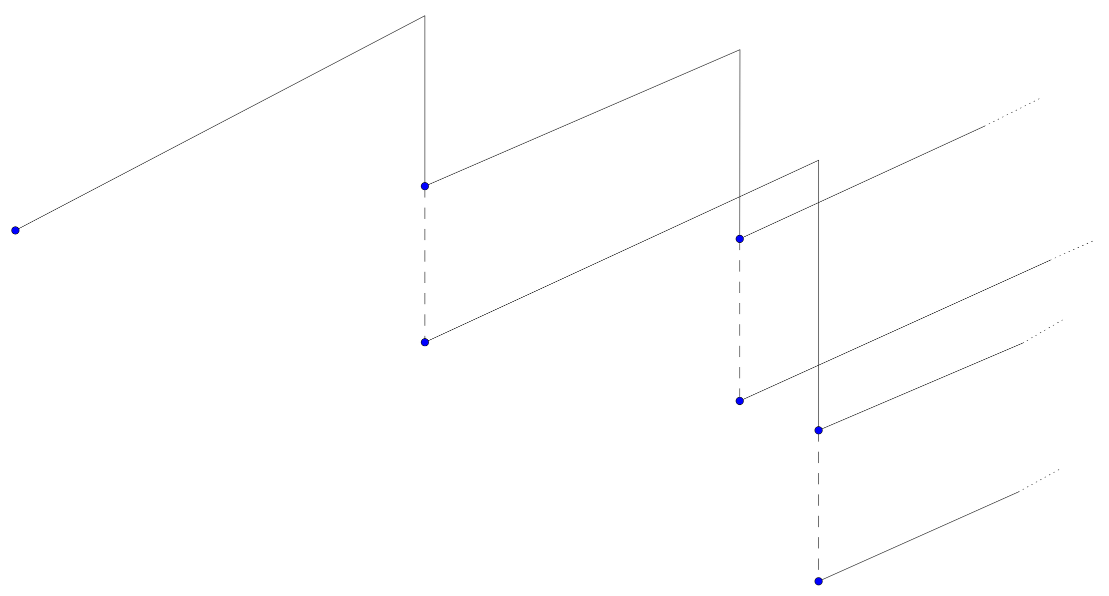


This graphic has a useful interpretation. By *identifying each particle with its largest child*, what we see above is a particle whose size remains constant for some exponential time, and then jumps by $\log y$. At the same time, a new particle is immigrated below at position $\log(1-y)$ relative to the pre-jump position. Each particle then independently runs the same process.

We see that $\mathcal{Z}$ is precisely a compound Poisson process, with Lévy measure $\Pi = K \circ \log^{-1}$, together with *immigration at its jumps*.

A more elaborate model

This suggests a generalisation. Let X be a Lévy process with only downward jumps, and denote its Lévy measure by Π. A more general model of fragmentation, $\mathcal{Z}$, can be regarded as follows: we start with a single particle, which runs the process X; and at every jump of X having size $\log y$, we immigrate an independent copy of $\mathcal{Z}$ at relative position $\log(1-y)$.



One can show that such a process does exist, and is locally finite (each compact set contains only finitely many particles.)

Let $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1} = \sum_{\text{particles } u} \delta_{\text{size of } u} \mathbb{1}_{\{u \text{ present at time } t\}}$. This is the *compensated fragmentation process*.

The measure $K := \Pi \circ \exp^{-1}$ is called the *dislocation measure*, and it satisfies $\int (1-y)^2 K(\mathrm{d}y) < \infty$.

A simple fragmentation equation

If K is a finite measure, the following equation describes the mean behaviour of the 'simple model', left.

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x \mapsto \int_{[\frac{1}{2},1)} \left\{ f(xy) + f(x(1-y)) - f(x) \right\} K(\mathrm{d}y) \right\rangle, \qquad f \in C_c^\infty(0,\infty),$$
$$\mu_0 = \delta_1$$

This equation still makes sense when K is infinite but satisfies $\int (1-y)\, K(\mathrm{d}y) < \infty$, and has been well-studied in the context of exchangeable fragmentations.

The growth-fragmentation equation

We study the following generalisation, which is well-defined for all K such that $\int (1-y)^2 K(\mathrm{d}y) < \infty$.

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x \mapsto axf'(x) + \int_{[\frac{1}{2},1)} \left\{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \right\} K(\mathrm{d}y) \right\rangle \qquad \text{(0-GF)}$$
$$\mu_0 = \delta_1$$

There are two 'growth' terms now present. The first is the term involving a; this represents a deterministic growth (or erosion, if $a < 0$) of all particles simultaneously. The second is the extra term in the integral; this represents a sort of compensation for the infinite rate of 'small fragmentations', where the larger child is close in size to the parent, and the smaller child has size close to zero; it is required for the weakening of the integral condition on K.

Theorem 1

Let

$$\kappa(q) = aq + \int_{[\frac{1}{2},1)} \left\{ y^q + (1-y)^q - 1 + q(1-y) \right\} K(\mathrm{d}y).$$

The unique solution of (0-GF) is given by

$$\langle \mu_t, f \rangle = \mathbb{E}_{\delta_1}\left[\sum_u f(\mathcal{Y}_u(t)) \right] = e^{t\kappa(\omega)} \mathbb{E}_1[\xi(t)^{-\omega} f(\xi(t))],$$

where $\mathcal{Y}$ is the compensated fragmentation process described on the left, and ξ is the exponential of a Lévy process with only downward jumps, having Laplace exponent $\kappa(\cdot + \omega) - \kappa(\omega)$; the paramter $\omega \in \operatorname{dom} \kappa$ can be freely chosen.

This theorem corresponds exactly with what was already shown by Haas for the case $\int (1-y)\, K(\mathrm{d}y) < \infty$, and the methods are very similar. However, there are some surprises in store for the self-similar equation, which we now describe.

The self-similar growth-fragmentation equation

Let $\alpha \in \mathbb{R}$. The self-similar version of (0-GF) is as follows:

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x \mapsto x^\alpha \left[axf'(x) + \int_{[\frac{1}{2},1)} \left\{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \right\} K(\mathrm{d}y) \right] \right\rangle. \qquad (\alpha\text{-GF})$$

The presence of the term x^α modifies the overall speed of evolution of a particle of size x. Thus, if $\alpha < 0$, then small particles (with size close to zero) grow and fragment rapidly, while larger particles have their behaviour slowed down; the opposite holds when $\alpha > 0$.

Theorem 2

We assume that certain *Malthusian hypotheses* are satisfied. This means that there exists some $\omega \in \operatorname{dom} \kappa$ such that $\kappa(\omega) = 0$, and either $\alpha < 0$ and $\kappa'(\omega) > 0$, or else $\alpha > 0$ and $\kappa'(\omega) < 0$. Under these conditions, we have the following results.

- There exists a solution (μ_t) to (α-GF), such that $\langle \mu_t, x \mapsto x^\omega \rangle \equiv 1$ and $\mu_0 = \delta_1$.
  - It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\omega} f(X_t)]$.
- There exists *another* solution (γ_t), such that $\langle \gamma_t, x \mapsto x^\omega \rangle \equiv 1$ for $t > 0$ but $\gamma_0 = 0$.
  - If $\alpha < 0$, it is given by $\langle \gamma_t, f \rangle = \mathbb{E}_0[X(t)^{-\omega} f(X_t)]$.
  - If $\alpha > 0$, it is given by $\langle \gamma_t, f \rangle = \mathbb{E}_{+\infty}[X(t)^{-\omega} f(X_t)]$.

Here, X is a positive, self-similar Markov process with index of self-similarity $-\alpha$, which is driven (through the Lamperti transform) by a Lévy process with Laplace exponent $\kappa(\cdot + \omega)$; the paramter ω is the Malthusian exponent appearing above.

This indicates that the solutions can exhibit *spontaneous generation of mass*, and precludes uniqueness.

Open questions

Some possible topics: existence of a minimal solution — study of the nonlinear semigroup and KPP equation — process variant of 'starting from zero mass' — study of biased mass functions '$\langle \mu_t, x \mapsto x^q \rangle$'...

J. Bertoin and A. R. Watson (2015) Probabilistic aspects of critical growth-fragmentation equations. Preprint, arXiv:1506.09187 [math.PR]. To appear in *Adv. Appl. Probab.*