APPLICATIONS OF LÉVY PROCESSES IN INSURANCE WITH OMEGA KILLING

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We consider two optimal stopping problems driven by Lévy processes with Omega killing, which can be understood as the so-called bankruptcy event for an insurance company. The main question is to decide when to stop the observation before bankruptcy happens in order to obtain the best reward. We work with stable processes and Cramér–Lundberg processes respectively under different choices of the gain (payoff) functions and time-dependent killing processes.

The first problem is the optimal stopping for an Omega-killed stable process. The gain function is similar to an American call option. We show that under certain conditions, the solution to the problem driven by a stable process can be transformed from the one driven by a special type of Lévy process. To see this, we first construct, from the original stable process, a positive self-similar Markov process (pssMp) and then, we generate, from this pssMp, a Lévy process via Lamperti transform, which is verified to be of the so-called double hypergeometric class. After solving the related optimal stopping problem under such double hypergeometric Lévy process, we show that these results are also valid and can be interpreted for the original problem with the stable process. We also share a remark on a similar optimal stopping problem while the gain function is of American put option style and a Markov additive process is considered with the help of Lamperti-Kiu transform.

In the second project, we study the optimal stopping problem driven by a Cramér-Lundberg process with piecewise constant killing intensity. The payoff function has a constant penalty p for negative values and is not continuous at zero, which makes it harder to apply the change of measure formula and to follow the classic verification steps of solving optimal stopping problems. Under some mild conditions w.r.t. penalty p in terms of the parameters for Cramér-Lundberg processes, the solutions are fully characterised where the optimal up-crossing thresholds are explicitly defined. The proofs consist of massive calculations based on existing explicit expressions for fluctuation functions of Cramér-Lundberg processes. By introducing a number of lemmas, we solve the optimal stopping problem. We also give numerical examples to illustrate our result and make discussions for future direction.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

This thesis deals with two optimal stopping problems with Omega killing, which can be interpreted as a non-immediate ruin event commonly referred to as bankruptcy event for an insurance company. In such problems, the insurer observes a random evolution of company wealth whose future cannot be predicted and the objective is to find stopping times so that the values of some given functionals can be optimised, or otherwise, bankruptcy occurs and the whole process is terminated. Lévy processes are often used to model the wealth of an insurance company or the capital flow for a portfolio of insurance products. Thanks to the known fluctuation identities of Lévy processes at first passage over a fixed level, analytically tractable formulas can be achieved when solving the related optimal stopping problems.

In our study, we consider a Lévy process X defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Also, we define an Omega clock with some non-negative density function ω as

$$\chi_t := \int_0^t \omega(X_s)\mathrm{d}s.$$

Let us extend the probability space above to $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_{>0})$ containing a random variable e, which is independent of X and exponentially distributed with parameter 1. Then, the time of killing denoted by T, is defined as

$$T := \chi^{-1}(\mathrm{e}) := \inf\{t \geq 0 \,|\, \chi_t \geq \mathrm{e}\}. \tag{1.1}$$

We are interested in the following optimal stopping problem:

$$v(x) = \sup_{\tau} \mathbb{E}_x \left[L_\tau\right], \tag{1.2}$$

where the supremum is taken over $\mathbb{F}-$adapted stopping times, $\mathbb{E}_x$ is the expectation under $\mathbb{P}_x$, which is the law of X given that $X_0 = x \in \mathbb{R}$ and

$$L_t := \mathbb{1}_{\{t<T\}} \left(e^{-rt} g(X_t)\right) + \mathbb{1}_{\{t>T\}} \left(e^{-rT} g\left(X_T\right)\right), \quad \text{for all } t \geq 0,$$

with $r \geq 0$ and g is the so-called *gain* or *payoff function*. For a discussion of why it is sufficient to consider $\mathbb{F}-$adapted stopping times only in (1.2), see [27]. So (1.2) is essentially about finding the optimal stopping strategy such that the insurance company can attain optimal expected wealth or the insurer can get the optimal expected value of the portfolio before the Omega clock rings, or otherwise, the insurer is left with a payoff (or fine) equal to $e^{-rT} g(X_T)$.

The study of optimal stopping problems with state-dependent killing has been popular in recent years. [21] and [9] consider the case of diffusion processes and [25] derive a Feynman–Kac formula which characterizes conditional expectations of functionals of killed time-inhomogeneous Lévy processes. Later, exit problems for different one-sided Lévy processes which are exponentially killed with a state-dependent killing intensity have been solved in [36] and [20]. In the case where the Lévy processes are (reflected) spectrally negative ones, the killing intensity depends on the present level of the process while in the case where Markov additive processes are studied, the killing intensity is bivariate, which depends on the present states of both the process and the environment. Respective resolvents are also analyzed in these two papers.

Applications of optimal stopping problem with state-dependent killing can be widely found in insurance and finance. In Insurance, Omega killing is often related to the event named “bankruptcy”. The concept of bankruptcy was first introduced in [1]. Later, [24] and [2] derive analytical results for the bankruptcy probability under the Brownian motion model and the compound Poisson risk model respectively. [26] considers the Cramér-Lundberg process and shows that this probability enjoys a linear relation with the classical ruin probability. In financial problem study, Omega killing can be for instance regarded as the default risk. [27] enriches the optimal stopping problem for a convertible bond by introducing such default risk. They look at different types of default intensities and present the analytical formulae for the no-arbitrage price of the convertible bonds by solving the free boundary problems respectively. The study of random maturity American options was introduced by [16], with the maturity date randomized and determined by the waiting time to a prespecified number of jumps of a standard Poisson process, which is independent of the underlying asset price process. [40] includes the Omega clock in the optimal stopping problem for an American call option so that the problem is under a random time-horizon setting. They show that for different choices of parameters in the Omega clock, optimal strategies differ as well. They obtain a complete characterization of optimal exercising thresholds in different scenarios, where it is either an up-crossing strategy or a two-sided exit one.

An optimal stopping problem similar to (1.2) was considered by [40] where they consider spectrally negative processes and piecewise constant omega intensity functions. Different from their work, we consider stable processes and Cramér-Lundberg processes, and the Omega clock is of a new type for the stable process problem. Usually, optimal stopping problems are often linked with one- and two-sided exit problems and with the assistance of a number of known fluctuation identities for Lévy processes, e.g., its overshoot above or below a certain level and the Laplace transform of the first-passage time, this type of exit problems have been well studied. See for example, [23], [4], [19], [37] deals with a general Lévy process. [17], [18], [5], [6] handles the case of spectrally negative processes while [46] considers reflected spectrally negative Lévy processes. Also, [22] computes the joint distribution of the exit time and value of a spectrally positive Lévy process for certain choice of interval and [11] derives an explicit formula for the Laplace transform of the distribution of the first time that a completely asymmetric stable process exits a finite interval. However, in our case, due to the existence of state-dependent killing, one- and two-sided exit strategies may not be optimal for some set of model parameters. Further, in comparison with most existing studies where problems are under exponential Lévy models, we look at an identity-type setting, which makes the analysis and computation more complicated.

Based on this framework, we study two problems with different choices of ω and gain function under stable process and Cramér Lundberg process respectively. The rest of this thesis is organised as follows.

In Chapter 2, we share a series of definitions and results for Lévy processes. Then, we give a brief introduction on two possible ways of solving an optimal stopping, i.e., the so-called martingale approach and Markovian approach. We also present crucial results of the celebrated Itô formula and its extensions, which will be used in later parts.

In Chapter 3, we deal with the optimal stopping for Omega-killed stable process. In this project, we show that an up-crossing strategy can be optimal under certain conditions and the existence of an optimal up-crossing stopping time depends on the shape of the gain function. We do not solve the problem under stable process in (1.2) directly. Instead, we construct, from the original stable process, a positive self-similar Markov process and then apply the Lamperti transform to generate a Lévy process, which belongs to the so-called double hypergeometric class. We solve the optimal stopping problem under the generated double hypergeometric Lévy process and prove that the solutions can be transformed to our problem under a stable process. We also make some discussions on a similar optimal stopping problem while the payoff function is of the American put option style.

In Chapter 4, we tackle the optimal stopping problem for Omega-killed Cramér Lundberg process. In this project, ω is of piecewise constant type and our gain function is a simple identity function for non-negative values with a penalty $p < 0$ for negative ones. Since g is not continuous at zero, it makes the whole problem more interesting as well as challenging. We show that under certain conditions on the parameters of Cramér Lundberg process as well as p, an up-crossing strategy is optimal for (1.2) with explicit solutions achieved. We also give numerical examples to show how value function behaves with respect to different choice of parameters. And finally, an outline of the results is given, based on which, we discuss the difficulties of solving certain scenarios and propose potential future directions.

Chapter 2

Preliminaries

In this chapter, we shall give some essential definitions and important results for Lévy processes and optimal stopping problems, which will be used in Chapter 3 and 4.

2.1 Lévy processes

In this section, we present a brief introduction to Lévy processes and its fluctuation theory, especially for the so-called Cramér–Lundberg risk process and stable process. All these results can be found in [10] and [30].

2.1.1 General Lévy process

Lévy process is named in honor of the outstanding work of Paul Lévy, a great French mathematician, who played an important role in bringing together an understanding and characterisation of processes with stationary independent increments. The definition of a Lévy process is as follows, see e.g., [30, Definition 1.1].

Definition 2.1 (Lévy process)**.** A process $X = (X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a Lévy process if it satisfies the following conditions:

1) The paths of X are $\mathbb{P}$-almost surely right-continuous with left limits;

2) $\mathbb{P}(X_0 = 0) = 1$;

3) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s};

4) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $(X_u)_{u\leq s}$. $\Diamond$

Throughout the study of Lévy processes, an instrumental role belongs to the notion of infinitely divisible distributions. It is known that any Lévy process can be associated with an infinitely divisible distribution, that is, for all $t \geq 0$,

$$\mathbb{E}\left[e^{i\theta X_t}\right] = e^{-t\Psi(\theta))}$$

where $\Psi(\theta)$ is called the characteristic exponent and defined by the following theorem (see [30, Theorem 1.6]).

Theorem 2.2 (Lévy-Khintchine formula)**.** *Consider $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. From this triple (a, σ^2, Π), define for each $\theta \in \mathbb{R}$,*

$$\Psi(\theta) := ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(\mathrm{d}x).$$

Then, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which a Lévy process is defined with characteristic exponent Ψ. On the other hand, every Lévy process has a unique characteristic exponent of the above form.

Note that for a killed Lévy process, an extra $q > 0$ which represents the killing rate will be added to Ψ. The objective Lévy process X is sent to the cemetery state ∂ at an exponential random time with rate q, which is otherwise independent of the path of X, and remains at ∂ forever.

According to the Lévy-Khintchine formula, any characteristic exponent Ψ belonging to an infinitely divisible distribution can be written in the form for all $\theta \in \mathbb{R}$

$$\Psi(\theta) = (\Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)})(\theta)$$

where

- $\Psi^{(1)}(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2$;
- $\Psi^{(2)}(\theta) = \Pi(\mathbb{R} \setminus (-1,1)) \int_{|x|\geq 1}(1 - e^{i\theta x})\frac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\setminus(-1,1))}$;
- $\Psi^{(3)}(\theta) = \int_{0<|x|<1}(1 - e^{i\theta x} + i\theta x)\Pi(\mathrm{d}x)$.

2.1.2 Spectrally Negative Lévy process and scale functions

From the decomposition of Lévy processes, we can see that a general Lévy process allows its jumps to be in two directions. Those which possess jumps in just one direction turn out to provide an obvious advantage for many calculations and analysis. Thus, in this subsection, we would like to offer some useful facts and fluctuation identities about spectrally negative processes (SNLP), i.e., Lévy processes with only downward jumps, and their scale functions. Note that SNLP does not include the negative of a subordinator. In particular, we present useful results of Cramér Lundberg process, a special type of SNLP, which will be used in Chapter 4.

When studying SNLPs, we usually make use of the so-called *Laplace exponent*, denoted by ψ, which fulfills the relation

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}\left[e^{\lambda X_t}\right] = -\Psi(-i\lambda), \quad \lambda \geq 0, \tag{2.1}$$

instead of the Lévy-Khintchine characteristic exponent. It is known that $\psi(\lambda)$ is finite if and only if $\int_{|x|\geq 1} e^{\lambda x}\Pi(\mathrm{d}x) < \infty$.

The following results Lemma 2.3 and Remark 2.4 from [30, Section 8.2] describe the path variation property and the law of change of measure for spectrally negative Lévy processes.

Lemma 2.3 (Path Variation of SNLP)**.** *Given the triple* (a, σ^2, Π)*, where the Lévy measure* Π *is concentrated on* $(-\infty, 0)$*, we have the Laplace exponent*

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \boldsymbol{1}_{(x>-1)})\Pi(\mathrm{d}x), \tag{2.2}$$

where $a \in \mathbb{R}$ *and* σ *is the Gaussian coefficient. Further, if* X *has paths of bounded variation, we can write it as*

$$\psi(\lambda) = \delta\lambda - \int_{(-\infty,0)} (1 - e^{\lambda x})\Pi(\mathrm{d}x),$$

where

$$\delta = -a - \int_{(-1,0)} x\Pi(\mathrm{d}x) \tag{2.3}$$

is strictly positive.

Note that ψ is a strictly convex function which is zero at zero and tends to infinity at infinity. For later reference we also introduce the function $\Phi : [0,\infty) \to [0,\infty)$ as the right inverse of ψ on $[0,\infty)$ such that for all $q \geq 0$

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Note that when $\psi'(0+) \geq 0$, it follows that $\theta = 0$ is the unique solution to $\psi(\theta) = 0$ on $[0,\infty)$. Otherwise, when $\psi'(0+) < 0$, there exist two solutions: 0 and $\Phi(0) > 0$.

Remark 2.4 (Exponential change of measure)**.** Consider a given SNLP X. For each $c \geq 0$, the process $\left(e^{cX_t - \psi(c)t}\right)_{t\geq 0}$ is a martingale and we can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{c(X_t - x) - \psi(c)t},\ t \geq 0,$$

under which X remains to be a SNLP with its Laplace exponent given by

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c), \ \text{for}\, \theta \geq -c. \qquad \triangle$$

Scale functions are firmly embedded in the study of spectrally negative Lévy processes and their applications. It is a fundamental aspect in most of known identities concerning exiting form a half-line and an interval. Below in Definition 2.5 we give the definition of the scale functions and in Theorem 2.6 one immediate example of a boundary crossing identity with the application of scale functions is presented. The proofs can be found in [30, Theorem 8.1].

Definition 2.5 ($q-$scale function)**.** For a given SNLP X, with Laplace exponent ψ, we define two families of functions indexed by $q \geq 0$, $W^{(q)} : \mathbb{R} \to [0,\infty)$ and $Z^{(q)} : \mathbb{R} \to [1,\infty)$, as follows. For each given $q \geq 0$, we have $W^{(q)}(x) = 0$ when $x < 0$ and otherwise on $[0,\infty)$, $W^{(q)}$ is a right continuous function and uniquely characterised by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\Psi(\beta) - q} \quad \text{for}\, \beta > \Phi(q).$$

Also, for $x \in \mathbb{R}$, we have

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y.$$

For convenience we shall always denote $W := W^{(0)}$ and $Z := Z^{(0)}$ with the observation that Z is equal to 1 for $q = 0$. Typically we shall refer to the functions $W^{(q)}$ and $Z^{(q)}$ as $q-$scale functions while W and Z are usually referred to as just the scale functions. $\diamondsuit$

The behaviour of the scale functions at 0 is of ultimate interest. As shown in [29, Lemma 3.1], $W^{(q)}(0) = 0$ if and only if X has paths of unbounded variation while $W^{(q)}(0) = 1/\delta$ when X has bounded variation.

Theorem 2.6 shows how important scale functions are for studying exit problems. In the theorem, we write $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote the law of X and its expectation given that $X_0 = x$, respectively. When $x = 0$, we simply write $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{E}_0 = \mathbb{E}$.

Theorem 2.6 (One- and two-sided exit formulae)**.** *Define*

$$\tau_a^+ = \inf\{t \geq 0 : X_t > a\} \quad \textit{and} \quad \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

(i) For any $x \in \mathbb{R}$ and $q \geq 0$,

$$\mathbb{E}\left[e^{-q\tau_x^+}\mathbb{1}_{\{\tau_x^+ < \infty\}}\right] = e^{-\Phi(q)x}, \tag{2.4}$$

and

$$\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^- < \infty\}}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \tag{2.5}$$

where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$.

(ii) For any $x \in \mathbb{R}$ and $q \geq 0$,

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbb{1}_{\{\tau_0^- > \tau_a^+\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.6}$$

and

$$\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^- < \tau_a^+\}}\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{2.7}$$

Example 2.7 (Cramér-Lundberg process)**.** The surplus for a homogeneous portfolio of insurance products held by an insurance company can be modelled by the so-called Cramér-Lundberg process $U = (U_t)_{t\geq 0}$ with

$$U_t = x + \delta t - \sum_{i=1}^{N_t} Y_i,$$

where x is the initial capital, $\delta > 0$ denotes the fixed premium rate, $(N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$ describing the number of claims until time t and $(Y_i)_{i\in\mathbb{N}}$, representing the claim sizes, is a sequence of positive, i.i.d. random variables with common law F, which is also independent of N. Denote its law by $\mathbb{P}_x$. We can see that such process U is nothing more than a compound Poisson process with drift of rate δ and is a special case of a SNLP. We see that the Lévy measure of U is $\Pi(\mathrm{d}x) = \lambda F(-\mathrm{d}x)$. If Y_i's follow an exponential distribution with parameter $\rho > 0$, then the Laplace exponent of U is

$$\psi(z) = \delta z - \frac{\lambda z}{\rho + z}, \quad z \geq -\rho,$$

and the $q-$scale function can be explicitly expressed as

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta x}}{\psi'(-\zeta)}, \quad x \geq 0,$$

where

$$\zeta = \frac{1}{2\delta}\left(\sqrt{(\lambda + q - \delta\rho)^2 + 4\delta q\rho} - (\lambda + q - \delta\rho)\right),$$

$$\Phi(q) = \frac{1}{2\delta}\left(\sqrt{(\lambda + q - \delta\rho)^2 + 4\delta q\rho} + (\lambda + q - \delta\rho)\right)$$

with $\Phi(q)$ defined in (2.3) and $-\zeta$ being the two solutions to the equation $\psi(z) = q$. See e.g. [29, Example 1.3] for details. $\lhd$

2.1.3 Stable process, positive self-similar Markov process and Lamperti transform

In this subsection, we shall briefly review definitions and results of stable process and the so-called positive self-similar Markov process (pssMp). We will also introduce a space-time transformation, namely the Lamperti transform from [35], through which the bijection between the class of pssMp and the class of killed Lévy processes can be expressed.

Definition 2.8 (Stable process)**.** A Lévy process $X = (X_t)_{t\geq 0}$ is called a *stable process* if it enjoys the *scaling property*, namely, that when started from $X_0 = 0$, the process $(cX_{tc^{-\alpha}})_{t\geq 0}$ has the same law as X for any $c > 0$. $\diamondsuit$

The parameter $\alpha \in (0, 2]$ is called the *index* of X and the case $\alpha = 2$ corresponds to Brownian motion, which we exclude. Stable processes can be described in terms of their Lévy-Khintchine formula as follows: [30, §§1.2.6 and 6.5.3]: $\sigma = 0$ and the Lévy measure Π is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{\{x<0\}}, \quad x \in \mathbb{R},$$

where $c_+, c_- \geq 0$ and $c_+ = c_-$ when $\alpha = 1$. It holds that a in Theorem 2.2 is equal to $(c_+ - c_-)/(\alpha - 1)$ when $\alpha \neq 1$ and we specify that $a = 0$ when $\alpha = 1$, which ensures that the symmetric Cauchy process is the only $1-$stable process we consider.

When $\alpha \in (0, 1) \cup (1, 2)$, we shall always parameterise the stable process such that

$$c_+ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} \quad \text{and} \quad c_- = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})}.$$

where the parameter ρ is called the *positivity parameter* of X which satisfies $\rho = \mathbb{P}_0(X_t \geq 0)$, and $\Gamma(\cdot)$ is the gamma function. For convenience, we write $\hat{\rho} = 1 - \rho$. See e.g. [28] and [45] for more details.

Introduced by [35], a $[0, \infty)$-valued strong Markov process Y is said to be a positive self-similar Markov process (pssMp) if there exists a constant $\alpha > 0$ such that, for any $y, c > 0$,

$$\text{the law of } (cY_{c^{-\alpha}t})_{t\geq 0} \text{ under } \mathbb{P}_y \text{ is equal to the law of } (Y_t)_{t\geq 0} \text{ under } \mathbb{P}_{cy},$$

where $\mathbb{P}_y$ is the law of Y when starting from y.

The work of Lamperti [35] provides a bijection between the class of Lévy processes killed at an independent and exponentially distributed time and the class of positive self-similar Markov processes, which can be expressed through a straightforward space-time transformation; [30, §13] offers a textbook treatment. For $t \geq 0$, define

$$S(t) = \int_0^t (Y_u)^{-\alpha} \, \mathrm{d}u,$$

which is continuous and strictly increasing until Y hits zero. Then the process $\xi = (\xi_t)_{t\geq 0}$ with

$$\xi_s = \log Y_{S^{-1}(t)}, \tag{2.8}$$

is a Lévy process. Such real-valued process ξ is often referred to as a *Lamperti-Lévy process*.

We can equivalently define a process T in terms of ξ as

$$T(s) = \int_0^s \exp(\alpha \xi_u) \mathrm{d}u, \quad s \geq 0.$$

Then, for any $s \geq 0$,

$$Y_s = \exp(\xi_{T^{-1}(s)}),$$

which shows that the Lamperti transform is indeed a bijection between the two classes of processes.

2.2 Optimal Stopping and Itô's formula

This section is based on [39] [38] and [43]. We will first exhibit basic results of general theory of optimal stopping in continuous time, and then review the central results of Itô's formula and some extensions of this celebrated formula for later use.

2.2.1 Optimal stopping problems

We will briefly introduce two approaches to solve an optimal stopping problem with one referred to [38, Corollary 2.9] and the other cited from [43, Theorem 3]. In the following part of this subsection, we consider a Markov process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with values in a state space $(E, \mathscr{B})$. It is assumed that the paths of X are right-continuous and left-continuous over stopping times and the filtration $(\mathscr{F}_t)_{t\geq 0}$ is right-continuous. Also, we shall use $\mathcal{T}$ to denote the set containing stopping times that are finite a.s. and $\overline{\mathcal{T}}$ to denote the class of all stopping times. Let us first take at the result from [38] where $E = \mathbb{R}^d$ for $d \geq 1$.

Given a measurable function $g : E \to \mathbb{R}$ satisfying the following condition

$$\mathbb{E}\left[\sup_{t\geq 0} |g(X_t)|\right] < \infty. \tag{2.9}$$

Then, we define the optimal stopping problem

$$v(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[g(X_\tau)\right] \quad \text{for } x \in E, \tag{2.10}$$

where $\mathbb{E}_x$ is the expectation w.r.t the probability measure $\mathbb{P}_x$ and the supremum is taken over all $\mathbb{F}-$adapted finite stopping times τ. Due to the Markov property of X, the state space naturally splits into the so-called *continuation set* C and the *stopping set* $D = E\backslash C$. Thus, to solve the optimal stopping problem (2.10) requires determining the sets C and D and finding an (explicit) expression for v. A general definition for C and D based on the fact that $v \geq g$ is as follows:

$$C = \{x \in E : v(x) > g(x)\}$$

and

$$D = \{x \in E : v(x) = g(x)\}.$$

Denote the first entry time τ_D of X into D as

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}.$$

If v is lower semicontinuous and g is upper semicontinuous, then it follows from [38, Corollary 2.9], τ_D is optimal in (2.10). Further by Remark 2.10 in the same book, if X_t has a limit X_∞ as $t \to \infty$, then $\mathcal{T}$ in (2.10) can be replaced by $\overline{\mathcal{T}}$ and τ_D remains to be optimal, i.e. in this case, we have

$$v(x) = \sup_{\tau\in\overline{\mathcal{T}}} \mathbb{E}_x\left[g(X_\tau)\right] = \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right], \quad \text{for } x \in E.$$

In the work of [43], E is not required to be $\mathbb{R}^d$ but simply be a semicompact set.[1] Given a $\mathscr{C}_0$-continuous[2] function g satisfying (2.9). Then, if the function g is upper semicontinuous,

[1] A semicompact is a locally separated space with a countable basis.

[2] The function $g(x)$ is referred to as $\mathscr{C}_0-$continuous if $\mathbb{P}_x\left(\liminf_{t\downarrow 0} g(X_t) \geq g(x)\right) = 1$ and $\mathbb{P}_x\left(\limsup_{t\downarrow 0} g(X_t) \leq g(x)\right) = 1$ hold for $x \in E$.

the time τ_D defined above is an optimal stopping time.

2.2.2 Itô's formula and its extensions

Now we turn our attention to the change of variable formulas. [39, §I, Theorem 54] provides the change of variable formula for finite variation (FV) processes with continuous paths. Here, we introduce an extension to FV process with right continuous paths, see e.g. [39, §II, Theorem 31]. For consistency with the literature we appeal to, we use the notation $\int_{0+}^{t} = \int_{(0,t]}$ to denote the integral over the half-open interval $(0, t]$.

Theorem 2.9 (Change of Variables)**.** *Let V be a right continuous FV process, and let f be such that its derivative f' exists and is continuous. Then $(f(V_t))_{t\geq 0}$ is an FV process and*

$$f(V_t) = f(V_0) + \int_{0+}^{t} f'(V_{s-})\mathrm{d}V_s + \sum_{0<s\leq t} \left(f(V_s) - f(V_{s-}) - f'(V_{s-})\Delta V_s\right),$$

where $\Delta X_s = X_s - X_{s-}$.

Theorem 2.9 is a formula for Stieltjes integrals while the famous Itô's formula below, which is a generalization of Theorem 2.9, is adaptable for stochastic integrals, see e.g. [39, §II, Theorem 32].

Theorem 2.10 (Itô's Formula)**.** *Consider a semimartingale X and a C^2 real function f. Then $f(X)$ is also a semimartingale and the following formula holds:*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_{0+}^{t} f'(X_{s-})\,\mathrm{d}X_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})\,\mathrm{d}\left[X, X\right]_s^c \\ &+ \sum_{0<s\leq t} \left\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right\}, \end{aligned}$$

where $[X, X]$ denotes quadratic variation of X and $[X, X]^c$ denotes the path-by-path continuous part of it.

Further, if X is continuous, then the corresponding Itô's formula for $f(X)$ is simplified as

$$f(X_t) = f(X_0) + \int_{0+}^{t} f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_{0+}^{t} f''(X_s)\,\mathrm{d}\left[X, X\right]_s.$$

Chapter 3

Optimal stopping for Omega-killed stable process

3.1 Introduction

Consider a stable Lévy process X with index $\alpha \in (0, 2)$ starting at $X_0 = x \in \mathbb{R} \setminus \{0\}$, to which we introduce a state-dependent killing, occuring at rate $\omega(X_t)$ at time t, where

$$\omega(x) = \begin{cases} k(-x)^{-\alpha}, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

for some parameter $k > 0$. In line with the actuarial literature, X may be regarded as the wealth of a company, and the killing is known as an Omega-clock. Since killing occurs at positive rate only when X is negative, it can be interpreted as a bankruptcy event for the company. Recall that the random killing time is denoted by T in(1.1). We are interested in solving the optimal stopping problem

$$v(x) = \sup_{\tau} \mathbb{E}_x \left[g(X_\tau) \mathbb{1}_{\{\tau < T\}} \right], \tag{3.1}$$

where the supremum is over all stopping times τ for the natural enlarged filtration generated by X (see [12, Definition 1.3.38]), and

$$g(x) = \begin{cases} (x^r - K)^+, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

for some $r \in \mathbb{R} \setminus \{0\}$ and $K > 0$. Note that at time T, a bankruptcy event happens and the asset is then not worth anything. So we set $g(X_T) = 0$ and together with the independence of e in (1.1), (3.1) is consistent with (1.2).

The main result of this work is the following theorem, which we initially present only in its broad strokes.

Theorem 3.1. *There exists* $\max(0, \alpha - 1) < \delta < \min(\alpha\rho, \alpha\hat{\rho})$*, uniquely characterised in terms of the parameters of the stable process and the killing coefficient* k*, such that the following holds.*

1. *When* $0 < r < \delta$*, the solution of the optimal stopping problem* (3.1) *is given by the upwards first passage time*

$$\tau^* := \inf\{t \geq 0 : X_t \geq b^*\},$$

where b^* *can be found explicitly.*

2. *When* $-(\delta - \alpha + 1) < r < 0$ *and* $\alpha \leq 1$*, the solution of the optimal stopping problem* (3.1) *is given by the first entrance time*

$$\tau^* := \inf\left\{t \geq 0 : 0 < X_t \leq \frac{1}{b^*}\right\}$$

where b^* *can again be found explicitly.*

3. *When* $r < -(\delta - \alpha + 1)$ *or* $r > \delta$*,* $v(x) = \infty$ *for all* x*.*

In both instances, the quantity b^* can be found in terms of the parameters of the stable process, the killing parameter k and the strike K, and there is an explicit expression for δ. The full version of this result appears as Theorem 3.15, once we have introduced the necessary notation.

The problem (3.1) is similar to a perpetual American option on the Omega-killed process. American option optimal stopping problems have been studied widely, which are usually linked to the first passage problem of a Lévy process. Most of the existing research set the underlying price process as a particular type of Lévy process, such that explicit solutions can be achieved. Remarkably, [37] provided a closed formula for the price of a perpetual American call option under a general Lévy process with constant killing rate, in terms of the overall supremum of it. Part of our setting for the problem is inspired by [40], which takes an in-depth look at the optimal stopping of an American call option under Omega-killed spectrally negative Lévy models, where the shape of the value function and optimal stopping strategies are provided for different choice of clock parameter. They show that certain parameters would result in the optimality of traditional up-crossing strategies while for other cases, two-sided exit strategies are the optimal choice. Unlike [40] where the underlying is an exponentiated Lévy process, our price process is a stable process which has two-sided jumps. Our main idea to solve the problem is to create a positive self-similar Markov process as the transfer station for the problem so that the solution to (3.1) can be achieved by solving the corresponding optimal stopping problem for a double hypergeometric Lévy process, which involves the application of Lamperti transform as well. Some of the terminologies we use are similar to [8] where an optimal prediction problem for positive self-similar Markov processes is studied.

The remaining part of the chapter is structured as follows. In Section 3.2, we construct, from the stable process, a Markov process called the killed path-censored stable process, which will be proved to be a positive self-similar Markov process by Proposition 3.2. Then, we introduce and apply the Lamperti transform to generate a Lévy process, the path structure and characteristic exponent of which will be studied. In Section 3.3, we show that the Lévy process defined in Section 3.2 belongs to the so-called double hypergeometric class and solves the related optimal stopping problem. Finally, Section 3.4 provides the proof of the main results for this chapter with some auxiliary results presented as well. We add Section 3.5 at last to give a remark on a similar optimal stopping problem while the payoff function is of American put option style.

3.2 The killed path-censored stable process and its Lamperti transform

Since the gain function g in the optimal stopping problem (3.1) is zero on $(-\infty, 0)$, it is natural to consider removing the path sections of X where it is negative. In this section, we will show that this gives rise to a positive self-similar Markov process, and identify its distribution using the Lamperti transform.

Let X be a stable process defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ where $\mathbb{F}$ is the natural enlarged filtration generated by X. X is parametrised by (α, ρ), that belongs to the following set of admissible parameters:

$$\mathscr{A}_{st} = \{(\alpha,\rho) : \alpha \in (0,1), \rho \in (0,1)\} \\ \cup \{(\alpha,\rho) : \alpha \in (1,2), \rho \in (1-1/\alpha, 1/\alpha)\} \cup \{(\alpha,\rho) = (1,1/2)\},$$

which encompasses (up to a multiplicative factor) all stable processes with the exception of Brownian motion, processes jumping only in one direction and symmetric Cauchy processes with non-zero drift. We write $\mathbb{P}_x$ for the law of the process started from x, and we will retain this notation for other stochastic processes wherever this is unambiguous.

Define the positive continuous additive functional

$$\chi_t = \int_0^t \omega(X_s)\,\mathrm{d}s, \quad t \geq 0,$$

where ω is as defined in the introduction. Note that though we require that the killing coefficient k is positive for our main results, in fact many of the intermediate results in Sections 3.2 and 3.3 also work when $k = 0$. Accordingly, we will give some remarks about this case along the way. Let e be an exponential random variable of rate 1, independent of X, and consistent with (1.1), the omega-clock killing time is defined by

$$T = \inf\{t \geq 0 : \chi_t > \mathrm{e}\}.$$

The *Omega-killed stable process* is given by

$$X_t^{\dagger} = X_t \mathbb{1}_{\{t<T\}}, \quad t \geq 0.$$

We note that here, 0 functions as a cemetery state for $X^\dagger$, which is a common convention with pssMps.

The role of 0 deserves special attention. Write $T_0 = \inf\{t \geq 0 : X_t = 0\}$ for its hitting time. When $\alpha \leq 1$, the process X cannot hit zero, whereas when $\alpha > 1$, T_0 is finite almost surely. However, one can also show that, if $k > 0$, $\chi_{T_0} = \infty$, which implies that $X^\dagger$ is always killed before reaching zero. Moreover, regardless of the value of α, when the process X is started from zero it is killed immediately, and so we can regard 0 as an absorbing state for $X^\dagger$; this is consistent with the convention for pssMps mentioned above. In the case $k = 0$, one should replace $X^\dagger$ with the definition $X^\dagger = X_t \mathbb{1}_{\{t < T \wedge T_0\}}$ to ensure that the state 0 is absorbing.

Let $C = (C_t)_{t \geq 0}$ be given by

$$C_t = \int_0^t \mathbb{1}_{\{X_s^\dagger \geq 0\}} \mathrm{d}s, \quad t \geq 0,$$

and call its right-continuous inverse C^{-1}. C counts the time that $X^\dagger$ spends above 0. The *killed path-censored stable process* Y is the stochastic process

$$Y_t = X^\dagger_{C_t^{-1}}, \quad t \geq 0.$$

The effect of the Markov time-change C^{-1} is to erase the negative components of $X^\dagger$ and glue the non-negative parts together at the endpoints, up until the time that $X^\dagger$ is killed during one of these negative components.

When $k = 0$, Y is the path-censored stable process defined in [33], and indeed many of the arguments below are analogous to ones in that work. However, the presence of killing when $k > 0$ introduces some interesting novel features.

Proposition 3.2. *The process Y is a positive self-similar Markov process with respect to the filtration $\mathbb{F} \circ C^{-1} = (\mathscr{F}_{C_t^{-1}})_{t \geq 0}$.*

Proof. The fact that the Markov property holds is a general fact about time-changed Markov processes [41, §III.21]. We prove the scaling property in two steps: first, we show that $X^\dagger$ is self-similar, and then that Y inherits this property.

1) $X^\dagger$ is self-similar; that is, the scaling property introduced in Definition 2.8 applies to it.

Fix $c > 0$. Let $\widetilde{X}_t = cX_{tc^{-\alpha}}$ and define the rescaled process $\widetilde{\chi}$ as

$$\begin{aligned}\widetilde{\chi}_t = \int_0^t \omega(\widetilde{X}_s)\,\mathrm{d}s &= \int_0^t kc^{-\alpha}\left(-(X_{c^{-\alpha}s})\right)_+^{-\alpha}\mathrm{d}s \\ &= \int_0^{c^{-\alpha}t} k(-X_u)_+^{-\alpha}\mathrm{d}u \\ &= \chi_{c^{-\alpha}t}\end{aligned}$$

Let $\widetilde{T} = \inf\{t : \widetilde{\chi}_t > \mathrm{e}\} = c^\alpha T$. The scaling property of $X^\dagger$ now follows, using in the third line the scaling of X:

$$\begin{aligned}\text{under } \mathbb{P}_x,\ (cX^\dagger_{c^{-\alpha}t})_{t\geq 0} &= \left(cX_{c^{-\alpha}t}\mathbb{1}_{\{c^{-\alpha}t<T\}}\right)_{t\geq 0} \\ &= \left(\widetilde{X}_t\mathbb{1}_{\{t<\widetilde{T}\}}\right)_{t\geq 0} \\ &\stackrel{d}{=} \left(X_t\mathbb{1}_{\{t<T\}}\right)_{t\geq 0} = X^\dagger \text{ under } \mathbb{P}_{cx}.\end{aligned}$$

2) Y is self-similar.

Let $\tilde{C}$ be the functional C applied to the process $(cX^\dagger_{tc^{-\alpha}})_{t\geq 0}$; a calculation similar to the one above yields that $\tilde{C}_t^{-1} = c^\alpha C^{-1}_{c^{-\alpha}t}$. We deduce the scaling property of Y:

$$\begin{aligned}\text{under } \mathbb{P}_x,\ (cY_{c^{-\alpha}t})_{t\geq 0} &= \left(cX^\dagger_{C^{-1}_{c^{-\alpha}t}}\right)_{t\geq 0} \\ &= \left(cX^\dagger_{c^{-\alpha}\tilde{C}_t^{-1}}\right)_{t\geq 0} \\ &\stackrel{d}{=} \left(X^\dagger_{C_t^{-1}}\right)_{t\geq 0} = Y \text{ under } \mathbb{P}_{cx},\end{aligned}$$

where we used step 1 in the third equality.

Since Y evidently has state space $[0, \infty)$, this completes the proof. $\square$

Our next aim is to obtain the characteristic function of the Lamperti-Lévy process ξ introduced in subsection 2.1.3, using the structure of Y in terms of gluing path sections of $X^\dagger$.

Define a stopping time

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\},$$

that the stable process X passes below zero for the first time. We will denote the first 'gluing time' of Y by σ_0; in fact, $\sigma_0 = \tau_0^-$, but the latter notation would be misleading for Y, since it actually stays positive at this time.

Lemma 3.3. *For any $x > 0$, the joint law of $(Y_{\sigma_0}, Y_{\sigma_0-})$ under $\mathbb{P}_x$ is equal to that of $(xY_{\sigma_0}, xY_{\sigma_0-})$ under $\mathbb{P}_1$.*

Proof. The proof is very similar to [45, Lemma 3.5], but as the situation is slightly different, we include it for completeness. Fix $c > 0$ and define the two rescaled processes $(\widetilde{X}_t)_{t\geq 0}$ by $\widetilde{X}_t = cX_{c^{-\alpha}t}$ and $(\widetilde{Y}_t)_{t\geq 0}$ by $\widetilde{Y}_t = cY_{c^{-\alpha}t}$. Let $\tilde{\tau}_0^- = \inf\{t \geq 0 : \tilde{X}_t < 0\}$ and introduce $\tilde{\sigma}_0$ which is the same as $\tilde{\tau}_0^-$ but under the sight of process Y. Then,

$$c^\alpha \tau_0^- = \inf\{c^\alpha t : t \geq 0, X_t < 0\} = \inf\{t \geq 0 : cX_{c^{-\alpha}t} < 0\} = \tilde{\tau}_0^-.$$

This implies that $c^\alpha \sigma_0 = \tilde{\sigma}_0$ which further gives that for every $c, x > 0$, the measures $\mathbb{P}_x(Y_{\sigma_0} \in \cdot)$ and $\mathbb{P}_{cx}(c^{-1}Y_{\sigma_0} \in \cdot)$ are equal. The lemma follows by setting $c = 1/x$. □

Denote by p the *killing probability* of Y at each gluing event, namely

$$p = \mathbb{P}_x\left(Y_{\sigma_0} = 0\right),$$

which we assert is independent of x. The following lemma gives the explicit expression for p.

Lemma 3.4 (Killing probability)**.** *The killing probability p is given by*

$$p = \frac{k}{c_+/\alpha + k}.$$

Proof. Recall that T is the time at which $X^\dagger$ is killed, and so C_T is the killing time of Y. Let $R = \inf\{t > \tau_0^- : X_t \geq 0\}$, the first return time of X above zero. In these terms, $p = \mathbb{P}_x(T \leq R)$.

Consider the dual process $\hat{X}$ with distribution $-X$, which is still a stable process (with different parameters.) Let $\hat{X}^*$ denote the process $\hat{X}$ sent to zero at the first time it passes below zero. It is well-known [14] that the Lamperti transform of the pssMp $\hat{X}^*$ is killed at exponential time of rate c_+/α, regardless of the value of $\hat{X}^*_0$, and said killing time corresponds (through the Lamperti time-change) to the first time that $\hat{X}$ passes below zero.

Writing $\hat{T}$ for the Lamperti time-change applied to $\hat{X}$, and as usual denoting by e_μ an exponential random variable with parameter μ independent of everything else, we have the following calculation:

$$\begin{aligned}
\mathbb{P}_x(R < T) &= \mathbb{P}_{-X_{\tau_0^-}}\left(\hat{T}(\mathrm{e}_{c_+/\alpha}) < \chi^{-1}(\mathrm{e})\right) \\
&= \mathbb{P}_{-X_{\tau_0^-}}\left[\left(\int_0^{\cdot} (\hat{X}_u)^{-\alpha}\mathrm{d}u\right)^{-1}(\mathrm{e}_{c_+/\alpha}) < \left(\int_0^{\cdot} k(\hat{X}_u)^{-\alpha}\mathrm{d}u\right)^{-1}(\mathrm{e})\right] \\
&= \mathbb{P}_{-X_{\tau_0^-}}\left[\left(\int_0^{\cdot} (\hat{X}_u)^{-\alpha}\mathrm{d}u\right)^{-1}(\mathrm{e}_{c_+/\alpha}) < \left(\int_0^{\cdot} (\hat{X}_u)^{-\alpha}\mathrm{d}u\right)^{-1}(\mathrm{e}_k)\right] \\
&= \frac{\frac{c_+}{\alpha}}{\frac{c_+}{\alpha} + k}.
\end{aligned}$$

Thus, the killing probability is

$$p = 1 - \mathbb{P}_x(R \le T) = \frac{k}{c_+/\alpha + k}.$$

This completes the proof. □

Let us use X^* to denote the stable process killed on exiting $[0, \infty)$, with the associated Lévy process in the Lamperti representation as described in (2.1.3) by ξ^*, whose characteristic exponent is shown in [14]. Together with the results above, we analyze the path structure of ξ and then present the explicit expression for its Wiener-Hopf factorisation.

Proposition 3.5 (Structure of ξ)**.** *The Lévy process ξ is the sum of two independent Lévy processes ξ^1 and ξ^2, which are characterised as follows:*

1. *The Lévy process ξ^1 has characteristic exponent*

   $$\Psi^1(\theta) = \Psi^*(\theta) - \frac{c_-}{\alpha}, \quad \theta \in \mathbb{R},$$

   where Ψ^* is the characteristic exponent of the process ξ^*.

2. *The process ξ^2 has has characteristic exponent*

   $$\Psi^2(\theta) = (1-p)\Psi^{cpp}(\theta) + p\frac{c_-}{\alpha},$$

   where $p = \frac{k}{c_+/\alpha + k}$ is the killing probability and Ψ^{cpp} is the characteristic exponent of

ξ^c, a compound Poisson process with jump rate c_-/α, which is expressed as

$$\Psi^{cpp}(\theta) = \frac{c_-}{\alpha}\left(1 - \frac{\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)\Gamma(1+i\theta)\Gamma(\alpha-i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha)}\right),$$

for $\theta \in \mathbb{R}$.

Proof. The proof is identical to that of Proposition 3.4 in [33] where the path-censored process is defined by $|X_t|\mathbb{1}_{\{t<T_0\}}$ for $t \geq 0$. In our problem, ξ is generated from a killed path-censored stable process due to the existence of the Omega clock. Hence, the proposition will be proved once we show that ξ is the sum of a process with the law of ξ^1 and a killed process ξ^2, which follows a similar step as in [33]. □

The structure of ξ gives the characteristic exponent of ξ. To give the explicit expression, we require the following result.

Lemma 3.6. *Let*

$$\delta = \frac{1}{2}\Big(\alpha - \frac{1}{\pi}\arccos\big(p\cos\pi(\alpha\rho-\alpha\hat{\rho}) + (1-p)\cos\pi\alpha\big)\Big).$$

Then, δ uniquely satisfies the conditions

$$\max(0,\alpha-1) \leq \delta < \min(\alpha\rho,\alpha\hat{\rho}) \tag{3.2}$$

and

$$(1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho} = \sin\pi(\alpha\rho-\delta)\sin\pi(\alpha\hat{\rho}-\delta). \tag{3.3}$$

The lower bound in (3.2) *holds with equality if and only if $k=0$.*

Proof. Using product-to-sum identities, condition (3.3) can be rewritten as follows:

$$(1-p)\big(\cos\pi(\alpha\rho-\alpha\hat{\rho}) - \cos\pi\alpha\big) = \cos\pi(\alpha\rho-\alpha\hat{\rho}) - \cos\pi s$$
$$\cos(\pi s) = p\cos\pi(\alpha\rho-\alpha\hat{\rho}) + (1-p)\cos\pi\alpha, \tag{3.4}$$

where $s = 2(\alpha/2-\delta)$, and the inequalities (3.2) are equivalent to $\max(\alpha\rho-\alpha\hat{\rho}, \alpha\hat{\rho}-\alpha\rho) < s \leq \min(\alpha, 2-\alpha)$.

We divide our analysis into two cases depending on the value of α. When $\alpha \in (0,1]$, we have that $-\alpha < \alpha\rho - \alpha\hat{\rho} < \alpha$. If $\rho \geq 1/2$, then taking

$$s = \frac{1}{\pi} \arccos\bigl(p \cos \pi(\alpha\rho - \alpha\hat{\rho}) + (1-p)\cos \pi\alpha\bigr)$$

yields $0 < \alpha\rho - \alpha\hat{\rho} < s \leq \alpha \leq 1$. Moreover, this is the unique value of s in the interval specified which satisfies (3.4). The analysis is similar when $\rho < 1/2$.

On the other hand, when $\alpha \in (1,2)$, we have instead $\alpha - 2 < \alpha\rho - \alpha\hat{\rho} < 2 - \alpha$. If $\rho \geq 1/2$, then taking s as above gives $0 < \alpha\rho - \alpha\hat{\rho} < s \leq 2 - \alpha < 1$; again, the uniqueness argument and the case $\rho < 1/2$ are similar. □

We note some special cases of the previous result: when $k = 0$, we have $p = 0$, and then $\delta = 0$ when $\alpha \leq 1$ and $\delta = \alpha - 1$ when $\alpha > 1$. This case is simply the path-censored stable process, with no killing, and many further calculations also simplify. When $p = 1$, which is not part of our parameter set but corresponds formally to $k = \infty$, that is, immediate killing when X goes below zero, we have $\delta = \min(\alpha\rho, \alpha\hat{\rho})$. When $\rho = 1/2$, the symmetric case, we have $\delta = \frac{1}{2}\bigl(\alpha - \frac{1}{\pi}\arccos\bigl(p + (1-p)\cos\pi\alpha\bigr)\bigr) = \frac{\alpha}{2} - \frac{1}{\pi}\arcsin\bigl(\sqrt{1-p}\,\sin(\pi\alpha/2)\bigr)$; this calculation corresponds to the one cited in [32] for the process denoted there $Y^{\natural}$.

The following result was announced, without proof, in [32] when $\rho = \frac{1}{2}$.

Corollary 3.7. *The characteristic exponent of ξ is expressed as*

$$\Psi(\theta) = \frac{\Gamma(\alpha - i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(1 + i\theta)\Gamma(1 - \alpha\rho + i\theta)}{\Gamma(\alpha - \delta - i\theta)\Gamma(\delta - i\theta)\Gamma(\delta + 1 - \alpha + i\theta)\Gamma(1 - \delta + i\theta)} \tag{3.5}$$

Proof. The beginning of the proof resembles that of [33, Theorem 5.3], but it then diverges due to the killing. By Proposition 3.5 above, we know that the characteristic exponent of ξ can be expressed as:

$$\begin{aligned}
\Psi(\theta) &= \Psi^*(\theta) + (1-p)\Psi^{cpp}(\theta) - (1-p)\frac{c_-}{\alpha} \\
&= \frac{\Gamma(\alpha - i\theta)\Gamma(1 + i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)\Gamma(1 - \alpha\hat{\rho} + i\theta)} \\
&\quad + (1-p)\frac{c_-}{\alpha} - (1-p)\frac{c_-}{\alpha}\frac{\Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(\alpha - i\theta)\Gamma(1 + i\theta)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)\Gamma(\alpha)} \\
&\quad - (1-p)\frac{c_-}{\alpha} \\
&= \Gamma(\alpha - i\theta)\Gamma(1 + i\theta)
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\frac{1}{\Gamma(\alpha\hat{\rho} - i\theta)\Gamma(1 - \alpha\hat{\rho} + i\theta)} - (1-p)\frac{\Gamma(\alpha\rho - i\theta)\Gamma(1 - \alpha\rho + i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}\right] \\
&= \frac{1}{\pi^2}\Gamma(\alpha - i\theta)\Gamma(1 + i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(1 - \alpha\rho + i\theta) \\
&\times \left[\sin\pi(\alpha\rho - i\theta)\sin\pi(\alpha\hat{\rho} - i\theta) - (1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho}\right], \qquad (3.6)
\end{aligned}
$$

where we use the expression for c_- in the third equation and apply the reflection formula to the forth equality.

Applying (3.3) gives that

$$
\begin{aligned}
&\sin\pi(\alpha\rho - i\theta)\sin\pi(\alpha\hat{\rho} - i\theta) - (1-p)\sin\pi\alpha\rho\sin\pi\alpha\hat{\rho} \\
&\quad = \frac{1}{2}\left[\cos\pi(\alpha\rho - \alpha\hat{\rho}) - \cos\pi(\alpha - 2i\theta) - \cos\pi(\alpha\rho - \alpha\hat{\rho}) + \cos\pi(\alpha - 2\delta)\right] \\
&\quad = \sin\pi(\alpha - \delta - i\theta)\sin\pi(\delta - i\theta) \\
&\quad = \frac{\pi^2}{\Gamma(\alpha - \delta - i\theta)\Gamma(1 - \alpha + \delta + i\theta)\Gamma(\delta - i\theta)\Gamma(1 - \delta + i\theta)}, \qquad (3.7)
\end{aligned}
$$

using product-to-sum and sum-to-product identities followed by the reflection formula.

Substituting (3.7) into (3.6) yields the expresssion in (3.5). □

3.3 Optimal stopping problems for the Lamperti-Lévy process ξ

Having identified the pssMp Y via its Lamperti transform ξ, we are in a position to solve a related optimal stopping problem for the latter process, which we will later translate into a solution to the original problem.

The solution to the optimal stopping problem for ξ will rely on the *Wiener-Hopf factorisation* of Lévy processes, which can briefly be described as follows. If Ψ is the characteristic exponent of a Lévy process ξ which is killed at rate $q \geq 0$, then there exists a unique factorisation of Ψ of the form

$$
\Psi(\theta) = \kappa(q, -i\theta)\hat{\kappa}(q, i\theta), \quad \theta \in \mathbb{R}, \qquad (3.8)
$$

where $\kappa(q,\cdot)$ and $\hat{\kappa}(q,\cdot)$ are Laplace exponents of two (possibly killed) subordinators, known as the ascending and descending ladder height processes. We say that ϕ is the Laplace exponent of a subordinator H if $\mathbb{E}e^{-\lambda H_t} = e^{-t\phi(\lambda)}$, and κ and $\hat{\kappa}$ are called the *Wiener-Hopf factors* of Ψ (or of ξ.)

These subordinators describe the way that ξ makes new maxima and minima, which goes some way to explaining their utility in the context of this problem.

Our first goal in this section is to characterise ξ by identifying it as a double hypergeometric Lévy process. This is a recently defined class of processes with explicit Wiener-Hopf factorisation. The process ξ is the second known example of a double hypergeometric process found 'in the wild', the other being the ricocheted stable process described by Budd [13].

3.3.1 Identification of the Lamperti transform ξ

Double hypergeometric processes, introduced in [32], are a family of Lévy processes with known Wiener-Hopf factorisation. The class can be characterised as follows. Let $\mathscr{O}$ be the set of all $(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}) \in [0, \infty)^4$ satisfying one of

1. For some $n \in \mathbb{N} \cup \{0\}$, $\mathtt{c} + n \le \mathtt{a} + n \le \mathtt{d} \le \mathtt{b} \le \mathtt{c} + n + 1$, or

2. For some $n \in \mathbb{N}$, $\mathtt{a} + n - 1 \le \mathtt{c} + n \le \mathtt{b} \le \mathtt{d} \le \mathtt{a} + n$.

When $(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}) \in \mathscr{O}$, the function

$$\mathrm{B}(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}; \lambda) := \frac{\Gamma(\lambda + \mathtt{a})\Gamma(\lambda + \mathtt{b})}{\Gamma(\lambda + \mathtt{c})\Gamma(\lambda + \mathtt{d})}, \quad \mathrm{Re}\,\lambda \ge 0,$$

is the Laplace exponent of a subordinator.

Moreover, it is shown in [32, Corollary 2.2] that, when $(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}), (\hat{\mathtt{a}}, \hat{\mathtt{b}}, \hat{\mathtt{c}}, \hat{\mathtt{d}}) \in \mathscr{O}$, the function

$$\Psi(\theta) = \mathrm{B}(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}; -i\theta)\mathrm{B}(\hat{\mathtt{a}}, \hat{\mathtt{b}}, \hat{\mathtt{c}}, \hat{\mathtt{d}}; i\theta), \quad \theta \in \mathbb{R},$$

is the characteristic exponent of a Lévy process in the *double hypergeometric class*.

Lemma 3.8. *The process ξ is a double hypergeometric Lévy process with parameters*

$$\mathtt{a} = \alpha\rho, \mathtt{b} = \alpha, \mathtt{c} = \delta, \mathtt{d} = \alpha - \delta,$$

and

$$\hat{\mathtt{a}} = 1 - \alpha\rho, \hat{\mathtt{b}} = 1, \hat{\mathtt{c}} = \delta + 1 - \alpha, \hat{\mathtt{d}} = 1 - \delta.$$

Accordingly, the Wiener-Hopf factors of ξ *are*

$$\kappa(q, z) = \frac{\Gamma(\alpha\rho + z)\Gamma(\alpha + z)}{\Gamma(\delta + z)\Gamma(\alpha - \delta + z)}, \tag{3.9}$$

and

$$\hat{\kappa}(q, z) = \frac{\Gamma(1 - \alpha\rho + z)\Gamma(1 + z)}{\Gamma(\delta + 1 - \alpha + z)\Gamma(1 - \delta + z)}, \tag{3.10}$$

where

$$q = \frac{\Gamma(\alpha\rho)\Gamma(\alpha)\Gamma(1 - \alpha\rho)}{\Gamma(\delta)\Gamma(\alpha - \delta)\Gamma(\delta + 1 - \alpha)\Gamma(1 - \delta)} = \frac{c_-}{\alpha} \frac{k}{k + \frac{c_+}{\alpha}}.$$

Proof. The interval for δ given in (3.2) implies that both $(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d})$ and $(\hat{\mathtt{a}}, \hat{\mathtt{b}}, \hat{\mathtt{c}}, \hat{\mathtt{d}})$ satisfy condition 1 for membership of $\mathscr{O}$, with $n = 0$.

It follows from Corollary 2.2 in [32] that the Lévy process ξ with characteristic exponent

$$\begin{aligned}\Psi(\theta) &= \mathrm{B}(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}; -i\theta)\mathrm{B}(\hat{\mathtt{a}}, \hat{\mathtt{b}}, \hat{\mathtt{c}}, \hat{\mathtt{d}}; i\theta), \\ &= \frac{\Gamma(\alpha\rho - i\theta)\Gamma(\alpha - i\theta)\Gamma(1 - \alpha\hat{\rho} + i\theta)\Gamma(1 + i\theta)}{\Gamma(\delta - i\theta)\Gamma(\alpha - \delta - i\theta)\Gamma(\delta + 1 - \alpha + i\theta)\Gamma(1 - \delta + i\theta)}\end{aligned}$$

exists as a member of the double hypergeometric class. Comparing with (3.5) shows that this identifies our process ξ, and the result of [32] yields the stated Wiener-Hopf factorisation.

□

When $k = 0$, the process ξ is a hypergeometric Lévy process in the simple ($\alpha \leq 1$) or extended ($\alpha > 1$) class, as described in [33, §5]. The expressions given in the above result for its Wiener-Hopf factors remain valid in this case, though they can be simplified further.

3.3.2 The solution for a perpetual call option

In this part, we derive the solutions for the optimal stopping problem

$$w(y) = \sup_{\sigma \in \mathscr{S}_{\mathbb{G}}} \mathbb{E}_y \left[\left(e^{r\xi_\sigma} - K \right)^+ \right], \tag{3.11}$$

where $r \in \mathbb{R}$, and $\mathscr{S}_{\mathbb{G}}$ indicates the set of all stopping times with respect to $\mathbb{G} = (\mathscr{G}_t)_{t \geq 0}$, the natural enlarged filtration of ξ. This is a perpetual American call option in which the underlying is the process $e^{r\xi}$, and has been addressed in [37].

The Wiener-Hopf factors κ and $\hat{\kappa}$ of ξ appear as components in the solution. We note that $\kappa(q, z)$ is a well-defined holomorphic function for $\operatorname{Re} z > -\alpha\rho$, and the same applies to $\hat{\kappa}(q, z)$ for $\operatorname{Re} z > \alpha\rho - 1$.

Theorem 3.9. *Consider the optimal stopping problem* (3.11).

1. *When $0 < r < \delta$, the solution is given by*

$$w(y) = \frac{1}{\mathbb{E}[e^{r\bar{\xi}_\zeta}]}\mathbb{E}\left[\left(e^{r(y+\bar{\xi}_\zeta)} - K\mathbb{E}\left[e^{r\bar{\xi}_\zeta}\right]\right)^+\right], \tag{3.12}$$

where ζ is an exponential random variable with parameter q, independent of ξ and $\bar{\xi}_t = \sup_{0\le s\le t}\xi_s$ with $\mathbb{E}\left[e^{r\bar{\xi}_\zeta}\right] = \frac{\kappa(q,0)}{\kappa(q,-r)}$. The optimal stopping time is given by

$$\sigma^* = \inf\{t \ge 0 : \xi_t \ge c^*\}$$

where

$$c^* = \frac{1}{r}\log K\frac{\kappa(q, 0)}{\kappa(q, -r)}.$$

2. *When $-(\delta + 1 - \alpha) < r < 0$, the solution is given by*

$$w(y) = \frac{1}{\mathbb{E}[e^{r\underline{\xi}_\zeta}]}\mathbb{E}\left[\left(e^{r(y+\underline{\xi}_\zeta)} - K\mathbb{E}\left[e^{r\underline{\xi}_\zeta}\right]\right)^+\right] \tag{3.13}$$

where $\underline{\xi}_t = \inf_{0\le s\le t}\xi_s$ with $\mathbb{E}\left[e^{r\underline{\xi}_\zeta}\right] = \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}$. The optimal stopping time is given by

$$\sigma^* = \inf\{t \ge 0 : \xi_t \le -c^*\}$$

where

$$c^* = \frac{1}{|r|}\log K\frac{\hat{\kappa}(q, 0)}{\hat{\kappa}(q, r)}.$$

3. *When $r > \delta$ or $r < -(\delta + 1 - \alpha)$, $w(y) = \infty$ for all y.*

Proof. Theorem 1 in [37] provides a solution for the valuation of the perpetual American call option for a general Lévy process, expressed in terms of the moment generating function of its supremum. The proof follows from an application of this result, the main prerequisite for which is to check the inequality $\mathbb{E}e^{r\xi_1} < 1$.

1. Take $r > 0$. Condition $\mathbb{E}e^{r\xi_1} < 1$ is equivalent to $\Psi(-ir) < 0$, where Ψ is the characteristic exponent of the killed Lévy process ξ.

   The first zero of $\Psi(-ir)$ occurs at δ, and so in this case $\mathbb{E}e^{r\xi_1} < 1$ if and only if $0 < r < \delta$.

   The value function w is now expressed in terms of the moment generating function of the overall supremum of $r\xi$, which is given by [30, Theorem 6.15] in terms of the Wiener-Hopf factor:
$$\mathbb{E}\left[e^{r\bar{\xi}_\varsigma}\right] = \frac{\kappa(q,0)}{\kappa(q,-r)}.$$
   The result follows from [37].

2. Take now $r < 0$. In this case, $\mathbb{E}e^{r\xi_1} < 1$ if and only if $0 < -r < \delta + 1 - \alpha$. Since $r < 0$, the value function is expressed in terms of
$$\mathbb{E}[e^{r\underline{\xi}_\varsigma}] = \frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)},$$
   and the result again follows from [37].

3. In this case, it holds that $\mathbb{E}[e^{r\xi_1}] > 1$, and so the result follows from [37, Theorem 1(c) and (d)], where Mordecki shows that arbitrarily large values can be obtained by stopping at deterministic times. □

In mathematics, the ordinary hypergeometric function ${}_2F_1(a,b;c;z)$ is a special function represented by the hypergeometric series. We can express the (3.11) in terms of this particular function as below.

Corollary 3.10. *Let $0 < r < \delta$. We can express*
$$w(y) = \kappa(q,-r)\int_0^\infty \left(e^{r(y+z)} - K\frac{\kappa(q,0)}{\kappa(q,-r)}\right)^+ u(z)\,\mathrm{d}z,$$
where
$$u(x) = \frac{1}{\Gamma(\alpha\rho)}e^{-\delta x}(1-e^{-x})^{\alpha\rho-1}_2F_1(\delta-\alpha\hat{\rho},\delta,\alpha\rho,1-e^{-x}), \quad x \geq 0,$$
and ${}_2F_1$ is the hypergeometric function.

Proof. The random variable $\bar{\xi}_\zeta$ has Laplace transform given by

$$\lambda \mapsto \frac{\kappa(q,0)}{\kappa(q,\lambda)}, \quad \operatorname{Re}\lambda \geq -\delta,$$

and the double beta subordinator with Laplace exponent $\kappa(q,\cdot)$ has a renewal density u whose Laplace transform is given by $\lambda \mapsto \frac{1}{\kappa(q,\lambda)}$. The function u is the convolution of the two functions

$$u_1(x) = \frac{1}{\Gamma(\delta)} e^{-(\alpha-\delta)x}(1-e^{-x})^{\delta-1}, \quad x \geq 0, \text{ and}$$
$$u_2(x) = \frac{1}{\Gamma(\alpha\rho-\delta)} e^{-\delta x}(1-e^{-x})^{\alpha\rho-\delta-1}, \quad x \geq 0,$$

which are the renewal densities of Lamperti-stable subordinators corresponding, respectively, to the Laplace exponents $\lambda \mapsto \frac{\Gamma(\alpha\rho+\lambda)}{\Gamma(\delta+\lambda)}$ and $\lambda \mapsto \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha-\delta+\lambda)}$. Computing the convolution of u_1 and u_2 yields the expression in the statement. □

A different expression for u appears in [32, Theorem 2.1], but in our particular parameter regime the one given above is simpler.

3.4 Solution of the optimal stopping problem

Now we have studied the characters and optimal stopping problem for the Lamperti-Lévy process and we need to transform the results for our original problem (3.1). Before giving our main result, which will encompass Theorem 3.1, we provide the following auxiliary results. In this section we return to the assumption $k > 0$.

Since we need to work with several filtrations and their associated stopping times, we introduce in this section the notation $\mathscr{S}_{\mathbb{H}}$, the set of stopping times associated with some naturally enlarged filtration $\mathbb{H}$.

Lemma 3.11. *For $x > 0$, it holds that*

$$\mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t\right)\mathbb{1}_{\{t<T\}}\right] = \mathbb{E}_{\log x}\left[\sup_{s\geq 0} g\left(e^{\xi_s}\right)\right].$$

Further, when $r \in (-(\delta+1-\alpha),\delta)\setminus\{0\}$, $\mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t\right)\mathbb{1}_{\{t<T\}}\right] < \infty$.

Proof. We begin by observing that the supremum of $\sup_{t\geq 0} g(X_t)\mathbb{1}_{\{t<T\}}$ can only occur at a time when X is positive. Since the time-change by C^{-1}, which yields Y, does not remove any such times, we see that

$$\mathbb{E}_x\left[\sup_t g(X_t)\mathbb{1}_{\{t<T\}}\right] = \mathbb{E}_x\left[\sup_t g(Y_t)\mathbb{1}_{\{t<C(T)\}}\right]. \tag{3.14}$$

Given a path of Y starting at x, the corresponding path of ξ starts at $\log x$ and is a deterministic space and time transformation, which implies that

$$\mathbb{E}_x\left[\sup_t g(Y_t)\mathbb{1}_{\{t<C(T)\}}\right] = \mathbb{E}_{\log x}\left[\sup_s g(e^{r\xi_s})\right], \tag{3.15}$$

and this gives the first claim in the statement.

For the second claim, we recall that by Theorem 1(a) in [37], $\mathbb{E}_0\left[\sup_s e^{r\xi_s}\right] < \infty$ is true if $\mathbb{E}_0\left[e^{r\xi_1}\right] < 1$ holds, and as outlined in the proof of Theorem 3.9, this can be reduced to the condition that $-(\delta+1-\alpha) < r < \delta$. This completes the proof. □

Lemma 3.12. *When $r \in (-(\delta+1-\alpha),\delta)\setminus\{0\}$, it holds that for $x<0$,*

$$\mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t\right)\mathbb{1}_{\{t<T\}}\right] < \infty.$$

Proof. Denote $\tau_0^+ = \inf\{t\geq 0 : X_t \geq 0\}$. Then, for all $x<0$,

$$\begin{aligned}\mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t\right)\mathbb{1}_{\{t<T\}}\right] &= \mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t^\dagger\right)\right]\\ &= \mathbb{E}_x\left[\sup_{t\geq 0} g\left(X_t^\dagger\right)\mathbb{1}_{\{\tau_0^+<T\}}\right]\\ &\leq \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^+}}\left[\sup_{t\geq 0} g\left(X_t^\dagger\right)\right]\right].\end{aligned}$$

Further, the inner expectation can be expressed using the scaling property as

$$\begin{aligned}\mathbb{E}_y\left[\sup_{t\geq 0} g(X_t^\dagger)\right]\Bigg|_{y=X_{\tau_0^+}} &= \mathbb{E}_1\left[\sup_{t\geq 0} g(y\cdot X_t^\dagger)\right]\Bigg|_{y=X_{\tau_0^+}}\\ &= y^r\mathbb{E}_1\left[\sup_{t\geq 0} g(X_t^\dagger)\right]\Bigg|_{y=X_{\tau_0^+}}.\end{aligned}$$

We know from Lemma 3.11 that $\mathbb{E}_1\left[\sup_{t\geq 0} g(X_t^\dagger)\right] < \infty$. Hence, it remains to show that $\mathbb{E}_x\left[\left(X_{\tau_0^+}\right)^r\right] < \infty$ for $x < 0$.

Based on the result in [42], we have for $x < 0$,

$$\begin{aligned}\mathbb{E}_x\left[\left(X_{\tau_0^+}\right)^r\right] &= \int_0^\infty y^r \frac{\sin \pi\alpha\rho}{\pi}(-x)^{\alpha\rho} y^{-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y \\ &= \frac{\sin \pi\alpha\rho}{\pi}(-x)^{\alpha\rho}\int_0^\infty y^{r-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y.\end{aligned}$$

By the condition on δ in (3.2), it holds that

$$r - \alpha\rho > -1 \text{ and } r - \alpha\rho - 1 < -1,$$

which follows that $\int_0^\infty y^{r-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y < \infty$.

This completes the proof. □

Based on the lemmas above, we have a theorem as following.

Theorem 3.13. *When $0 < r < \delta$, there exists a set $D \subset (0,\infty)$ such that*

$$\tau_D := \inf\{t \geq 0 : X_t \in D\}$$

is an optimal stopping time in (3.1).

Proof. Consider a 2-dimensional optimal stopping problem defined as

$$V(x,a) = \sup_{\tau\in\mathscr{S}_{\mathbb{F}}} \mathbb{E}_{x,a}\left[G(X_\tau,\chi_\tau)\right], \tag{3.16}$$

where $G(x,a) = e^{-a}g(x)$ and we note that the bivariate process (X,A) with state space $\mathbb{R}\times[0,\infty)$ is Markovian with respect to the natural enlarged filtration $\mathbb{F}$ of X. The notation here is that $\mathbb{E}_{x,a}[F(X_t,\chi_t)] := \mathbb{E}_x[F(X_t,\chi_t+a)]$ for any measurable bounded F. This is equivalent to the problem (3.1), but viewed from a 2-dimensional perspective.

Let

$$S = \{(x,a) : V(x,a) = G(x,a)\} = \{(x,a) : V(x,0) = g(x)\}.$$

According to [38, Corollary 2.9 and Remark 2.10], the first entrance time τ_S of (X,χ) into S is optimal for (3.16) when the following four points are verified.

1) $\mathbb{E}_{x,a}\left[\sup_t G(X_t, \chi_t)\right] < \infty$.

Lemma 3.11 tells that $\mathbb{E}_x\left[\sup_t g(X_t)\mathbb{1}_{\{t<T\}}\right] < \infty$. Also, we have

$$\mathbb{E}_x\left[\sup_{t\geq 0} g(X_t)\mathbb{1}_{\{t<T\}}\right] \geq \mathbb{E}_x\left[\sup_{t\geq 0} g(X_t)\mathbb{E}\left[\mathbb{1}_{\{t<T\}} \,|\, \mathscr{F}_t\right]\right]$$
$$= \mathbb{E}_x\left[\sup_{t\geq 0} e^{-\chi_t} g(X_t)\right] \geq \mathbb{E}_{x,a}\left[\sup_{t\geq 0} e^{-\chi_t} g(X_t)\right],$$

where we apply the tower property in the first line. Thus, for all $x \in \mathbb{R}$ and $a \geq 0$, $\mathbb{E}_{x,a}\left[\sup_t G(X_t, \chi_t)\right] < \infty$.

2) G is upper semicontinuous.

In fact, G is even continuous, and hence automatically upper semicontinuous.

3) V is lower semicontinuous.

Fix (x, a) and, for $\epsilon > 0$, let τ_ϵ be some stopping time satisfying

$$\mathbb{E}_{x,a}\left[G(X_{\tau_\epsilon}, \chi_{\tau_\epsilon})\right] \geq V(x, a) - \epsilon. \tag{3.17}$$

Let $\{(x_n, a_n)\,;\, n \geq 0\}$ be any sequence which converges to (x, a) when n tends to infinity. Then,

$$V(x_n, a_n) \geq \mathbb{E}_{x_n,a_n}\left[G\left(X_{\tau_\epsilon}, \chi_{\tau_\epsilon}\right)\right] = \mathbb{E}\left[G\left(x_n + X_{\tau_\epsilon}, a_n + \chi^{(x_n)}_{\tau_\epsilon}\right)\right], \tag{3.18}$$

where $\mathbb{E} = \mathbb{E}_{0,0}$ and

$$\chi^{(x)}_t = \int_0^t \omega(X_s + x)\mathrm{d}s.$$

Applying Fatou's lemma in (3.18) gives

$$\liminf_{n\to\infty} V(x_n, a_n) \geq \mathbb{E}\left[\liminf_{n\to\infty} G\left(x_n + X_{\tau_\epsilon}, a_n + \chi^{(x_n)}_{\tau_\epsilon}\right)\right], \tag{3.19}$$

We consider $G(x_n + X_{\tau_\epsilon}, a_n + \chi^{(x_n)}_{\tau_\epsilon})$ pathwise, and look at two cases. First, if $X_{s-} + x \neq 0$ and $X_s + x \neq 0$ for $s \in [0, \tau_\epsilon]$, then $|X_s + x| > c$ for $s \in [0, \tau_\epsilon]$ for some positive (path-dependent) constant c. From this it follows that, for sufficiently large n, $|X_s + x_n| > c$ also holds. This gives the bound $\omega(x_n + X_s) \leq kc^{-\alpha}$, and by the dominated convergence theorem, we have

$$\liminf_{n\to\infty} \chi_{\tau_\epsilon}^{(x_n)} = \chi_{\tau_\epsilon}^{(x)}$$

which gives

$$\liminf_{n\to\infty} G\left(x_n + X_{\tau_\epsilon}, a_n + \chi_{\tau_\epsilon}^{(x_n)}\right) = G\left(x + X_{\tau_\epsilon}, a + \chi_{\tau_\epsilon}^{(x)}\right).$$

If, on the other hand, there exists $s \leq \tau_\epsilon$ such that $X_{s-} + x = 0$ or $X_s + x = 0$, which can only occur if $\alpha > 1$ since otherwise points are polar for X [10, §VIII], then $\chi_{\tau_\epsilon}^{(x)} = \infty$, and we have that

$$\liminf_{n\to\infty} e^{-\chi_{\tau_\epsilon}^{(x_n)}} \geq 0 = e^{-\chi_{\tau_\epsilon}^{(x)}}.$$

From this it again follows that

$$\liminf_{n\to\infty} G\left(x_n + X_{\tau_\epsilon}, a_n + \chi_{\tau_\epsilon}^{(x_n)}\right) \geq G\left(x + X_{\tau_\epsilon}, a + \chi_{\tau_\epsilon}^{(x)}\right).$$

This analysis together with (3.19) gives us

$$\liminf_{n\to\infty} V(x_n, a_n) \geq \mathbb{E}\left[G\left(x + X_{\tau_\epsilon}, a + \chi_{\tau_\epsilon}^{(x)}\right)\right].$$

Finally, applying (3.17) and letting $\epsilon \to 0$, we have

$$\liminf_{n\to\infty} V(x_n, a_n) \geq V(x, a),$$

which means that function V is lower semi-continuous.

4) $\lim_{t\to\infty} G(X_t, \chi_t)$ exists and equals zero.

We calculate

$$\begin{aligned}
\lim_{t\to\infty} G(X_t, \chi_t) &= \lim_{t\to\infty} e^{-\chi_t} g(X_t) \\
&= \lim_{t\to\infty} \mathbb{E}\left[\mathbb{1}_{\{t<T\}} g(X_t) \mid \mathscr{F}_\infty\right] \\
&= \mathbb{E}\left[\lim_{t\to\infty} \mathbb{1}_{\{t<T\}} g(X_t) \mid \mathscr{F}_\infty\right]
\end{aligned}$$

$$= 0,$$

where the third equality results from the dominated convergence theorem, applicable due to Lemma 3.11. It then follows that $\lim_{t\to\infty} G(X_t, \chi_t) = 0$.

We have concluded, by Remark 2.10 in [38], that τ_S is optimal for the two-dimensional problem (3.16). If we define

$$D = \{x : V(x, 0) = g(x)\},$$

then it immediately follows that τ_D is optimal for the equivalent problem (3.1).

To show that D may be chosen as a subset of $(0, \infty)$, set $D' = D \cap (0, \infty)$, and then consider the following calculation:

$$\begin{aligned}
\mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}] &= \mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}\mathbb{1}_{\{X_{\tau_D}>0\}}] + \mathbb{E}_x[g(X_{\tau_D})\mathbb{1}_{\{\tau_D<T\}}\mathbb{1}_{\{X_{\tau_D}\le 0\}}] \\
&= \mathbb{E}_x[g(X_{\tau_{D'}})\mathbb{1}_{\{\tau_{D'}<T\}}\mathbb{1}_{\{\tau_D=\tau_{D'}\}}] + 0 \\
&\le \mathbb{E}_x[g(X_{\tau_{D'}})\mathbb{1}_{\{\tau_{D'}<T\}}].
\end{aligned}$$

Hence, if $D \not\subset (0, \infty)$, we can replace it with D' and obtain at least as good a value; indeed, since τ_D is optimal, the values obtained from τ_D and $\tau_{D'}$ must be equal. □

Theorem 3.14. *When $-(\delta + 1 - \alpha) < r < 0$ and $\alpha \le 1$, there exists a set $D \subset (0, \infty)$ such that*

$$\tau_D := \inf\{t \ge 0 : X_t \in D\}$$

is an optimal stopping time in (3.1).

Proof. We use a different way compared with Theorem 3.14 to prove the argument, which mainly refers to [43, Theorem 3].

As we have mentioned in Section 3.2, when $\alpha \le 1$, it holds that $T_0 = \infty$ a.s., which means that we can treat X as having state space $E = \mathbb{R} \setminus \{0\}$. Hence, $g(x)$ is continuous on E.

Then by [43, Theorem 3], it follows that τ_D is optimal for (3.1) with

$$D = \{x : v(x) = g(x)\},$$

and the argument of $D \subset (0,\infty)$ can be achieved via the same way as in Theorem 3.14. $\square$

Our main result, which implies the version given in the introduction, is as follows. Recall that T is the killing time of the process X.

Theorem 3.15. *Let $x \in \mathbb{R}\setminus\{0\}$. The solution of the optimal stopping problem* (3.1) *is given as follows.*

1. *If $0 < r < \delta$, then the optimal stopping time is given by*

$$\tau^* = \inf\{t \geq 0 : X_t \geq b^*\}$$

where

$$b^* = \left(K\frac{\kappa(q,0)}{\kappa(q,-r)}\right)^{1/r}.$$

Moreover $\mathbb{E}_1\left[\overline{X}_T^r\right] < \infty$ and the optimal value is given for $x > 0$ by

$$v(x) = \frac{1}{\mathbb{E}_1[\overline{X}_T^r]}\mathbb{E}_1\left[\left((x\overline{X}_T)^r - K\mathbb{E}_1\left[\overline{X}_T^r\right]\right)^+\right], \tag{3.20}$$

where $\overline{X}_t = \sup_{s\leq t} X_s$.

2. *If $-(\delta+1-\alpha) < r < 0$ and $\alpha \leq 1$, then the optimal stopping time is given by*

$$\tau^* = \inf\{t \geq 0 : 0 < X_t \leq 1/b^*\}$$

where

$$b^* = \left(K\frac{\hat{\kappa}(q,0)}{\hat{\kappa}(q,r)}\right)^{1/|r|}.$$

Moreover, $\mathbb{E}_1[(\underline{Y}_T)^r] < \infty$ and the optimal value is given for $x > 0$ by

$$v(x) = \frac{1}{\mathbb{E}_1[\underline{Y}_T{}^r]}\mathbb{E}_1\left[((x\underline{Y}_T)^r - K\mathbb{E}_1[\underline{Y}_T{}^r])^+\right], \tag{3.21}$$

where $\underline{Y}_t = \inf_{s\leq t} Y_s = \inf_{s\leq t} X_{C_s^{-1}}$.

3. *If $r > \delta$ or $r < -(\delta+1-\alpha)$, then $v(x) = \infty$.*

Proof. Since $g(x) = 0$ for $x \leq 0$, it is never optimal to stop when X is negative. The processes X and Y have the same range when restricted to $(0,\infty)$, so the optimal stopping problem (3.1) is equivalent to

$$v(x) = \sup_{\tau' \in \mathscr{S}_{\mathbb{F}^X \circ C^{-1}}} \mathbb{E}_x \left[g\left(Y_{\tau'}\right)\right], \tag{3.22}$$

where we recall $\mathbb{F} \circ C^{-1} = \left(\mathscr{F}_{C_t^{-1}}\right)_{t\geq 0}$. Moreover, as already outlined, the pssMp Y corresponds to ξ under a time and space change, which implies that

$$v(y) = \sup_{\tau'' \in \mathscr{S}_{\mathbb{F}\circ C^{-1} \circ T}} \mathbb{E}_y \left[g\left(e^{\xi_{\tau''}}\right)\right], \tag{3.23}$$

where $y = \log x$ and here again we have $\mathbb{F} \circ C^{-1} \circ T = \left(\mathscr{F}_{C^{-1}_{T(t)}}\right)_{t\geq 0}$.

We know from Theorem 3.14 that the hitting time τ_D of some set $D \subset (0, \infty)$ is optimal for (3.1), and hence that the first passage time σ_H of ξ into set H is optimal for (3.23), where $H = \{\log x : x \in D\}$.

The solution in Theorem 3.9 showed that the solution of the problem (3.11) for ξ is given by a hitting time σ^*. In (3.23), we optimise over $\mathscr{S}_{\mathbb{F}\circ C^{-1}\circ T}$, and in (3.11) we optimise over $\mathscr{S}_{\mathbb{G}}$. The former set of stopping times is larger, since the filtration contains information about the times that X spends below zero. Since $\sigma_H \in \mathscr{S}_{\mathbb{G}}$, it follows that it is also optimal for (3.11). Comparing the form of σ^* with σ_H and hence with the original stopping region D, we obtain that

$$D = \left[\left(K\frac{\kappa(q,0)}{\kappa(q,-r)}\right)^{1/r}, \infty\right) \text{ if } 0 < r < \delta,$$

and

$$D = \left(0, \left(\frac{\hat{\kappa}(q,-r)}{K\hat{\kappa}(q,0)}\right)^{1/|r|}\right] \text{ if } -(\delta + 1 - \alpha) < r < 0.$$

For part 3, we turn to the corresponding part of Theorem 3.9, where it is shown that arbitrarily large values of w can be obtained by stopping at a deterministic time, say t_0. Since time t_0 for ξ corresponds to time $C^{-1}_{T_{t_0}}$ for X, this is a viable time at which to stop X. It follows that $v(x) = \infty$ for $x > 0$. When $x < 0$, one can first wait for X to pass above zero, which happens without being killed with positive probability, and then act as above, again attaining unbounded values. □

Remark 3.16. Though the semi-explicit expressions for v given in the preceding theorem are only valid when $x > 0$, one can still express the value function in other cases. Clearly $v(0) = 0$. When $x < 0$, $X^\dagger$ is started below zero and we should wait for it to either be killed (with probability p independent of x) or jump back above (with probability $1-p$). Let $\tau_0^+ = \inf\{t \geq 0 : X_t^\dagger \geq 0\}$. Then, for all $x < 0$,

$$\begin{aligned} v(x) &= \mathbb{E}_x[v(X^\dagger_{\tau_0^+})] \\ &= (1-p)\int_0^\infty v(y) \frac{\sin \pi\alpha\rho}{\pi}(-x)^{\alpha\rho} y^{-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y, \end{aligned}$$

where the integral expression comes from [42]. In conjunction with the expression for w given in Corollary 3.10, this can be used to write v as a double integral suitable for numerical computation. $\triangle$

3.5 Remark on a variant stopping problem

In this section, we briefly describe how a gain function akin to that of a put option requires a different analysis, despite the superficial similarities. We consider the optimal stopping problem

$$v(x) = \sup_\tau \mathbb{E}_x\left[g(X_\tau)\mathbb{1}_{\{\tau<T\}}\right], \qquad g(x) = (K-x)^+, \tag{3.24}$$

where $K \in \mathbb{R}$. Since $g(x)$ may be positive for $x < 0$, it no longer makes sense to erase the sojourns of X in $(-\infty, 0)$. Instead, we may describe the problem using the so-called Lamperti-Kiu transform. This gives $X^\dagger$ in terms of a Markov additive process (MAP), which is a process (ξ, J) on $\mathbb{R} \times \{\pm 1\}$ obtained by

$$\xi_t = \log|X^\dagger_{T(t)}| \text{ and } J_t = \operatorname{sgn} X^\dagger_{T(t)}, \quad t \geq 0,$$

where T is a time-change.

The process (ξ, J) corresponding to $X^\dagger$ will be killed at a rate $\omega(\xi_t, J_t)$, where

$$\omega(y,j) = \begin{cases} 0, & j = 1, \\ k, & j = -1, \end{cases}$$

and the problem (3.24) will correspond to

$$v(x) = v(y,j) = \sup_\sigma \mathbb{E}_{y,j}\left[g(\xi_\sigma, J_\sigma)\mathbb{1}_{\{t<\zeta\}}\right],$$

where $(y, j) = (\log |x|, \text{sgn} x)$, ζ is the killing time of (ξ, J) and

$$g(y, j) = \begin{cases} (K - e^y)^+, & j = 1, \\ (K + e^y)^+, & j = -1. \end{cases}$$

The MAP (ξ, J) can be described explicitly in terms of its matrix exponent. Unfortunately, though this translation to a MAP problem is relatively simple to describe, two new issues arise. The first is that the theory of optimal stopping is much less developed for these processes, outside of the spectrally negative case [15]. The second is that the presence of J-dependent killing means that the matrix Wiener-Hopf factorisation of (ξ, J), which is known when $k = 0$ [31, 34], is no longer evident.

Chapter 4

The optimal time to liquidate a portfolio of insurance products in the presence of bankruptcy

4.1 Introduction

Let U be a Cramér-Lundberg process starting at $U_0 = x \in \mathbb{R}$, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ where $\mathbb{F}$ is the natural enlarged filtration generated by U. For any $t \geq 0$,

$$U_t = x + \delta t - \sum_{i=1}^{N_t} \xi_i, \tag{4.1}$$

where $\delta > 0$ is the drift, ξ_i are i.i.d. r.v. which are exponentially distributed with parameter $\rho > 0$ and N is an independent Poisson process with intensity $\lambda > 0$. U can be used to describe the capital flow for a portfolio of insurance products where $x > 0$ is the initial capital, δ denotes the premium rate and ξ_i's represent the claim sizes with N counting the number of claims. See e.g. [3]. Note that U is a spectrally negative Lévy process with bounded variation paths and throughout this chapter, we will denote $\nu(\mathrm{d}x) := \Pi(-\mathrm{d}x) = \lambda\rho e^{-\rho x}\mathbb{1}_{\{x>0\}}\mathrm{d}x$ where Π is the Lévy measure of U.

Define a payoff function g with penalty for negative values as

$$g(x) = \begin{cases} x & \text{if}\, x \geq 0 \\ p & \text{if}\, x < 0, \end{cases} \tag{4.2}$$

where $p < 0$ is the penalty parameter.

Further, by setting the killing density, introduced in Chapter 1,as

$$\omega(x) = q\mathbb{1}_{\{x<0\}},$$

with $q > 0$, we create an Omega clock that 'ticks' at a constant rate but only if the capital level is strictly negative. We are interested in the following optimal stopping problem:

$$v(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r(\tau\wedge T)} g(U_{\tau\wedge T}) \right] = \sup_{\tau} \mathbb{E}_x \left[L^*_\tau \right], \tag{4.3}$$

where T is defined in (1.1) and

$$L^*_t := e^{-A_t} g(U_t) + \int_0^t q\mathbb{1}_{\{U_s<0\}} e^{-A_s} g(U_s) \mathrm{d}s, \quad t \geq 0$$

with $A_t := rt + \chi_t = rt + \int_0^t \omega(U_s)\mathrm{d}s$. Here the second equality in (4.3) results from taking expectation with respect to the independent exponentially distributed random variable that appears in the Omega clock. We will show shortly below in (4.4) that $L^*_\infty := \lim_{t\to\infty} L^*_t$ exists which allows us to consider stopping times that can also take the value ∞.

The main results are presented in Theorem 4.12and Theorem 4.13, where we prove that when p, δ, r, λ and ρ satisfy certain conditions, an up-crossing strategy with the stopping region $[a, \infty)$ is optimal for (4.3); further, we show that smooth pasting principle holds when $a > 0$ and fails when $a = 0$. Also, for the cases where an up-crossing strategy has not yet been proved to be optimal, we give numerical examples and corresponding analysis in section 4.6.

Many of the existing studies under Cramér-Lundberg model are related to insurance. Problem with state-dependent killing in insurance was introduced in [1], where the concept of bankruptcy was established. The insurance company is allowed to continue doing business with a negative surplus until bankruptcy occurs. The concept of bankruptcy is exactly the same as the concept of Omega killing time we introduced in Chapter 1. The bankruptcy probability and the expectation of a discounted penalty at the time of killing are widely studied. See [1] and [24] where the surplus process is modeled by a Brownian motion and [2] under a compound Poisson risk model, for example.

For our problem (4.3), a possible interpretation is as follows. As mentioned above, U

represents a capital process corresponding to a portfolio of insurance products. The problem (4.3) is essentially about choosing the optimal time to liquidate the portfolio, that is, to transfer all insurance policies to another company or otherwise terminate all policies. Then, the value function v represents the largest expected value that the insurer can obtain from the portfolio in this model where liquidation is possible. When U goes below zero, the insurer has limited time to restore the capital level back to positive levels. If this does not happen quickly enough, i.e. if the Omega clock 'rings' at time T, then a regulator steps in to take over the portfolio and leaves the company with the fine p. Note that we do not allow the insurer to liquidate the portfolio when the capital U is negative. This means that while U is below zero, the only possible payoff for the insurer is the penalty p when the Omega clock rings. This explains why we set $g(x) = p$ for $x < 0$. Nevertheless, for convenience, we can simply optimise over all stopping times in the optimal stopping problem (4.3) since it can anyhow not be optimal to stop when U is negative and g attains its global minimum. Indeed this is confirmed by our main results. Also, note that for $x \geq 0$, the difference $v(x) - g(x) = v(x) - x$ can be seen as a measure of expected profitability of the portfolio in this model. In particular, if $v(x) - x = 0$, then the insurer is best off stopping immediately, i.e. not to sell these policies in the first place.

The remainder of this chapter is structured as follows. In Section 4.2, we present some analytical expressions, including the Itô's decomposition of L^*, the value function for an up-crossing strategy denoted by v_a. Then, in Section 4.3, we introduce a partition of the parameter space which corresponds to different optimal strategies as shown in the main results. Preliminary results with proofs are provided in Section 4.4, which will be used to prove the main theorems. Based on the calculations and lemmas, we state the main results of this chapter in Section 4.5 and give the detailed proofs there. Finally, in Section 4.6, we outline the results and share discussions with some numerical examples.

4.2 Some analytical expressions

In this section, we first find an alternative way to express L^* by applying the change of variable theorem. Then in Section 4.2.2, we derive an expression for the value function under an up-crossing strategy, in terms of the well-known scale functions. In Section 4.2.3, we do some analysis related to the smooth pasting property for later use. And at last, we prove some results for the candidate optimal value process in Section 4.2.4.

4.2.1 A decomposition of L^*

We begin with a quick observation related to L^*. Recall that for any $t \geq 0$,

$$L^*_t := e^{-A_t} g(U_t) + \int_0^t q \mathbb{1}_{\{U_s<0\}} e^{-A_s} g(U_s) \mathrm{d}s.$$

It is clear that

$$L^*_t \geq p + pq \int_0^\infty e^{-rs} \mathrm{d}s = p\left(1 + \frac{q}{r}\right),$$

and under $\mathbb{P}_x$ with $x \in \mathbb{R}$ given, we have $U_t \leq x + \delta t$, from which it follows

$$L^*_t \leq e^{-rt}\left(x + \delta t\right) \leq C_x, \quad \text{for all } t \geq 0,$$

where $C_x < \infty$ depends only on x. Hence, L^* is a bounded process under any $\mathbb{P}_x$. Furthermore, considering the lower bound on payoff function g and the upper bound on U_t, it is also clear that for any $x \in \mathbb{R}$,

$$L^*_\infty := \lim_{t\to\infty} L^*_t = pq \int_0^\infty \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \quad \mathbb{P}_x - \text{a.s.} \tag{4.4}$$

Based on this, we have the following lemma.

Lemma 4.1. *For any $x \in \mathbb{R}$, we have $\mathbb{P}_x$–a.s. that*

$$L^*_t = L^*_0 + M_t + \int_0^t e^{-A_s} k(U_s) \mathrm{d}s - p \int_0^t e^{-A_s} \mathrm{d}\hat{N}_s \quad \text{for all } t \geq 0, \tag{4.5}$$

where $\hat{N}$ is a piecewise constant process that has a unit jump size each time U hits zero, M is a UI-martingale and

$$k(z) = \begin{cases} -rp & z < 0 \\ \delta - \frac{\lambda}{\rho} - rz + \lambda\left(p + \frac{1}{\rho}\right) e^{-\rho z} & z \geq 0. \end{cases} \tag{4.6}$$

Proof. Fix some $x \in \mathbb{R}$. Define a new payoff function $g^{(n)}$ for $n = 1, 2, ...,$ as

$$g^{(n)}(x) = \begin{cases} g(x) & x \geq 0 \text{ or } x < -\frac{1}{n} \\ -pnx & -\frac{1}{n} \leq x < 0. \end{cases} \tag{4.7}$$

Also, define a new process $L^{(n)}$ by replacing g in L^* with $g^{(n)}$, that is for $t \geq 0$

$$L_t^{(n)} = e^{-A_t} g^{(n)}(U_t) + q \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} g^{(n)}(U_s) \mathrm{d}s. \tag{4.8}$$

According to similar analysis as we did for L^* at the start of this subsection, we can see that $p\left(1 + \frac{q}{r}\right) \leq L_t^{(n)} \leq e^{-rt}(x + \delta t) \leq C_x < \infty$ for $t \geq 0$.

Although $g^{(n)}$ is not C^1 on $\mathbb{R}$, the standard change of variables theorem still holds for $L^{(n)}$. This can be proved by approximation from the change of variables formula in Theorem 2.9 where the objective function is assumed to be continuous.[1] Hence, by applying the change of variable formula to $L^{(n)}$, we have

$$\begin{aligned}
L_t^{(n)} &= L_0^{(n)} + M_t^{(n)} - \int_0^t e^{-A_s}\left(r + q\mathbb{1}_{\{U_s<0\}}\right) g^{(n)}(U_s)\mathrm{d}s + \delta \int_0^t e^{-A_s} g^{(n),\prime}(U_s)\mathrm{d}s \\
&\quad + \int_{[0,t]} \int_{(0,\infty)} e^{-A_s}\left(g^{(n)}(U_s - y) - g^{(n)}(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s + \int_0^t q\mathbb{1}_{\{U_s<0\}} e^{-A_s} g^{(n)}(U_s)\mathrm{d}s \\
&= L_0^{(n)} + M_t^{(n)} + \int_0^t e^{-A_s} k^{(n)}(U_s)\mathrm{d}s \\
&= L_0^{(n)} + M_t^{(n)} \\
&\quad + \int_0^t \mathbb{1}_{\{U_s \geq 0\}} e^{-A_s} k^{(n)}(U_s)\mathrm{d}s + \int_0^t \mathbb{1}_{\left\{U_s < -\frac{1}{n}\right\}} e^{-A_s} k^{(n)}(U_s)\mathrm{d}s \\
&\quad + \int_0^t \mathbb{1}_{\left\{U_s \in \left[-\frac{1}{n}, 0\right)\right\}} e^{-A_s} k^{(n)}(U_s)\mathrm{d}s,
\end{aligned} \tag{4.9}$$

where

$$M_t^{(n)} = \int_{[0,t]} \int_{(0,\infty)} e^{-A_s}\left(g^{(n)}(U_s - y) - g^{(n)}(U_s)\right) N(\mathrm{d}s \times \mathrm{d}y)$$

---

[1] Personal communication with Dr Ronnie Loeffen. More precisely, if a function f is only piecewise C^1 with a locally bounded derivative, then the change of variables formula from Theorem 2.9 still holds. To see this e.g. as follows. If x_0 is a point where f' is not continuous, then one can define a C^1 function $w^{(m)}$ such that $w^{(m)}(x) = f(x)$ when $x \in \mathbb{R} \setminus (x_0 - 1/m, x_0 + 1/m)$ while $w^{(m)}$ and $w^{(m),\prime}$ stay bounded on $(x_0 - 1/m, x_0 + 1/m)$. Then, Theorem 2.9 can be applied to f with f replaced by $w^{(m)}$ and the indicators: $\mathbb{1}_{\{U_s \in (x_0 - 1/m, x_0 + 1/m)\}}$ and $\mathbb{1}_{\{U_s \in \mathbb{R} \setminus (x_0 - 1/m, x_0 + 1/m)\}}$ will be added to the integral term with the former one equal to 0 and the latter equal to 1 when $m \to \infty$. Same approximation method can be used when analyzing $Z^{(a)}$ below in Section 4.2.4.

$$-\int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g^{(n)}(U_s-y)-g^{(n)}(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s \quad (4.10)$$

is a local martingale process, and

$$k^{(n)}(z)=\delta g^{(n),\prime}(z)-rg^{(n)}(z)+\int_{(0,\infty)}\left(g^{(n)}(z-y)-g^{(n)}(z)\right)\nu(\mathrm{d}y).$$

Then we break the remaining proof into four steps as below.

1) Determine $\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s\geq 0\}}e^{-A_s}k^{(n)}(U_s)\mathrm{d}s$ for $t\geq 0$.

When $z\geq 0$, we have

$$\begin{aligned}
k^{(n)}(z) &= \delta-(r+\lambda)z\\
&\quad+\int_0^z (z-y)\,\nu(\mathrm{d}y)+\int_z^{z+\frac{1}{n}} -pn(z-y)\nu(\mathrm{d}y)+\int_{z+\frac{1}{n}}^\infty p\Pi(\mathrm{d}y)\\
&=\delta-rz+\frac{\lambda}{\rho}\left(e^{-\rho z}-1\right)+\int_z^{z+\frac{1}{n}} -pn(z-y)\nu(\mathrm{d}y)+p\lambda e^{-\rho(z+\frac{1}{n})},
\end{aligned}$$

which shows that

$$-rz-\frac{\lambda}{\rho}+2p\lambda\leq k^{(n)}(z)\leq\delta.$$

As for any $t\geq 0$, U_t is bounded above by $x+\delta t$, it follows by dominated convergence theorem that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s\geq 0\}}e^{-A_s}k^{(n)}(U_s)\mathrm{d}s\\
&=\int_0^t \mathbb{1}_{\{U_s\geq 0\}}e^{-A_s}\lim_{n\to\infty}k^{(n)}(U_s)\mathrm{d}s\\
&=\int_0^t \mathbb{1}_{\{U_s\geq 0\}}e^{-A_s}\left(\delta-\frac{\lambda}{\rho}-rU_s+\lambda\left(p+\frac{1}{\rho}\right)e^{-\rho U_s}\right)\mathrm{d}s.
\end{aligned}$$

2) Determine $\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s<-1/n\}}e^{-A_s}k^{(n)}(U_s)\mathrm{d}s$ for $t\geq 0$.

When $z\leq -1/n$, we have $k^{(n)}(z)=-rp$, which again, by dominated convergence theorem, gives that

$$\begin{aligned}
\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s<-1/n\}}e^{-A_s}k^{(n)}(U_s)\mathrm{d}s &= \int_0^t e^{-A_s}\lim_{n\to\infty}\left(\mathbb{1}_{\{U_s<-1/n\}}k^{(n)}(U_s)\right)\mathrm{d}s\\
&=\int_0^t \mathbb{1}_{\{U_s<0\}}e^{-A_s}(-rp)\mathrm{d}s
\end{aligned}$$

3) Determine $\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s\in[-1/n,0)\}}e^{-A_s}k^{(n)}(U_s)\mathrm{d}s$ for $t\geq 0$.

When $z\in[-1/n,0)$, we have

$$k^{(n)}(z) = -\delta pn + rpnz + \int_{(0,\infty)} \left(g^{(n)}(z-y) - g^{(n)}(z)\right)\nu(\mathrm{d}y)$$

where $nz\in[-1,0)$ and $g^{(n)}(z-y)-g^{(n)}(z)\in(p,0]$, which using dominated convergence theorem gives that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s\in[-1/n,0)\}}e^{-A_s}\left(k^{(n)}(U_s)+\delta pn\right)\mathrm{d}s\\
&=\int_0^t e^{-A_s}\\
&\quad\lim_{n\to\infty}\left[\mathbb{1}_{\{U_s\in[-1/n,0)\}}\left(rpnU_s+\int_{(0,\infty)}\left(g^{(n)}(U_s-y)-g^{(n)}(U_s)\right)\nu(\mathrm{d}y)\right)\right]\mathrm{d}s\\
&=0.
\end{aligned}$$

So the only thing we need to check here is the value of

$$\lim_{n\to\infty}\int_0^t \mathbb{1}_{\{U_s\in[-1/n,0)\}}e^{-A_s}(-\delta pn)\mathrm{d}s.$$

Let us fix some $t\geq 0$, and a path $s\mapsto U_s$ which has on $[0,t]$ only finitely many jumps and finitely many hitting times of 0 a.s.. Denote by $T_1,\dots,T_k$ the hitting times of 0 with $T_0=0$ and by $J_1,\dots,J_l$ the jump times of the path.

Now, consider all the jumps in U so that $U_{J_i}<0$ and note their distance to 0. Because there are only finitely many jumps, the minimal distance is positive i.e.

$$\epsilon_1 := \inf\{|U_{J_i}|\,|\,U_{J_i}<0\}>0.$$

Next consider all the points below 0 just before a jump happens i.e. $U_{J_i-}<0$ and their distance to 0, which must be positive due to the same reason, i.e.

$$\epsilon_2 := \inf\{|U_{J_i-}|\,|\,U_{J_i-}<0\}>0.$$

Let us now take n large enough, in particular so that $1/n<\min\{\epsilon_1,\epsilon_2\}$ and then there is no interference of jumps whenever $U_s\in(-1/n,0)$. More precisely, U_s can only enter this interval by drifting in from below i.e. by hitting $-1/n$, then it keeps drifting without experiencing a jump until it exits the interval again by hitting 0. That is to say, the set on the time axis $\{s \le t \,|\, U_s \in (-1/n, 0)\}$ consists of the union of disjoint time intervals $(T_i - 1/(n\delta), T_i)$ and we can then write

$$n\int_0^t \mathbb{1}_{\{U_s\in(-1/n,0)\}} e^{-A_s}\mathrm{d}s = n\sum_{i=1}^{k}\int_{T_i-1/(n\delta)}^{T_i} e^{-A_s}\mathrm{d}s.$$

Further, for any i

$$n\int_{T_i-1/(n\delta)}^{T_i} e^{-A_s}\mathrm{d}s = e^{-A_{T_i}} n\int_{T_i-1/(n\delta)}^{T_i} e^{-(A_s-A_{T_i})}\mathrm{d}s$$

where the path $s\mapsto e^{-(A_s-A_{T_i})}$ is smooth with a value equal to 1 in $s=T_i$, that is,

$$\lim_{n\to\infty} n\int_{T_i-1/(n\delta)}^{T_i} e^{-(A_s-A_{T_i})}\mathrm{d}s = \lim_{n\to\infty} n\int_{T_i-1/(n\delta)}^{T_i} 1\mathrm{d}s = \frac{1}{\delta}.$$

So it follows that

$$\lim_{n\to\infty} n\int_0^t \mathbb{1}_{\{U_s\in(-1/n,0)\}} e^{-A_s}\mathrm{d}s = \frac{1}{\delta}\sum_{i=1}^{k} e^{-A_{T_i}} = \frac{1}{\delta}\int_0^t e^{-A_s}\mathrm{d}\hat{N}_s$$

where $\hat{N}$ is defined as the piecewise constant process that has a jump of size 1 each time U hits 0.

4) $\mathbb{E}_x\left[\int_0^\infty e^{-A_s}\mathrm{d}\hat{N}_s\right] < \infty$.

So far we have computed the limits of the integrals in (4.9). To see the behavior of $M^{(n)}$, we determine $\mathbb{E}_x\left[\int_0^\infty e^{-A_s}\mathrm{d}\hat{N}_s\right]$ first.

We set $T_k = \infty$ if path does not hit 0 after the $(k-1)-$th hitting. Then,

$$\begin{aligned}\mathbb{E}_x\left[\int_0^\infty e^{-A_s}\mathrm{d}\hat{N}_s\right] &= \mathbb{E}_x\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{T_k<\infty\}} e^{-A_{T_k}}\right]\\ &= \sum_{k=1}^{\infty} \mathbb{E}_x\left[\mathbb{1}_{\{T_k<\infty\}} e^{-A_{T_k}}\right], \qquad (4.11)\end{aligned}$$

where we apply the Monotone convergence theorem to the second equation.

Define $q_0 := \mathbb{P}_0(T_1 < \infty)$. Recall that ψ denotes the Laplace exponent of U and the following holds:

- $\psi'(0) > 0$ resp. $\psi'(0) < 0 \iff U$ drift to ∞ resp. $-\infty$ a.s $\Longrightarrow q_0 < 1$;
- $\psi'(0) = 0 \iff \limsup_{t\to\infty} U_t = -\liminf_{t\to\infty} U_t = \infty \implies q_0 = 1$.

See e.g. [30, §§8.1].

When $\psi'(0) \neq 0$, by homogeneity of U, we have

$$\mathbb{P}_x(T_k < \infty) \leq q_0^{k-1},$$

which shows that

$$(4.11) \leq \sum_{k=1}^{\infty} \mathbb{E}_x\left[\mathbb{1}_{\{T_k<\infty\}}\right] = \sum_{k=1}^{\infty} \mathbb{P}_x(T_k < \infty) \leq \sum_{k=1}^{\infty} q_0^{k-1} = \frac{1}{1-q_0}. \tag{4.12}$$

When $\psi'(0) = 0$, we have $T_k < \infty$ a.s. for all $k \geq 1$ and

$$(4.11) = \sum_{k=1}^{\infty} \mathbb{E}_x\left[e^{-A_{T_k}}\right],$$

with

$$\begin{aligned}\mathbb{E}_x\left[e^{-A_{T_k}}\right] &= \mathbb{E}_x\left[e^{-(A_{T_0}+\sum_{i=1}^{k}(A_{T_i}-A_{T_{i-1}}))}\right]\\ &= e^{-A_{T_0}}\mathbb{E}_x\left[e^{A_{T_1}-A_{T_0}}\right]\left(\mathbb{E}_x\left[e^{A_{T_2}-A_{T_1}}\right]\right)^{k-1},\end{aligned} \tag{4.13}$$

where the second equation results from the fact that $A_{T_i} - A_{T_{i-1}}$ is independent of each other for $i \geq 1$ and is identical for $i \geq 2$.

Since $\mathbb{E}_x\left[e^{A_{T_2}-A_{T_1}}\right] \in (0,1)$, by applying the geometric series formula to (4.13), we have (4.11) $< \infty$ in this case as well.

Hence, we can conclude that $\mathbb{E}_x\left[\int_0^\infty e^{-A_s}\mathrm{d}\hat{N}_s\right] < \infty$.

5) $M := \lim_{n\to\infty} M^{(n)}$ exists and is a martingale.

As a result of 1)–4), when letting $n \to \infty$, by dominated convergence theorem, we have

$$\int_0^t e^{-A_s}k^{(n)}(U_s)\mathrm{d}s \to \int_0^t e^{-A_s}k(U_s)\mathrm{d}s - p\int_0^t e^{-A_s}\mathrm{d}\hat{N}_s, \tag{4.14}$$

where

$$k(z)=\begin{cases} -rp & z<0 \\ \delta-\frac{\lambda}{\rho}-rz+\lambda\left(p+\frac{1}{\rho}\right)e^{-\rho z} & z\geq 0. \end{cases}$$

Also, (4.8) shows that $L^{(n)}$ is decreasing in n, which by applying monotone convergence theorem gives that $\lim_{n\to\infty} L_t^{(n)} \to L_t^*$.

Since for $t\geq 0$,

$$M_t^{(n)} = L_t^{(n)} - L_0^{(n)} - \int_0^t e^{-A_s} k^{(n)}(U_s)\mathrm{d}s, \quad t\geq 0,$$

by (4.14) and the existence of $\lim_{t\to\infty} L_t^{(n)}$ and $\lim_{n\to\infty} L_0^{(n)}$, it follows that $M_t^{(n)}$ has a limit (a.s.) as well, i.e., $M_t := \lim_{n\to\infty} M_t^{(n)}$ exists. Also because that $|g^{(n)}(U_s-y)-g^{(n)}(U_s)|\leq \delta s - p$ for any $s\geq 0$, it follows by Dominated convergence theorem that

$$\begin{aligned} M_t = &\int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g(U_s-y)-g(U_s)\right)N(\mathrm{d}s\times\mathrm{d}y) \\ &- \int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g(U_s-y)-g(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s, \end{aligned} \tag{4.15}$$

which according to [30, Corollary 4.6], is a martingale.

6) M in (4.15) is uniformly integrable.

First note that for $y\geq 0$, $p-z\leq g(z-y)-g(z)\leq 0$. Then together with the fact that $A_t\geq rt$ and $U_t\leq x+\delta t$, it follows

$$\begin{aligned} &\sup_t \left|\int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g(U_s-y)-g(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s\right| \\ &= \sup_t \int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g(U_s)-g(U_s-y)\right)\nu(\mathrm{d}y)\mathrm{d}s \\ &= \int_{[0,\infty)}\int_{(0,\infty)} e^{-A_s}\left(g(U_s)-g(U_s-y)\right)\nu(\mathrm{d}y)\mathrm{d}s \\ &\leq \int_0^\infty e^{-rs}(x+\delta s-p)\mathrm{d}s \\ &= \frac{x-p}{r}+\frac{\delta}{r^2}. \end{aligned} \tag{4.16}$$

Further, [30, Theorem 4.4] provides that

$$\mathbb{E}\left[\int_{[0,\infty)}\int_{(0,\infty)} e^{-A_s}\left(g(U_s)-g(U_s-y)\right)N(\mathrm{d}s\times\mathrm{d}y)\right]$$
$$=\mathbb{E}\left[\int_{[0,\infty)}\int_{(0,\infty)} e^{-A_s}\left(g(U_s)-g(U_s-y)\right)\nu(\mathrm{d}y)\mathrm{d}s\right], \quad (4.17)$$

which together with (4.16) indicates

$$\mathbb{E}\left[\sup_t\left|\int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(g(U_s-y)-g(U_s)\right)N(\mathrm{d}s\times\mathrm{d}y)\right|\right]<\infty.$$

Hence, according to [39, Theorem 52], M is a uniformly integrable martingale.

This completes the proof. □

Lemma 4.2. *For $x\in\mathbb{R}$, problem* (4.3) *is equivalent to*

$$v(x)=g(x)+\sup_\tau \mathbb{E}_x\left[\int_0^\tau e^{-A_s}k(U_s)\mathrm{d}s-p\int_0^\tau e^{-A_s}\mathrm{d}\hat{N}_s\right]. \quad (4.18)$$

Proof. First note that for any stopping time τ, as M is a UI-martingale, it follows by Doob's optional stopping theorem that

$$\mathbb{E}_x\left[M_\tau\right]=\mathbb{E}_x\left[M_0\right]=0.$$

Then the result automatically follows from (4.3) and Lemma 4.1. □

4.2.2 Value function for an up-crossing strategy

In this section, we make computations related to an up-crossing strategy with the motivation that it will be important later on. As presented in Example 2.7, the expressions for the Laplace exponent ψ and scale functions of a Cramér-Lundberg process are quite explicit. Hence, we will make use of these nice expressions to do fundamental computations, which will be used for later analysis. From now on, we use $-\eta$ and $\Phi(r)$ to denote the two real solutions to the equation $\psi(z)=r$, which satisfy

$$-\eta<0<\Phi(r),$$

and their explicit expressions can be found in Example 2.7 . Define a stopping time

$$\tau_a^+ = \inf\{t \geq 0 \,:\, U_t \geq a\}, \quad a > 0,$$

with $\inf \varnothing = \infty$. We denote the corresponding value function by

$$v_a(x) = \mathbb{E}_x \left[L^*_{\tau_a^+} \right] = \mathbb{E}_x \left[e^{-A_{\tau_a^+}} U_{\tau_a^+} + \int_0^{\tau_a^+} pq \mathbb{1}_{(U_s<0)} e^{-A_s} \mathrm{d}s \right], \quad x \in \mathbb{R},\, a > 0. \quad (4.19)$$

Let us introduce a function $\mathcal{I}^{(r,q)} : \mathbb{R} \to (0, \infty)$ defined in [40] which will be used below:

$$\mathcal{I}^{(r,q)}(x) := \int_{(0,\infty)} e^{-\Phi(r+q)u} W^{(r)}(u+x)\mathrm{d}u, \quad \text{for all } x \in \mathbb{R}. \quad (4.20)$$

which satisfies

$$\mathcal{I}^{(r,q)}(x) = e^{\Phi(r+q)x}/q \quad (4.21)$$

for all $x \leq 0$. Also, $\mathcal{I}^{(r,q)}$ is continuously twice differentiable over $(0, \infty)$ and this holds for a general spectrally negative Lévy process. See [40] for proof.

Also, to present the result below, we make use of a two variable extension $Z^{(q)}(x, \theta)$ of the scale function, introduced by [7, Definiton 5.8], which is defined as

$$Z^{(q)}(x, \theta) = e^{\theta x} + (q - \psi(\theta)) \int_0^x e^{\theta(x-y)} W^{(q)}(y)\mathrm{d}y, \quad \theta \in \mathbb{C}, \quad (4.22)$$

satisfying $Z^{(q)}(x, \theta) = e^{\theta x}$ for $x \leq 0$.

Then, we give the following result of v_a. Note that even though we work with a Cramér-Lundberg process, expressions below in Lemma 4.3 remain valid when U is any spectrally negative Lévy process.

Lemma 4.3. *For $a > 0$, we have*

$$v_a(x) = \begin{cases} a \cdot \dfrac{e^{\Phi(r+q)x}}{q\mathcal{I}^{(r,q)}(a)} + pqh^-(x; a) & x \leq 0 \\ a \cdot \dfrac{\mathcal{I}^{(r,q)}(x)}{\mathcal{I}^{(r,q)}(a)} + pqh^+(x; a) & 0 < x < a \\ x & x \geq a, \end{cases} \quad (4.23)$$

where

$$h^-(x;a) = \frac{1}{r+q}\left(1 - e^{\Phi(r+q)x}\cdot\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right)$$

and

$$h^+(x;a) = \frac{1}{r+q}\left(Z^{(r)}(x) - Z^{(r)}(x,\theta)\cdot\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right),$$

with $Z^{(r)}(x,\theta)$ *defined in* (4.22) *and* $\theta = \Phi(r+q)$.

Proof. In the first place, immediately by Proposition 4.1 in [40], we have

$$\mathbb{E}_x\left[e^{-A_{\tau_a^+}}U_{\tau_a^+}\right] = a\cdot\frac{\mathcal{I}^{(r,q)}(x)}{\mathcal{I}^{(r,q)}(a)}, \quad \text{for all } x < a. \tag{4.24}$$

Then, (4.19) can be decomposed into

$$v_a(x) = \begin{cases} a\cdot\frac{\mathcal{I}^{(r,q)}(x)}{\mathcal{I}^{(r,q)}(a)} + pqh(x;a) & x < a \\ g(x) & x \geq a, \end{cases} \tag{4.25}$$

where

$$h(x;a) := \mathbb{E}_x\left[\int_0^{\tau_a^+} \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s\right] = \begin{cases} h^-(x;a) & x \leq 0 \\ h^+(x;a) & 0 < x < a. \end{cases} \tag{4.26}$$

Below we break into six steps to complete the proof.

1) $h(\cdot,a)$ is continuous on $(-\infty,a)$.

   We can write

$$h(x;a) = \mathbb{E}_0\left[\int_0^{\tau_{a-x}^+} \mathbb{1}_{\{x+U_s<0\}}e^{-A_s^{(x)}}\mathrm{d}s\right] = \mathbb{E}_0\left[\int_0^{\infty} \mathbb{1}_{\{s\leq\tau_{a-x}^+\}}\mathbb{1}_{\{x+U_s<0\}}e^{-A_s^{(x)}}\mathrm{d}s\right]$$

   where $A_t^{(x)} = rt + \int_0^t \omega(U_s+x)\mathrm{d}s$.

   Fix some $x \in (-\infty,a)$ and let $\{x_k; k = 1,2,...\}$ be a sequence which converges to x as k tends to infinity. Then, it holds a.s. that for $s \geq 0$,

$$\mathbb{1}_{\{s\leq\tau_{a-x_k}^+\}}\mathbb{1}_{\{x_k+U_s<0\}}e^{-A_s^{(x_k)}} \to \mathbb{1}_{\{s\leq\tau_{a-x}^+\}}\mathbb{1}_{\{x+U_s<0\}}e^{-A_s^{(x)}}$$

   when $k \to \infty$.

   Since $\int_0^\infty e^{-A_s^{(x)}}\mathrm{d}s \leq \int_0^\infty e^{-rs}\mathrm{d}s = \frac{1}{r}$, by dominated convergence theorem we also have that $h(x_k;a) \to h(x;a)$ as $k \to \infty$.

2) An expression for $h^-(x;a)$.

Denote $\tau_0^- = \inf\{t > 0 : U_t < 0\}$. When $x \in (-\infty, 0)$, by the strong Markov property of X, we have

$$h^-(x;a) = \mathbb{E}_x\left[\int_0^{\tau_0^+} e^{-(r+q)s}\mathrm{d}s + e^{-(r+q)\tau_0^+}h(0;a)\right]$$
$$= \mathbb{E}_x\left[\int_0^{\tau_0^+} e^{-(r+q)s}\mathrm{d}s\right] + h(0;a)\mathbb{E}_x\left[e^{-(r+q)\tau_0^+}\mathbb{1}_{\{\tau_0^+<\infty\}}\right]. \quad (4.27)$$

By Theorem 2.6, we have

$$\mathbb{E}_x\left[e^{-q\tau_0^+}\mathbb{1}_{(\tau_0^+<\infty)}\right] = e^{\Phi(q)x}$$

which can be used to replace the second term in (4.27). For the first term, we have

$$\mathbb{E}_x\left[\int_0^{\tau_0^+} e^{-(r+q)s}\mathrm{d}s\right] = \frac{1}{r+q}\left(1 - \mathbb{E}_x\left[e^{-(r+q)\tau_0^+}\right]\right).$$

Hence, an expression for $h^-(\cdot;a)$ is

$$h^-(x;a) = \frac{1}{r+q} + \left(h(0;a) - \frac{1}{r+q}\right)e^{\Phi(r+q)x}. \quad (4.28)$$

3) An expression for $h^+(x;a)$.

When $x \in (0, a]$, it follows by the Markov property of U and A that

$$\begin{aligned}
&h^+(x;a)\\
&= \mathbb{E}_x\left[\int_0^{\tau_a^+} \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s\right]\\
&= \mathbb{E}_x\left[\mathbb{1}_{(\tau_a^+<\tau_0^-)}\int_0^{\tau_a^+} \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s\right] + \mathbb{E}_x\left[\mathbb{1}_{(\tau_0^-<\tau_a^+)}\int_0^{\tau_a^+} \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s\right]\\
&= \mathbb{E}_x\left[\mathbb{1}_{(\tau_0^-<\tau_a^+)}e^{-r\tau_0^-}\mathbb{E}_{U_{\tau_0^-}}\left[\int_0^{\tau_a^+} \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s\right]\right]\\
&= \mathbb{E}_x\left[\mathbb{1}_{(\tau_0^-<\tau_a^+)}e^{-r\tau_0^-}h^-(U_{\tau_0^-};a)\right].
\end{aligned}$$

Then by plugging in the expression for $h^-(\cdot; a)$ in (4.28), we have

$$\begin{aligned} & h^+(x;a) \\ &= \frac{1}{r+q}\mathbb{E}_x\left[\mathbb{1}_{(\tau_0^- < \tau_a^+)} e^{-r\tau_0^-}\right] \\ &\quad + \left(h(0;a) - \frac{1}{r+q}\right)\mathbb{E}_x\left[\mathbb{1}_{(\tau_0^- < \tau_a^+)} e^{-r\tau_0^- + \Phi(r+q)U_{\tau_0^-}}\right] \\ &= \frac{1}{r+q}\left(Z^{(r)}(x) - Z^{(r)}(a)\cdot\frac{W^{(r)}(x)}{W^{(r)}(a)}\right) \\ &\quad + \left(h(0;a) - \frac{1}{r+q}\right)\left(Z^{(r)}\left(x, \Phi(r+q)\right) - Z^{(r)}\left(a, \Phi(r+q)\right)\cdot\frac{W^{(r)}(x)}{W^{(r)}(a)}\right) \end{aligned} \tag{4.29}$$

where the second equation refers to [7, Definition 5.8].

4) Conclusion.

Consider equation (4.29). By 1), it holds that for $x \downarrow 0$

$$h^+(x;a) = h(x;a) \to h(0;a).$$

Hence, we get from (4.29) with this limit an expression for $h(0;a)$:

$$h(0;a) = \frac{1}{r+q}\left(1 - \frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right). \tag{4.30}$$

By plugging (4.30) in (4.28) and (4.29), we have

$$h^-(x;a) = \frac{1}{r+q}\left(1 - e^{\Phi(r+q)x}\cdot\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right) \tag{4.31}$$

and

$$h^+(x;a) = \frac{1}{r+q}\left(Z^{(r)}(x) - Z^{(r)}(x,\theta)\cdot\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right). \tag{4.32}$$

By plugging (4.31) and (4.32) into (4.26) and then into (4.25), we complete the proof.

□

A computation for v_a when $a = 0$ can be made based on the result in Lemma 4.3.

Considering (4.28) with $h(0;a)=0$, we can get that

$$v_0(x)=\mathbb{E}_x[L^*_{\tau_0^+}]=\begin{cases}\dfrac{pq}{r+q}\left(1-e^{\Phi(r+q)x}\right) & \text{if } x<0\\ x & \text{if } x\geq 0.\end{cases} \tag{4.33}$$

Note that since we are dealing with Cramér-Lundberg processes, scale functions and $\mathcal{I}^{(r,q)}$ can be then expressed quite explicitly in Lemma 4.4 below.

Lemma 4.4. *For* $r,q>0$, *it holds that*

$$\mathcal{I}^{(r,q)}(x)=\frac{e^{\Phi(r)x}}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)}+\frac{e^{-\eta x}}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)},\quad \textit{for } x\geq 0, \tag{4.34}$$

with

$$\mathcal{I}^{(r,q)}(0)=\frac{1}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)}+\frac{1}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}=\frac{1}{q}. \tag{4.35}$$

Also, for $x,r,q>0$,

$$Z^{(r)}(x)=1+r\left(\frac{e^{\Phi(r)x}-1}{\psi'(\Phi(r))\Phi(r)}-\frac{e^{-\eta x}-1}{\psi'(-\eta)\eta}\right); \tag{4.36}$$

$$Z^{(r)}(x,\Phi(r+q))=q\mathcal{I}^{(r,q)}(x). \tag{4.37}$$

Proof. The expression for $\mathcal{I}^{(r,q)}(x)$ can be achieved directly by plugging the explicit expressions for scale function (see (2.7)) into (4.20), i.e.

$$\begin{aligned}\mathcal{I}^{(r,q)}(x)&=\int_{(0,\infty)}e^{-\Phi(r+q)u}\left(\frac{e^{\Phi(r)(x+u)}}{\psi'(\Phi(r))}+\frac{e^{-\eta(x+u)}}{\psi'(-\eta)}\right)\mathrm{d}u\\&=\frac{e^{\Phi(r)x}}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)}+\frac{e^{-\eta x}}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}.\end{aligned} \tag{4.38}$$

Note that (4.38) holds for any $x\geq 0$. Also, it follows from the definition of $\mathcal{I}^{(r,q)}$ in (4.20) that $\mathcal{I}^{(r,q)}(0)=1/q$. As a result, we have

$$\mathcal{I}^{(r,q)}(0)=\frac{1}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)}+\frac{1}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}=\frac{1}{q}.$$

The expression for $Z^{(r)}(x)$ is an immediate result using the explicit expression for $r-$scale function $W^{(r)}$ shown in Example 2.7.

For $Z^{(r)}(x, \Phi(r+q))$, we first have that

$$\begin{aligned}
&Z^{(r)}(x, \Phi(r+q)) \\
&= q\left(\frac{e^{\Phi(r)x}}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)} + \frac{e^{-\eta x}}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}\right) \\
&\quad + e^{\Phi(r+q)x}\left[1 - q\left(\frac{1}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)} + \frac{1}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}\right)\right], \quad (4.39)
\end{aligned}$$

which according to the calculations on $\mathcal{I}^{(r,q)}$ above can be simplified as

$$\begin{aligned}
Z^{(r)}(x, \Phi(r+q)) &= q\left(\frac{e^{\Phi(r)x}}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)} + \frac{e^{-\eta x}}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)}\right) \\
&= q\mathcal{I}^{(r,q)}(x).
\end{aligned}$$

□

Now in the following corollary, we give the nice expressions for v_a.

Corollary 4.5. *When $a > 0$, v_a in* (4.23) *can be explicitly expressed as*

$$v_a(x) = \begin{cases} \frac{pq}{r+q} + C_1^{(a)} e^{\Phi(r+q)x} & x \le 0 \\ C_2^{(a)} e^{\Phi(r)x} + C_3^{(a)} e^{-\eta x} & 0 < x < a \\ x & x \ge a, \end{cases} \quad (4.40)$$

where

$$\begin{aligned}
C_1^{(a)} &= \frac{a}{q\mathcal{I}^{(r,q)}(a)} - \frac{pq}{r+q}\frac{Z^{(r)}(a)}{Z^{(r)}(a, \Phi(r+q))}; \\
C_2^{(a)} &= \frac{a}{\mathcal{I}^{(r,q)}(a)}\frac{1}{\psi'(\Phi(r))\left(\Phi(r+q)-\Phi(r)\right)} + \frac{pqr}{(r+q)\psi'(\Phi(r))\Phi(r)} \\
&\quad + \frac{pq}{r+q}\frac{Z^{(r)}(a)}{\mathcal{I}^{(r,q)}(a)}\frac{1}{(\Phi(r)-\Phi(r+q))\psi'(\Phi(r))} \\
C_3^{(a)} &= \frac{a}{\mathcal{I}^{(r,q)}(a)}\frac{1}{\psi'(-\eta)\left(\Phi(r+q)+\eta\right)} - \frac{pqr}{(r+q)\psi'(-\eta)\eta} \\
&\quad - \frac{pq}{r+q}\frac{Z^{(r)}(a)}{\mathcal{I}^{(r,q)}(a)}\frac{1}{(\eta+\Phi(r+q))\psi'(-\eta)}.
\end{aligned}$$

In the special case where v_a satisfies $v_a'(a-) = 1$ in addition to $v_a(a) = a$, i.e. smooth pasting holds which we will also discuss later, we have the simpler expressions

$$C_2^{(a)} = e^{-\Phi(r)a}\frac{a\eta + 1}{\Phi(r)+\eta};$$

$$C_3^{(a)} = e^{\eta a} \frac{a\Phi(r) - 1}{\Phi(r) + \eta}.$$

Proof. Then, replacing $\mathcal{I}^{(r,q)}$ and scale functions in (4.23) with their explicit expressions shown in Lemma 4.4 and above gives the expression for v_a.

When v_a satisfies smooth pasting condition, we have

$$\begin{cases} v_a(a) = C_2^{(a)} e^{\Phi(r)a} + C_3^{(a)} e^{-\eta a} = a \\ v_a'(a-) = C_2^{(a)} \Phi(r) e^{\Phi(r)a} - C_3^{(a)} \eta e^{-\eta a} = 1, \end{cases}$$

which together provide simper expressions for $C_2^{(a)}$ and $C_3^{(a)}$. □

We can see from (4.40) that v_a is C^∞ on $(-\infty, 0) \cup (0, a) \cup (a, \infty)$ and continuous on $\mathbb{R}$ with finite left and right derivatives of all orders in $x = 0$ and $x = a$.

4.2.3 First-order condition versus smooth fit

Later we will see that an up-crossing strategy is optimal for certain parameter regimes. This subsection contains some useful results for that later analysis.

Proposition 4.6 (Derivative of v_a with respect to a)**.** *Given x, for any positive $a \geq x$, we have*

$$\frac{\partial}{\partial a} v_a(x) = \mathcal{I}^{(r,q)}(x) t(a),$$

where

$$t(a) = \frac{\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),\prime}(a) - \frac{pq}{r+q}\left(\mathcal{I}^{(r,q)}(a) Z^{(r),\prime}(a) - Z^{(r)}(a)\mathcal{I}^{(r,q),\prime}(a)\right)}{\left(\mathcal{I}^{(r,q)}(a)\right)^2}.$$

Proof. For $a \geq x$ in the case that $a > 0$, by taking the derivative of v_a in (4.23) with respect to a, we have

$$\begin{aligned} \frac{\partial}{\partial a} v_a(x) = \mathcal{I}^{(r,q)}(x) \cdot \frac{\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),\prime}(a)}{\left(\mathcal{I}^{(r,q)}(a)\right)^2} \\ - \frac{pqZ^{(r)}(x,\theta)}{r+q} \cdot \frac{Z^{(r),\prime}(a) Z^{(r)}(a,\theta) - Z^{(r),\prime}(a,\theta) Z^{(r)}(a)}{\left(Z^{(r)}(a,\theta)\right)^2}. \end{aligned} \tag{4.41}$$

Then, by Corollary 4.5, (4.41) can be written as

$$\frac{\partial}{\partial a} v_a(x) = \mathcal{I}^{(r,q)}(x) \cdot t(a)$$

where

$$\begin{aligned} t(a) &= \left[\frac{\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),'}(a)}{\left(\mathcal{I}^{(r,q)}(a)\right)^2} - \frac{pq^2}{r+q} \frac{Z^{(r),'}(a)Z^{(r)}(a,\theta) - Z^{(r),'}(a,\theta)Z^{(r)}(a)}{\left(Z^{(r)}(a,\theta)\right)^2} \right] \\ &= \frac{\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),'}(a) - \frac{pq}{r+q}\left(\mathcal{I}^{(r,q)}(a)Z^{(r),'}(a) - Z^{(r)}(a)\mathcal{I}^{(r,q),'}(a)\right)}{\left(\mathcal{I}^{(r,q)}(a)\right)^2}. \end{aligned}$$

□

Proposition 4.6 indicates that if an up-crossing strategy is optimal, then the optimal choice for the up-crossing level a is independent of x and should be the root of

$$\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),'}(a) - \frac{pq}{r+q}\left(\mathcal{I}^{(r,q)}(a)Z^{(r),'}(a) - Z^{(r)}(a)\mathcal{I}^{(r,q),'}(a)\right) = 0. \tag{4.42}$$

It turns out that the solution to (4.42) also has the property that the corresponding value function v_a connects smoothly with g in $x = a$, in the sense that

$$v_a(a-) = v_a(a) = g(a) = a \quad \text{and} \quad v_a^{'}(a-) = g^{'}(a) = 1$$

as we will see next.

Note that $v_a(a-) = g(a)$ is true for any $a > 0$ if U is regular upwards (See e.g. [30, Definition 6.4]) like the Cramér-Lundberg process. And in such cases, a well-known rule of thumb in optimal stopping is that the optimal choice for a is determined by smooth pasting, see e.g. [38].

By taking the first derivative of $v_a(x)$ with respect to x and letting $x \uparrow a$, we have

$$\begin{aligned} v_a^{'}(a-) &= \frac{a}{\mathcal{I}^{(r,q)}(a)}\mathcal{I}^{(r,q),'}(a-) + \frac{pq}{r+q}\left(Z^{(r),'}(a) - Z^{(r),'}(a,\theta)\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right) \\ &= \frac{a}{\mathcal{I}^{(r,q)}(a)}\mathcal{I}^{(r,q),'}(a-) + \frac{pq}{r+q}\left(Z^{(r),'}(a) - \mathcal{I}^{(r,q),'}(a)\frac{Z^{(r)}(a)}{\mathcal{I}^{(r,q)}(a)}\right). \end{aligned}$$

The smooth pasting condition requires that $1 - v_a^{'}(a-) = 0$, that is

$$\mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),'}(a-) - \frac{pq}{r+q}\left(Z^{(r),'}(a)\mathcal{I}^{(r,q)}(a) - \mathcal{I}^{(r,q),'}(a)Z^{(r)}(a)\right) = 0,$$

which is exactly the same as (4.42).

Based on this, let us define the function $J:(0,\infty)\to\mathbb{R}$ as

$$J(a) = 1 - v_a^{'}(a-) \tag{4.43}$$

which charaterizes the possible optimal level if a root for (4.43) exists on $[0,\infty)$. Below we look at the behaviour of J at zero and $+\infty$.

Lemma 4.7. *For $a>0$, J defined in* (4.43) *can be explicitly expressed as*

$$J(a) = 1 - C_2^{(a)}\Phi(r)e^{\Phi(r)a} + C_3^{(a)}\eta e^{-\eta a} \tag{4.44}$$

where $C_2^{(a)}$ and $C_3^{(a)}$ are given in Corollary 4.5.

Further, it holds that

$$J(0+) \coloneqq \lim_{a\downarrow 0} J(a) = \frac{1}{q} - p\left(\frac{1}{\delta} - \frac{\Phi(r+q)}{r+q}\right), \tag{4.45}$$

and

$$\lim_{a\to\infty} J(a) = -\infty. \tag{4.46}$$

Proof. When $a>0$, we have

$$J(a) = \mathcal{I}^{(r,q)}(a) - a\mathcal{I}^{(r,q),'}(a) - \frac{pq}{r+q}\left(\mathcal{I}^{(r,q)}(a)Z^{(r),'}(a) - Z^{(r)}(a)\mathcal{I}^{(r,q),'}(a)\right), \tag{4.47}$$

which by expressions in Example 2.7, we have

$$\begin{aligned}
J(0+) &= \mathcal{I}^{(r,q)}(0) - \frac{pq}{r+q}\left(\mathcal{I}^{(r,q)}(0)Z^{(r),'}(0) - Z^{(r)}(0)\mathcal{I}^{(r,q),'}(0+)\right)\\
&= \frac{1}{q} - \frac{pq}{r+q}\left(\frac{rW^{(r)}(0)}{q} - \frac{\Phi(r+q)}{q} + W^{(r)}(0)\right)\\
&= \frac{1}{q} - \frac{pq}{r+q}\left(\frac{r}{\delta q} - \frac{\Phi(r+q)}{q} + \frac{1}{\delta}\right)\\
&= \frac{1}{q} - p\left(\frac{1}{\delta} - \frac{\Phi(r+q)}{r+q}\right).
\end{aligned}$$

Further, let (4.47) divided by $\mathcal{I}^{(r,q)}(a)$ and $a\to\infty$, we have

$$\lim_{a\to\infty}\frac{J(a)}{\mathcal{I}^{(r,q)}(a)} = \lim_{a\to\infty}\left\{1 - a\frac{\mathcal{I}^{(r,q),'}(a)}{\mathcal{I}^{(r,q)}(a)} - \frac{pq}{r+q}\left(Z^{(r),'}(a) - Z^{(r)}(a)\frac{\mathcal{I}^{(r,q),'}(a)}{\mathcal{I}^{(r,q)}(a)}\right)\right\}$$

$$
\begin{aligned}
&= \lim_{a\to\infty} \left\{ 1 - a\overline{\Lambda}(a) - \frac{pq}{r+q}\left(Z^{(r),\prime}(a) - Z^{(r)}(a)\overline{\Lambda}(a) \right) \right\} \\
&= \lim_{a\to\infty} \left\{ 1 - a\overline{\Lambda}(a) - \frac{pq}{r+q}\left(rW^{(r)}(a) - Z^{(r)}(a)\overline{\Lambda}(a) \right) \right\} \\
&= \lim_{a\to\infty} \left\{ 1 - a\overline{\Lambda}(a) - \frac{pq}{r+q}W^{(r)}(a)\left(r - \frac{Z^{(r)}(a)}{W^{(r)}(a)}\overline{\Lambda}(a) \right) \right\}
\end{aligned}
$$

where $\overline{\Lambda}(a) = \frac{\mathcal{I}^{(r,q),\prime}(a)}{\mathcal{I}^{(r,q)}(a)}$ is strictly decreasing over $[0,\infty)$ with $\overline{\Lambda}(\infty) = \Phi(r)$ (see [40, Lemma 4.2]). Also, (2.7) shows that $\frac{Z^{(r)}(a)}{W^{(r)}(a)}$ is decreasing in a on $(0,\infty)$ and [29, Lemma 3.3] gives that

$$
\lim_{a\to\infty} Z^{(r)}(a)/W^{(r)}(a) = r/\Phi(r).
$$

Then, combining all these together, we can see that $\lim_{a\to\infty} J(a) = -\infty$. □

Note that although we do not use Proposition 4.6 later on, it is a nice result that may well still be useful in solving the cases we haven't been able to solve yet.

4.2.4 Analysis of a candidate value process

If we use an up-crossing strategy with level a as our stopping time, then the corresponding value process is the process $Z^{(a)} = \left(Z_t^{(a)}\right)_{t\geq 0}$ defined as

$$
Z_t^{(a)} = e^{-rt}v_a(U_t) + pq\int_0^t \mathbb{1}_{\{U_s<0\}}e^{-A_s}\mathrm{d}s \quad \text{for all } t \geq 0. \tag{4.48}
$$

In this subsection, we look at some useful properties of this process.

By the same justification as in Lemma 4.1, we can apply Itô's formula to $Z^{(a)}$, which gives that

$$
\begin{aligned}
Z_t^{(a)} &= Z_0^{(a)} - \int_0^t e^{-A_s}\left(r + q\mathbb{1}_{(U_s<0)}\right)v_a(U_s)\mathrm{d}s + \delta\int_0^t e^{-A_s}v_a'(U_s)\mathrm{d}s \\
&\quad + \int_0^t\int_0^\infty e^{-A_s}\left(v_a(U_s-y) - v_a(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s + pq\int_0^t \mathbb{1}_{(U_s<0)}e^{-A_s}\mathrm{d}s \\
&\quad + \widetilde{M}_t \\
&= Z_0 + \int_0^t e^{-A_s}k_a(U_s)\mathrm{d}s + \widetilde{M}_t,
\end{aligned}
$$

where

$$k_a(z) \coloneqq \delta v_a'(z) - \left(r + q\mathbb{1}_{(z<0)}\right) v_a(z) + pq\mathbb{1}_{(z<0)} + \int_0^\infty \left(v_a(z-y) - v_a(z)\right)\nu(\mathrm{d}y) \tag{4.49}$$

and

$$\widetilde{M}_t = \int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(v_a(U_s - y) - v_a(U_s)\right) N(\mathrm{d}s \times \mathrm{d}y) - \int_{[0,t]}\int_{(0,\infty)} e^{-A_s}\left(v_a(U_s - y) - v_a(U_s)\right)\nu(\mathrm{d}y)\mathrm{d}s$$

is a martingale which can also be verified following same steps as in Lemma 4.1.

Below we provide a result related to k_a.

Lemma 4.8. *For any $a > 0$, it holds that*

$$k_a(z) = \delta - rz - \frac{\lambda}{\rho} + B_a e^{-\rho z} \quad \textit{for } z > a, \tag{4.50}$$

where B_a is a constant which depends on a.

Also, for $a = 0$, we have

$$k_0(z) = \delta - rz - \frac{\lambda}{\rho} + \lambda e^{-\rho z}\left(\frac{1}{\rho} + \frac{pq}{r+q}\frac{\Phi(r+q)}{\Phi(r+q)+\rho}\right) \quad \textit{for } z > 0. \tag{4.51}$$

Proof. For $z > a$ with $a > 0$ given and fixed, by plugging the expression (4.23) for v_a into (4.49), we have

$$\begin{aligned}
k_a(z) &= \delta - rz - z\int_0^\infty \lambda\rho e^{-\rho y}\mathrm{d}y + \int_0^\infty v_a(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y \\
&= \delta - (r+\lambda)z + \int_0^{z-a}(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y \\
&\quad + \int_{z-a}^{z} v_a(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y + \int_z^\infty v_a(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y \\
&= \delta - rz - \frac{\lambda}{\rho} + \left(\frac{\lambda}{\rho} - a\lambda\right)e^{\rho a}e^{-\rho z} \\
&\quad + \int_z^\infty \left(\frac{a}{q\mathcal{I}^{(r,q)}(a)}e^{\Phi(r+q)(z-y)} + \frac{pq}{r+q}\left(1 - \frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}e^{\theta(z-y)}\right)\right)\lambda\rho e^{-\rho y}\mathrm{d}y \\
&\quad + \int_{z-a}^{z}\frac{a}{\mathcal{I}^{(r,q)}(a)}\mathcal{I}^{(r,q)}(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y
\end{aligned}$$

$$+\int_{z-a}^{z}\frac{pq}{r+q}\left(Z^{(r)}(z-y)-\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}Z^{(r)}(z-y,\theta)\right)\lambda\rho e^{-\rho y}\mathrm{d}y,$$

where $\theta=\Phi(r+q)$.

By calculating the integrals separately, we have

$$\int_{z}^{\infty}e^{\theta z}e^{-(\theta+\rho)y}\mathrm{d}y=\frac{1}{\theta+\rho}e^{-\rho z};$$
$$\int_{z}^{\infty}\left(1-\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}e^{\theta(z-y)}\right)e^{-\rho y}\mathrm{d}y=\left(\frac{1}{\rho}-\frac{1}{\theta+\rho}\frac{Z^{(r)}(a)}{Z^{(r)}(a,\theta)}\right)e^{-\rho z};$$
$$\int_{z-1}^{z}\mathcal{I}^{(r,q)}(z-y)e^{-\rho y}\mathrm{d}y=\left[\frac{A_1}{\Phi(r)+\rho}\left(e^{(\rho+\Phi(r))a}-1\right)+\frac{A_2}{\rho-\eta}\left(e^{(\rho-\eta)a}-1\right)\right]e^{-\rho z},$$

where $A_1=\frac{1}{\psi'(\Phi(r))(\theta-\Phi(r))}$ and $A_2=\frac{1}{\psi'(-\eta)(\theta+\eta)}$;

$$\begin{aligned}\int_{z-a}^{z}Z^{(r)}(z-y)e^{-\rho y}\mathrm{d}y=&\left(1-r\left(\frac{1}{\psi'(\Phi(r))\Phi(r)}-\frac{1}{\psi'(-\eta)\eta}\right)\right)\frac{1}{\rho}\left(e^{\rho a}-1\right)e^{-\rho z}\\&+\frac{r}{\psi'(\Phi(r))\Phi(r)\left(\Phi(r)+\rho\right)}\left(e^{a(\Phi(r)+\rho)}-1\right)e^{-\rho z}\\&-\frac{r}{\psi'(-\eta)\eta\left(\rho-\eta\right)}\left(e^{a(\rho-\eta)}-1\right)e^{-\rho z};\end{aligned}$$

$$\begin{aligned}&\int_{z-a}^{z}Z^{(r)}(z-y,\theta)e^{-\rho y}\mathrm{d}y\\=&\left(1+q\left(\frac{1}{\psi'(\Phi(r))\left(\Phi(r)-\theta\right)}-\frac{1}{\psi'(-\eta)\left(\theta+\eta\right)}\right)\right)\frac{1}{\theta+\rho}\left(e^{(\theta+\rho)a}-1\right)e^{-\rho z}\\&+\frac{q}{\psi'(-\eta)(\theta+\eta)(\rho-\eta)}\left(e^{a(\rho-\eta)}-1\right)e^{-\rho z}\\&-\frac{q}{\psi'(\Phi(r))(\Phi(r)-\theta)(\rho+\Phi(r))}\left(e^{a(\Phi(r)+\rho)}-1\right)e^{-\rho z}.\end{aligned}$$

From the calculations of these integrals, we can conclude that

$$k_a(z)=\delta-rz-\frac{\lambda}{\rho}+B_a e^{-\rho z},$$

where B_a is a constant in terms of the those derived above and it depends on a.

When $a=0$, we simply need to replace the expression for v_a in (4.49) with v_0 defined in (4.33), which gives that for any $z\geq 0$,

$$k_0(z)=\delta-rz-\frac{\lambda}{\rho}+\lambda e^{-\rho z}\left(\frac{1}{\rho}+\frac{pq}{r+q}\frac{\Phi(r+q)}{\Phi(r+q)+\rho}\right).\qquad\square$$

One thing worth pointing out is that the reason that the expression (4.50) for k_a is so relatively simple is the fact that U is a Cramér-Lundberg process with exponential jumps.

Lemma 4.9. *For $z \in \mathbb{R}$, the stopped process $\left(Z^{(a)}_{t\wedge\tau_a^+}\right)_{t\geq 0}$ is a $\mathbb{P}_z$−martingale.*

Proof. Define a new process $H = (H_t)_{t\geq 0}$ as $H_t = \mathbb{E}_x\left[M|\mathscr{F}_t\right]$ where M is a $\mathscr{F}_\infty$-measurable and integrable random variable, then it is well known that H is a martingale. Based on this, we prove the result as follows.

For $x \in \mathbb{R}$, let $M = e^{-A_{\tau_a^+}} v_a(U_{\tau_a^+}) + pq\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s$ and by plugging this into H we have

$$
\begin{aligned}
H_t &= \mathbb{E}_x\left[e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right) + pq\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s\middle|\mathscr{F}_t\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+\leq t\}}\left(e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right) + pq\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s\right)\middle|\mathscr{F}_t\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+> t\}}\left(e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right) + pq\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s\right)\middle|\mathscr{F}_t\right]. \qquad (4.52)
\end{aligned}
$$

The first expectation in the second equation of (4.52) can easily be simplified as

$$
\mathbb{1}_{\{\tau_a^+\leq t\}}\left(e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right) + pq\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s\right) \qquad (4.53)
$$

since it is $\mathscr{F}_t$−measurable.

For the second expectation, we split the calculation in two parts. For the first one,

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+> t\}} e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right)\middle|\mathscr{F}_t\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+> t\}} e^{-r\left(t+(\tau_a^+-t)\right)-q\left(\int_0^t \mathbb{1}_{\{U_s<0\}}\mathrm{d}s + \int_t^{\tau_a^+} \mathbb{1}_{\{U_s<0\}}\mathrm{d}s\right)} v_a\left(U_{\tau_a^+}\right)\middle|\mathscr{F}_t\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+> t\}} e^{-A_t}\cdot e^{-r\left(\tau_a^+-t\right)-q\int_t^{\tau_a^+} \mathbb{1}_{\{U_s<0\}}\mathrm{d}s} v_a\left(U_{\tau_a^+}\right)\middle|\mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau_a^+> t\}} e^{-A_t}\mathbb{E}_{U_t}\left[e^{-A_{\tau_a^+}} v_a\left(U_{\tau_a^+}\right)\right], \qquad (4.54)
\end{aligned}
$$

where the last equation comes from the Markov property of U and A.

Then, for the second one, we derive that

$$
\mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+> t\}}\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s\middle|\mathscr{F}_t\right]
$$

$$
\begin{aligned}
&= \mathbb{E}_x \left[\mathbb{1}_{\{\tau_a^+ > t\}} \left(\int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s + \int_t^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right) \middle| \mathscr{F}_t \right] \\
&= \mathbb{1}_{\{\tau_a^+ > t\}} \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s + \mathbb{E}_x \left[\mathbb{1}_{\{\tau_a^+ > t\}} \int_t^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \middle| \mathscr{F}_t \right] \\
&= \mathbb{1}_{\{\tau_a^+ > t\}} \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s + \mathbb{E}_x \left[\mathbb{1}_{\{\tau_a^+ > t\}} e^{-A_t} \int_t^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-(A_t - A_s)} \mathrm{d}s \middle| \mathscr{F}_t \right] \\
&= \mathbb{1}_{\{\tau_a^+ > t\}} \left[\int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s + e^{-A_t} \mathbb{E}_{U_t} \left[\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right] \right], \qquad (4.55)
\end{aligned}
$$

where again we make use of the Markov property.

Following by plugging (4.53), (4.54) and (4.55) into (4.52), we have

$$
\begin{aligned}
H_t = & \ \mathbb{1}_{\{\tau_a^+ \le t\}} \left(e^{-A_{\tau_a^+}} v_a \left(U_{\tau_a^+} \right) + pq \int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right) \\
& + \mathbb{1}_{\{\tau_a^+ > t\}} e^{-A_t} \mathbb{E}_{U_t} \left[e^{-A_{\tau_a^+}} v_a \left(U_{\tau_a^+} \right) \right] \\
& + pq \mathbb{1}_{\{\tau_a^+ > t\}} \left[\int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s + e^{-A_t} \mathbb{E}_{U_t} \left[\int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right] \right] \\
= & \ \mathbb{1}_{\{\tau_a^+ \le t\}} \left(e^{-A_{\tau_a^+}} v_a \left(U_{\tau_a^+} \right) + pq \int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right) \\
& + \mathbb{1}_{\{\tau_a^+ > t\}} e^{-A_t} \mathbb{E}_{U_t} \left[e^{-A_{\tau_a^+}} v_a \left(U_{\tau_a^+} \right) + pq \int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right] \\
& + pq \mathbb{1}_{\{\tau_a^+ > t\}} \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \\
= & \ \mathbb{1}_{\{\tau_a^+ \le t\}} \left(e^{-A_{\tau_a^+}} v_a \left(U_{\tau_a^+} \right) + pq \int_0^{\tau_a^+} \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right) \\
& + \mathbb{1}_{\{\tau_a^+ > t\}} \left(e^{-A_t} v_a (U_t) + pq \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s \right) \\
= & \ Z^{(a)}_{t \wedge \tau_a^+},
\end{aligned}
$$

which is a martingale as we discussed at that beginning.

This completes the proof. □

4.3 Partitioning the parameter space

So far we have defined three important functions: drift function k of the process L^* in (4.6), smooth pasting function J in (4.43) and drift function k_a of the process $Z^{(a)}$ in Lemma 4.8. In this section we discuss a partition of the parameter space. We focus on expressing the parts in terms of the penalty parameter $p \in (-\infty, 0)$, and base it on the sign of the maximum of k and k_0.

First we consider the condition

$$k(x) = \delta - \frac{\lambda}{\rho} - rx + \lambda e^{-\rho x}\left(\frac{1}{\rho} + p\right) \leq 0 \quad \text{for all } x \geq 0. \tag{4.56}$$

Note that, uniform in $x \geq 0$, k is increasing in p and that for $p > -\delta/\lambda$ we have $k(0) > 0$. Hence $\widehat{p} := \widehat{p}(\delta, \lambda, \rho, r) \in (-\infty, -\delta/\lambda]$ exists so that (4.56) holds iff $p \in (-\infty, \widehat{p}]$.

To specify $\widehat{p}$, let us introduce x_{max} to denote the maximiser of k. It can be seen from (4.6) that $k(x_{\text{max}})$ is a continuous and increasing function of p and $k(\infty) = -\infty$. Then, we make the following partition and analysis:

- for $p < -(r/\lambda + 1)/\rho$, k is concave with $k'(0) > 0$ and hence

$$x_{\text{max}} = -\frac{1}{\rho}\log\left(-\frac{r}{\lambda(1+\rho p)}\right), \quad k(x_{\text{max}}) = \delta - \frac{r+\lambda}{\rho} + \frac{r}{\rho}\log\left(-\frac{r}{\lambda(1+\rho p)}\right);$$

- for $-(r/\lambda + 1)/\rho \leq p < -1/\rho$, k is concave with $k'(0) \leq 0$ and hence

$$x_{\text{max}} = 0, \quad k(x_{\text{max}}) = \delta + \lambda p;$$

- for $p \geq -1/\rho$, k is convex and decreasing hence again

$$x_{\text{max}} = 0, \quad k(x_{\text{max}}) = \delta + \lambda p.$$

It follows that $\widehat{p}$ is the unique root on $(-\infty, 0)$ of the mapping

$$p \mapsto \begin{cases} \delta - \frac{r+\lambda}{\rho} + \frac{r}{\rho}\log\left(-\frac{r}{\lambda(1+\rho p)}\right) & p < -(r/\lambda+1)/\rho \\ \delta + \lambda p & p \geq -(r/\lambda+1)/\rho \end{cases}$$

which yields

$$\widehat{p} = \widehat{p}(\delta, \lambda, \rho, r) = \begin{cases} -\frac{\delta}{\lambda} & \delta - (r+\lambda)/\rho \leq 0 \\ -\frac{1}{\rho}\left(\frac{r}{\lambda} e^{(\delta\rho - \lambda - r)/r} + 1\right) & \delta - (r+\lambda)/\rho > 0. \end{cases} \tag{4.57}$$

Next let us consider k_0 defined in (4.51). Note that k_0 is identical to k except that p is multiplied by a factor

$$\frac{q}{r+q} \frac{\Phi(r+q)}{\Phi(r+q)+\rho} \in (0,1).$$

So under the analysis with k above, we can immediately get that $k_0(x) \leq 0$ for all $x \geq 0$ holds iff

$$\frac{q}{r+q} \frac{\Phi(r+q)}{\Phi(r+q)+\rho} p \leq \widehat{p}$$

i.e. iff $p \in (-\infty, \widehat{p}_0]$ where

$$\widehat{p}_0 = \widehat{p}_0(\delta, \lambda, \rho, r) := \frac{r+q}{q} \frac{\Phi(r+q)+\rho}{\Phi(r+q)} \widehat{p} \in (-\infty, 0) \tag{4.58}$$

with $\widehat{p}$ as defined in (4.57). Note that $\widehat{p}_0 < \widehat{p}$.

4.4 Preliminary results

In this section, we provide several preliminary results which will be used in Section 4.5 to prove the main results of this chapter.

Lemma 4.10. *The set $D := \{x \,|\, v(x) = g(x)\}$ is closed and is the optimal stopping region in the sense that*

$$\tau^* := \{t \geq 0 \,|\, U_t \in D\}$$

is an optimal stopping time for v.

Proof. Consider a 3-dimensional optimal stopping problem defined as

$$V(x,m,n) = \sup_{\tau} \mathbb{E}_{x,m,n}\left[G(U_\tau, A_\tau, I_\tau)\right], \tag{4.59}$$

where $I_t = \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s$ and $G(x,m,n) = e^{-m} g(x) + pqn$. This trivariate process (U, A, I) is Markovian with respect to the filtration generated by U. Also, problem V in

(4.59) can regarded as the same problem in (4.3) under a 3-dimensional perspective with $m = n = 0$. The resulting optimal stopping region is a subset of the state space $\mathbb{R} \times [0,\infty) \times [0,\infty)$ of this trivariate process. Let

$$D = \{(x,m,n)\,|\,V(x,m,n) = G(x,m,n)\}.$$

According to [38, Corollary 2.9 and Remark 2.10], the first entrance time τ_D of (U, A, I) into D is optimal for (4.59) when the following three points are verified.

1) $\mathbb{E}_{x,m.n}\left[\sup_t G(U_t, A_t, I_t)\right] < \infty$.

   As shown at the beginning of subsection 4.2.1, for any $t \geq 0$, L_t^* is a bounded process under $\mathbb{P}_x$ and when $t \to \infty$, we have $L_\infty^* = pq\int_0^\infty \mathbb{1}_{\{U_s<0\}} e^{-A_s}\mathrm{d}s$ $\mathbb{P}_x$−a.s., which gives that $\mathbb{E}_{x,m.n}\left[\sup_t G(U_t, A_t, I_t)\right] < \infty$.

2) G is upper semicontinuous.

   The upper semicontinuous property of G automatically holds due to the fact that g is upper semicontinuous and the discounting and integral terms are continuous.

3) V is lower semicontinuous. Now fix (x,m,n) and let $\epsilon > 0$. Let τ_ϵ be an $\epsilon/2$ optimal stopping time i.e.

$$\begin{aligned}\mathbb{E}_{x,m,n}\left[G(U_{\tau_\epsilon}, A_{\tau_\epsilon}, I_{\tau_\epsilon})\right] &= \mathbb{E}\left[G(x + U_{\tau_\epsilon}, m + A^{(x)}_{\tau_\epsilon}, n + I^{(x)}_{\tau_\epsilon})\right] \\ &\geq V(x,m,n) - \frac{\epsilon}{2}, \end{aligned} \tag{4.60}$$

   where $\mathbb{E} = \mathbb{E}_{0,0,0}$, $A_t^{(x)} = rt + q\int_0^t \mathbb{1}_{\{x+U_s<0\}}\mathrm{d}s$ and $I_t^{(x)} = \int_0^t \mathbb{1}_{\{x+U_s<0\}} e^{-A_s^{(x)}}\mathrm{d}s$.

   Let us also define for $l > 0$ the stopping time

$$\sigma_{\epsilon,l} = \begin{cases} \tau_\epsilon & \text{if } x + U_{\tau_\epsilon} \neq 0 \\ \inf\{t > \tau_\epsilon \,|\, x + U_t \geq l\} & \text{if } x + U_{\tau_\epsilon} = 0. \end{cases}$$

   Since U is regular upwards, we have that $\sigma_{\epsilon,l} \downarrow \tau_\epsilon$ as $l \downarrow 0$. Further since G is right continuous in its first argument and continuous in its other two arguments, and the processes $A^{(x)}$ and $I^{(x)}$ are continuous, it also follows that

$$\lim_{l\downarrow 0} G\left(x+U_{\sigma_{\epsilon,l}}, m+A^{(x)}_{\sigma_{\epsilon,l}}, n+I^{(x)}_{\sigma_{\epsilon,l}}\right) = G\left(x+U_{\tau_\epsilon}, m+A^{(x)}_{\tau_\epsilon}, n+I^{(x)}_{\tau_\epsilon}\right)$$

and using the integrability condition in 1) we also get, using dominated convergence

$$\lim_{l\downarrow 0} \mathbb{E}\left[G\left(x+U_{\sigma_{\epsilon,l}}, m+A^{(x)}_{\sigma_{\epsilon,l}}, n+I^{(x)}_{\sigma_{\epsilon,l}}\right)\right] = \mathbb{E}\left[G\left(x+U_{\tau_\epsilon}, m+A^{(x)}_{\tau_\epsilon}, n+I^{(x)}_{\tau_\epsilon}\right)\right].$$

Hence an $\widehat{l} > 0$ small enough exists so that with $\sigma_\epsilon := \sigma_{\epsilon,\widehat{l}}$ we have that

$$\left|\mathbb{E}\left[G\left(x+U_{\sigma_\epsilon}, m+A^{(x)}_{\sigma_\epsilon}, n+I^{(x)}_{\sigma_\epsilon}\right)\right] - \mathbb{E}\left[G\left(x+U_{\tau_\epsilon}, m+A^{(x)}_{\tau_\epsilon}, n+I^{(x)}_{\tau_\epsilon}\right)\right]\right| \leq \frac{\epsilon}{2}. \tag{4.61}$$

Hence we now have that, the first inequality by (4.61) and the second by (4.60)

$$\begin{aligned}\mathbb{E}\left[G\left(x+U_{\sigma_\epsilon}, m+A^{(x)}_{\sigma_\epsilon}, n+I^{(x)}_{\sigma_\epsilon}\right)\right] &\geq \mathbb{E}\left[G\left(x+U_{\tau_\epsilon}, m+A^{(x)}_{\tau_\epsilon}, n+I^{(x)}_{\tau_\epsilon}\right)\right] - \frac{\epsilon}{2}\\ &\geq V(x,m,n) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = V(x,m,n) - \epsilon.\end{aligned}$$

Next, let $\{(x_k, m_k, n_k)\,;\, k \geq 0\}$ be any sequence which converges to (x,m,n) when k tends to infinity. Then,

$$\begin{aligned}V(x_k,m_k,n_k) &\geq \mathbb{E}_{x_k,m_k,n_k}\left[G(U_{\sigma_\epsilon}, A_{\sigma_\epsilon}, I_{\sigma_\epsilon})\right]\\ &= \mathbb{E}\left[G(x_k+U_{\sigma_\epsilon}, m_k+A^{(x_k)}_{\sigma_\epsilon}, n_k+I^{(x_k)}_{\sigma_\epsilon}))\right],\end{aligned} \tag{4.62}$$

where the inequality comes directly from the definition of V.

By letting $k \to \infty$ in (4.62), it follows that

$$\begin{aligned}\liminf_{k\to\infty} V(x_k,m_k,n_k) &\geq \liminf_{k\to\infty} \mathbb{E}\left[G(x_k+U_{\sigma_\epsilon}, m_k+A^{(x_k)}_{\sigma_\epsilon}, n_k+I^{(x_k)}_{\sigma_\epsilon})\right]\\ &\geq \mathbb{E}\left[\liminf_{k\to\infty} G(x_k+U_{\sigma_\epsilon}, m_k+A^{(x_k)}_{\sigma_\epsilon}, n_k+I^{(x_k)}_{\sigma_\epsilon})\right]\\ &= \mathbb{E}\left[G(x+U_{\sigma_\epsilon}, m+A^{(x)}_{\sigma_\epsilon}, n+I^{(x)}_{\sigma_\epsilon})\right]\\ &= \mathbb{E}_{x,m,n}\left[G(U_{\sigma_\epsilon}, A_{\sigma_\epsilon}, I_{\sigma_\epsilon})\right],\end{aligned}$$

where second inequality results from Fatou's Lemma and the first equation is due to the continuity of G by construction of σ_ϵ which let us avoid the discontinuity G has when its first argument crosses zero. So together with (4.60) and letting $\epsilon \to 0$, we have

$$\liminf_{(x_k,m_k,n_k)\to(x,m,n)} V(x_k,m_k,n_k) \geq V(x,m,n),$$

which shows that the function V is lower semi-continuous.

Then, according to [38, Corollary 2.9 and Remark 2.10], τ_D is optimal for (4.59). Further, for any (x,m,n), by the Markov property, we have

$$\begin{aligned} V(x,m,n) &= \sup_\tau \mathbb{E}_{x,0,0}\left[G\left(U_\tau, m+A_\tau, n+I_\tau\right)\right] \\ &= \sup_\tau \mathbb{E}_{x,0,0}\left[e^{-(m+A_\tau)}g(U_\tau) + pq\int_0^\tau \mathbb{1}_{\{U_s<0\}}e^{-(m+A_s)}\mathrm{d}s + n\right] \\ &= e^{-m}\sup_\tau \mathbb{E}_{x,0,0}\left[e^{-A_\tau}g(U_\tau) + pq\int_0^\tau \mathbb{1}_{\{U_s<0\}}e^{-A_s}\mathrm{d}s\right] + n \\ &= e^{-m}V(x,0,0) + n = e^{-m}v(x) + n. \end{aligned}$$

Hence, the stopping region S can be written as

$$S = \{(x,m,n)\,|\,e^{-m}v(x) + n = e^{-m}g(x) + n\} = \{(x,m,n)\,|\,v(x) = g(x)\}.$$

So for any (x,m,n), the optimal stopping time for $V(x,m,n)$ is

$$\tau^* := \inf\{t \geq 0\,|\,U_t \in D\},$$

where $D = \{x\,|\,v(x) = g(x)\}$.

In particular, τ^* is also optimal for $V(x,0,0) = v(x)$. This completes the proof. □

Lemma 4.11. *If* $k(x_0) > 0$*, then* $x_0 \notin D$*.*

Proof. When $k(x_0) > 0$, due to the right-continuity of k and the fact that k is positive on $(-\infty,0)$, we can find some interval (x_0-h, x_0+h) on which $k > 0$ with h small enough. Then define the first exit time from this interval as τ_h, i.e. $\tau_h = \inf\{t \geq 0\,|\,U_t \notin (x_0-h, x_0+h)\}$. It follows by the expression for value function v in (4.18) that

$$v(x_0) \geq g(x_0) + \mathbb{E}_{x_0}\left[\int_0^{\tau_h} e^{-A_s}k(U_s)\mathrm{d}s - p\int_0^{\tau_h} e^{-A_s}\mathrm{d}\hat{N}_s\right] > g(x_0),$$

where the first inequality comes from the definition of v and the second one results from the fact that k is positive on $(x_0 - h, x_0 + h)$. Thus, $x_0 \notin D$. $\square$

4.5 Main results and proofs

Recall that for our optimal stopping problem (see (4.3))

$$v(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[L^*_\tau \right],$$

$D = \{x \mid v(x) = g(x)\}$ is the optimal stopping region (see Lemma 4.10) and v_a is the value function corresponding to an up-crossing strategy with level a. We have achieved a simple expression for v_a as shown in (4.23) and for v_0 in (4.33). Also, the drift function k of the process L^* and k_a of the process $Z^{(a)}$ are defined in (4.6) and (4.50) respectively.

Now we present the two main results below.

Theorem 4.12. *Let $p \in (-\delta/\lambda, 0)$. Denote by z_1 the unique strictly positive root of k. Let a^* denote the largest root of J in $[z_1, \infty)$. We have that $D = [a^*, \infty)$ i.e. $v = v_{a^*}$. Furthermore we have smooth pasting: $v'(a^*-) = g'(a^*) = 1$.*

Proof. The proof is broken down into a number of points as below.

1) D is a subset of $[z_1, \infty)$ and D is not empty.

   From Lemma 4.10 and Lemma 4.11 it follows that $D \subseteq [z_1, \infty)$.

   Further, assume that D is empty. Then for any $x > 0$, we have

   $$v(x) = \mathbb{E}_x \left[L^*_\infty \right] = pq \mathbb{E}_x \left[\int_0^\infty \mathbb{1}_{\{U_s < 0\}} e^{-A_s} \mathrm{d}s \right] < 0 < g(x),$$

   which contradicts $v \geq g$. Hence, D is not an empty set.

2) Define $a^* := \inf D$. Then a^* is a root of J defined in (4.43).

   Again from Lemma 4.10 and Lemma 4.11, we have that $a^* \in D$, which together with step 1) gives that $a^* \in [z_1, \infty)$. Hence, $v_{a^*}(x) = v(x) \geq g(x)$ for all $x \leq a^*$.

   Then, as $v_{a^*}(a^*) = g(a^*)$ by definition of the stopping set, together with that v_{a^*} is C^1

on $(0,a)$ shown in (4.40), it follows that

$$v_{a^*}^{'}(a^*-) \leq g^{'}(a^*) = 1.$$

Assume that $v_{a^*}^{'}(a^*-) < 1$, which equivalently means $J(a^*) > 0$. Then by the continuity of J on $(0,\infty)$ which can be seen from Lemma 4.7, there exists some $c, h > 0$ such that

$$J(a) \geq c, \quad \text{for all } a \in (a^*, a^*+h),$$

which means that $v_{a}^{'}(a-) \leq 1-c$ for $a \in (a^*, a^*+h)$. By mean value theorem, there exists $\xi_a \in (a^*, a)$ so that $v_a(a) - v_a(a^*) = (a-a^*)v_a'(\xi_a)$, which gives

$$v_a(a^*) = a - (a-a^*)v_a'(\xi_a).$$

So for $a > a^*$, $v_a(a^*) \leq v(a^*) = a^*$ gives that $v_a'(\xi_a) \geq 1$.

Again, by mean value theorem and the fact that v_a is C^2 on $(0,a)$ shown in (4.40), there exists some $\hat{\xi}_a \in (\xi_a, a)$ such that

$$\begin{aligned} v_a''(\hat{\xi}_a) &= \frac{v_a'(a-) - v_a'(\xi_a)}{a-\xi_a} \\ &\leq \frac{1-c-1}{a-\xi_a} = -\frac{c}{a-\xi_a}. \end{aligned}$$

As $a \downarrow a^*$, $\xi_a \downarrow a^*$ and $\hat{\xi}_a \downarrow a^*$, it follows that

$$a - \xi_a \downarrow 0 \quad \text{and} \quad v_a''(\hat{\xi}_a) \to -\infty,$$

which contradicts the fact shown in (4.40) that v_a'' is bounded away from $-\infty$ uniformly in a on some interval around a^*.

Hence, it follows that $v_{a^*}^{'}(a^*-) = 1$, i.e. a^* is a root of J.

3) $v_{a^*} \geq g$ on $\mathbb{R}$.

Note that $v_{a^*}(x) = g(x)$ for all $x \geq a^*$ while for $x < a^*$ we know that $v_{a^*}(x) = v(x) \geq g(x)$ from step 2), which shows that $v_{a^*} \geq g$ on $\mathbb{R}$.

4) $Z^{(a^*)}$ is uniformly integrable.

Recall from (4.48) that for any $t \geq 0$,

$$Z_t^{(a)} = e^{-rt} v_a(U_t) + pq \int_0^t \mathbb{1}_{\{U_s<0\}} e^{-A_s} \mathrm{d}s.$$

Then, by (4.40), it is clear that

$$\begin{aligned} Z_t^{(a^*)} &\geq e^{-rt} \left(\frac{pq}{r+q} + C_1^{(a^*)} e^{\Phi(r+q)x} \mathbb{1}_{\{x\leq 0\}} \right) + pq \int_0^\infty \mathbb{1}_{\{U_s<0\}} e^{-rs} \mathrm{d}s \\ &\geq \frac{pq}{r+q} + \frac{pq}{r} \end{aligned} \tag{4.63}$$

where the second line comes from the fact that $C_1^{(a^*)} > 0$ as can be seen in (4.40). Also, under $\mathbb{P}_x$ with $x \in \mathbb{R}$ given, it is true that

$$v_a(x) \leq \max\{B^{(a)}, x\},$$

where $B^{(a)} = \max_{z\leq a} v_a(z) < \infty$. Then, it holds for any $t \geq 0$ that

$$Z_t^{(a^*)} \leq e^{-rt} \max\{B^{(a^*)}, x + \delta t\} \leq \widetilde{B}_x^{(a^*)} \tag{4.64}$$

due to that $U_t \leq x + \delta t$ for any $t \geq 0$ and here $\widetilde{B}_x^{(a^*)} < \infty$ depends only on x and a^*. So it follows from (4.63) and (4.64) that $Z^{(a^*)}$ is uniformly integrable.

5) $Z^{(a^*)}$ is a supermartingale.

For the supermartingale property, recall the drift function k_{a^*} from (4.49) and (4.50). We have proved in Lemma 4.9 that the stopped process $(Z_{t\wedge\tau_{a^*}^+}^{(a^*)})_{t\geq 0}$ is a $\mathbb{P}_z$-martingale for all $z \in \mathbb{R}$ and hence necessarily $k_{a^*}(z) = 0$ for $z < a^*$. Due to the continuity and piecewise C^1 property of v_{a^*} seen from (4.40) and in particular, the fact that v_{a^*} connects with g in a C^1 fashion as shown in step 2), it then holds that k_{a^*} defined by (4.49) is continuous except at zero. Hence $k_{a^*}(a^*+) = k_{a^*}(a^*) = k_{a^*}(a^*-) = 0$. Further it is clear from (4.50) that $k_{a^*}(\infty) = -\infty$, and that k_{a^*} is either concave or convex. Then if $k'_{a^*}(a^*+) \leq 0$, it holds that $k_{a^*} \leq 0$ on $[a^*, \infty)$, from which it follows that $Z^{(a^*)}$ is a supermartingale. Hence, we will show that $k'_{a^*}(a^*+) \leq 0$ below.

For $0 < z < a^*$

$$0 = k'_{a^*}(z) = \delta v''_{a^*}(z) - (r+\lambda)v'_{a^*}(z) + \int_0^\infty v'_{a^*}(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y,$$

where differentiation inside the integral can be justified since v_{a^*} is continuous and piecewise C^1 and v'_{a^*} is bounded (see e.g. [44]). Then taking the limit for $z \uparrow a^*$ we get using dominated convergence and smooth pasting that

$$0 = \delta v''_{a^*}(a^*-) - (r+\lambda) + \int_0^\infty v'_{a^*}(a^*-y)\lambda\rho e^{-\rho y}\mathrm{d}y. \tag{4.65}$$

Similarly for $z > a^*$ we have $v_{a^*}(z) = z$ and hence

$$\begin{aligned} k'_{a^*}(z) &= \delta v''_{a^*}(z) - (r+\lambda)v'_{a^*}(z) + \int_0^\infty v'_{a^*}(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y \\ &= -(r+\lambda) + \int_0^\infty v'_{a^*}(z-y)\lambda\rho e^{-\rho y}\mathrm{d}y. \end{aligned}$$

Further, taking the limit for $z \downarrow a^*$ yields

$$k'_{a^*}(a^*+) = -(r+\lambda) + \int_0^\infty v'_{a^*}(a^*-y)\lambda\rho e^{-\rho y}\mathrm{d}y$$

and it follows from (4.65) that

$$k'_{a^*}(a^*+) = -\delta v''_{a^*}(a^*-).$$

Hence it is enough to show that $v''_{a^*}(a^*-) \geq 0$. To see this, note that for $0 < x < a^*$ we have

$$\int_x^{a^*}\int_y^{a^*} v''_{a^*}(u)\mathrm{d}u\mathrm{d}y = v_{a^*}(x) - x.$$

Assume that $v''_{a^*}(a^*-) < 0$. Then for x close enough to a^*, we have $v''_{a^*} < 0$ on (x, a^*) and hence also $v_{a^*}(x) - x < 0$, which contradicts the fact given by step 2) that for any $x < a^*$ we have $v_{a^*}(x) = v(x) \geq x$. Hence, $v''_{a^*}(a^*-) \geq 0$, i.e. $k'_{a^*}(a^*+) \leq 0$ and thus $Z^{(a^*)}$ is a supermartingale.

6) $D = [a^*, \infty)$.

We have shown that $v_{a^*} \geq g$ and $Z^{(a^*)}$ is a UI-supermartingale in steps 3) – 5). Then for any $x \in \mathbb{R}$ and any stopping time τ we have that

$$v_{a^*}(x) = \mathbb{E}_x\left[Z_0^{(a^*)}\right] \geq \mathbb{E}_x\left[Z_\tau^{(a^*)}\right] \geq \mathbb{E}_x\left[L_\tau^*\right]$$

where optional sampling theorem is applied in the first inequality. This together with step 2) implies that $v_{a^*}(x) \geq v(x)$ i.e. indeed $v_{a^*}(x) = v(x)$ and $D = [a^*, \infty)$.

7) a^* is the largest root of J.

   Suppose that J has a larger root $a_0 > a^*$. As shown in Lemma 4.7 that J is a mixture of exponential functions and $J(\infty) = -\infty$, there would exist an $a_1 \in (a^*, a_0)$ so that $J(a_1) > 0$. Then by a similar argument as in step 2), we can derive a contradiction, which shows that a^* must be the largest root of J.

This completes the proof. □

In Theorem 4.12, a unique strictly positive root exists for k while in the following theorem, we show that when such root does not exist, an up-crossing strategy is still optimal when the parameters satisfy certain conditions.

Theorem 4.13. *In the following two cases, an up-crossing strategy is also optimal for* (4.3).

(i) If $p \leq \widehat{p}_0$ then $D = [0, \infty)$ i.e. $v = v_0$. In this case smooth pasting fails to hold i.e. $v_0'(0-) \neq g'(0+) = 1$ unless $p = -\frac{r+q}{q\Phi(r+q)}$.

(ii) If $p \in (\widehat{p}_0, \widehat{p}]$ and $\delta - (r + \lambda)/\rho \leq 0$ then $D = [a^*, \infty)$ i.e. $v = v_{a^*}$ where $a^* > 0$ denotes the largest root of J in $(0, \infty)$. Furthermore smooth pasting holds i.e. $v'(a^*-) = g'(a^*) = 1$.

Proof. Ad (i). In this case, the verification argument in Theorem 4.12 can be applied. It readily follows from (4.33) that $v_0(x) \geq g(x)$ for all $x \in \mathbb{R}$ and by assumption $k_0(x) \leq 0$ for all $x \geq 0$. Hence the same arguments used in steps 3)–6) in the proof of Theorem 4.12 show that $v = v_0$.

Ad (ii). By Lemma 4.10, Lemma 4.11 and the same argument as in point step 1) of the proof of Theorem 4.12, we know that D is a non-empty, closed set and $D \subsetneq [0, \infty)$. Again, we set $a^* = \inf D \geq 0$.

First we show that $J(0+) > 0$. We have

$$\widehat{p}_0 = \frac{r+q}{q}\frac{\Phi(r+q)+\rho}{\Phi(r+q)}\widehat{p} = -\frac{r+q}{q}\frac{\Phi(r+q)+\rho}{\Phi(r+q)}\frac{\delta}{\lambda}.$$

Since

$$r+q = \psi(\Phi(r+q)) = \delta\Phi(r+q) - \frac{\lambda\Phi(r+q)}{\rho+\Phi(r+q)}$$

we note that $\delta\Phi(r+q) - (r+q) > 0$ and we can further simplify

$$\widehat{p}_0 = -\frac{1}{q}\frac{1}{\Phi(r+q)/(r+q) - 1/\delta},$$

which shows that since $p > \widehat{p}_0$ we have together with equation (4.45) that

$$J(0+) = \frac{1}{q} - p\left(\frac{1}{\delta} - \frac{\Phi(r+q)}{r+q}\right) > 0.$$

Next we show that $a^* > 0$. If it were true that $a^* = 0$ then we would have $v(0) = g(0) = 0$. But since $J(0+) > 0$, $1 - v_a'(a-)$ has a positive limit as $a \downarrow 0$ i.e. $v_a'(a-)$ has a limit less than 1. Since $v_a(a) = g(a) = a$, for a close enough to 0 we have that $v_a(0) > 0$ which contradicts with $v_a(0) \le v(0) = g(0) = 0$.

Now we can follow steps 2)–7) from the proof of Theorem 4.12 to arrive at the result.

□

4.6 Discussion and future work

Based on the partition from Section 4.3, we make an overview of the results from the above Section 4.5 in terms of the optimal stopping region as follows:

Range of p	$\delta - (r+\lambda)/\rho \le 0$	$\delta - (r+\lambda)/\rho > 0$
$(-\delta/\lambda, 0)$	$[a^*, \infty)$ where $a^* > 0$	$[a^*, \infty)$ where $a^* > 0$
$(\widehat{p}, -\delta/\lambda]$	—	$[a^*, \infty)$ ***where*** $a^* > 0$
$(\widehat{p}_0, \widehat{p}]$	$[a^*, \infty)$ where $a^* > 0$	$[a^*, \infty)$ ***where*** $a^* > 0$ ***for*** p ***closer to*** $\widehat{p}_0$***; or*** $[a^*, b^*] \cup [c^*, \infty)$ ***where*** $0 \le a^* \le b^* < c^*$ ***for*** p ***closer to*** $\widehat{p}$***.***
$(-\infty, \widehat{p}_0]$	$[0, \infty)$	$[0, \infty)$

Here, the symbol '—' means that the corresponding case in the cell does not exist, while those in bold italics denote the cases that we are not yet solved but have made rational guesses which can be seen shortly below. We can see from above that there are seven possible scenarios and for the case $\delta - (r + \lambda)/\rho \leq 0$, we have fully solved the problem; while when $\delta - (r + \lambda)/\rho > 0$, we are left with two cases to figure out. In the following part, we make discussions together with numerical work on each of these scenarios.

In the numerical examples below, we work with a Cramér-Lundberg process U, as defined in (4.1), with parameters

$$\lambda = 0.4, \rho = 1, q = 0.1, r = 0.1,$$

and we change the value of δ and p to distinguish possible cases.

4.6.1 When $\delta - (r + \lambda)/\rho \leq 0$.

In this subsection, we set $\delta = 0.5$ so that $\delta - (r + \lambda)/\rho \leq 0$ holds. We provide three examples where p belongs to $(-\delta/\lambda, 0)$, $(\widehat{p}_0, \widehat{p}]$ and $(-\infty, \widehat{p}_0]$ respectively.

Example 4.14. When $p = -1 \in (-\delta/\lambda, 0)$.

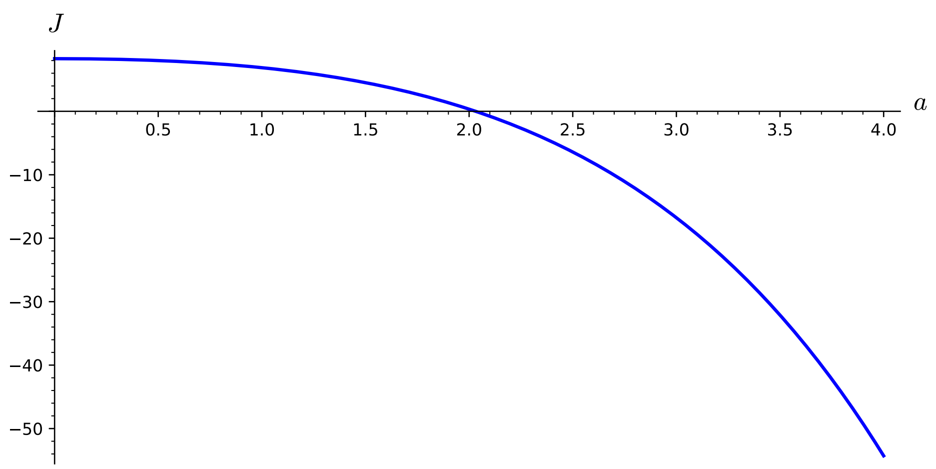

Figure 4.1: A plot of J when $\delta = 0.5$ and $p = -1$.

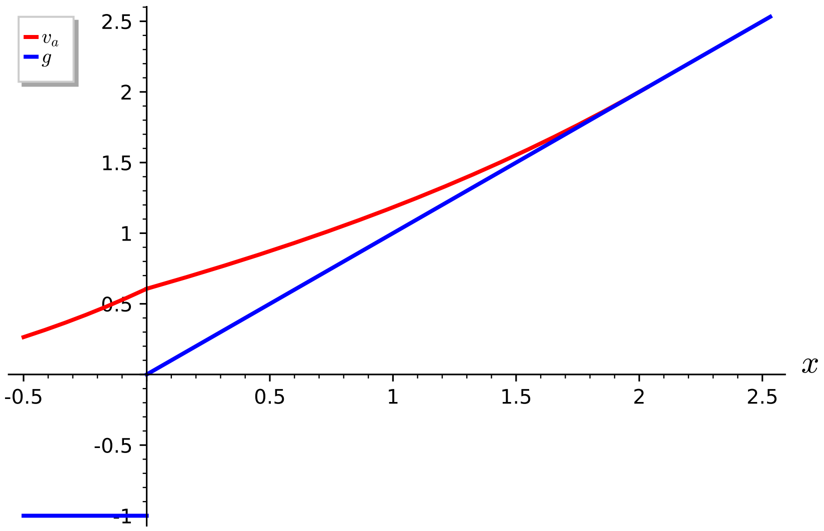

Figure 4.2: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.5$ and $p = -1$. Here $a^* = 2.0318$.

As shown in Figure 4.1 that J is actually monotone and has a unique root $a^* = 2.0318$ here, which by Theorem 4.12 is the optimal level. We can see from Figure 4.2 that smooth pasting holds in $x = a^*$ and as we mentioned before, v_{a^*} is not C^1 across $x = 0$. $\triangleleft$

Example 4.15. When $p = -5 \in (\widehat{p}_0, \widehat{p}]$.

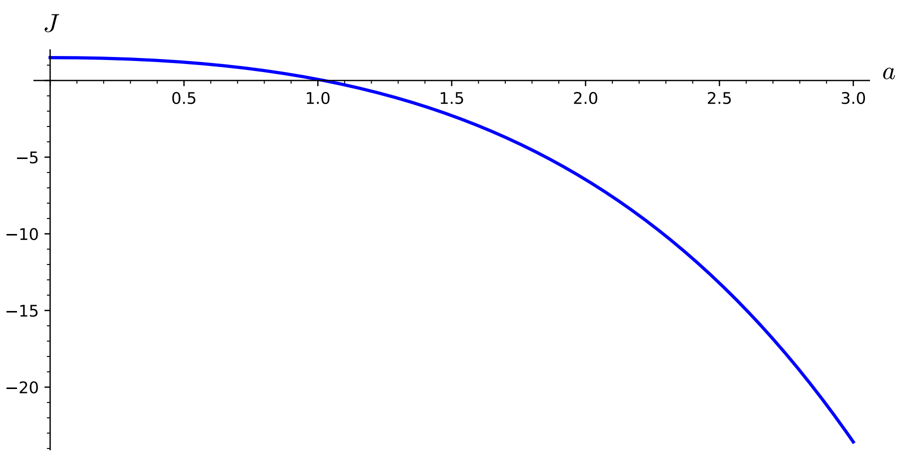

Figure 4.3: A plot of J when $\delta = 0.5$ and $p = -5$.

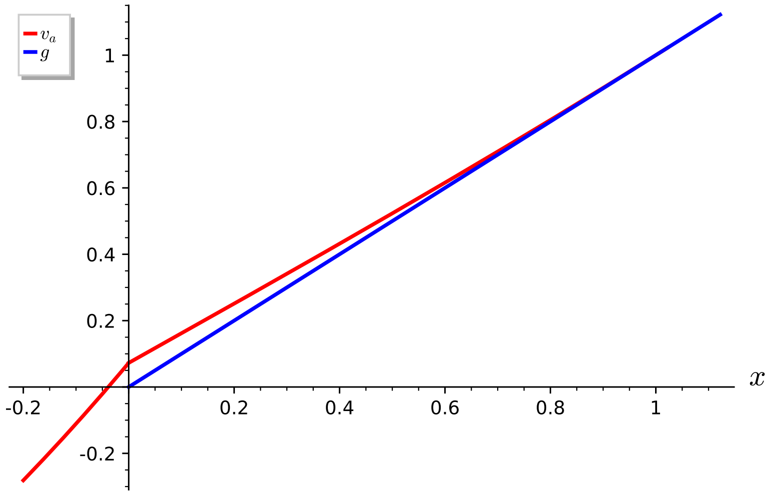

Figure 4.4: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.5$ and $p = -5$. Here $a^* = 1.022$.

According to case (ii) in Theorem 4.13, same comments can be made here as for Figure 4.1 and Figure 4.2. One thing worth to obeserve here is that the value function in Figure 4.4 is below the one from Figure 4.2, which is due to the fact that v is uniformly decreasing in p. $\triangleleft$

Example 4.16. When $p = -15 \in (-\infty, \widehat{p}_0]$.

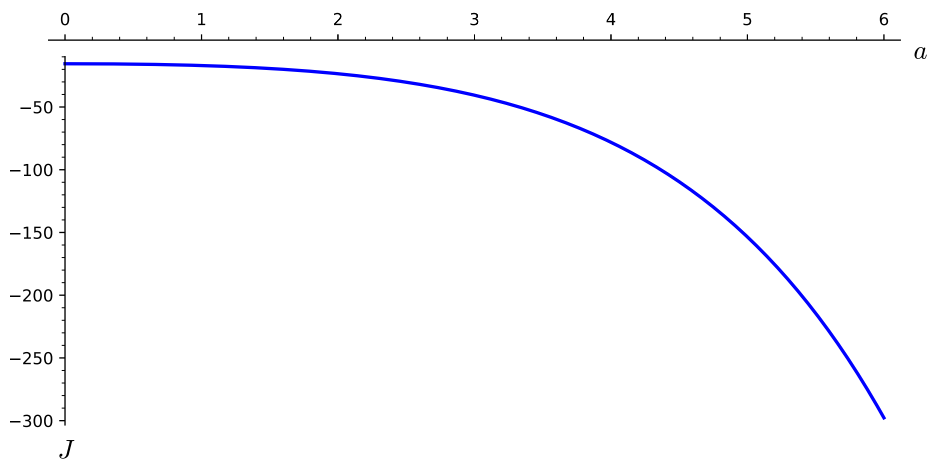

Figure 4.5: A plot of J when $\delta = 0.5$ and $p = -15$.

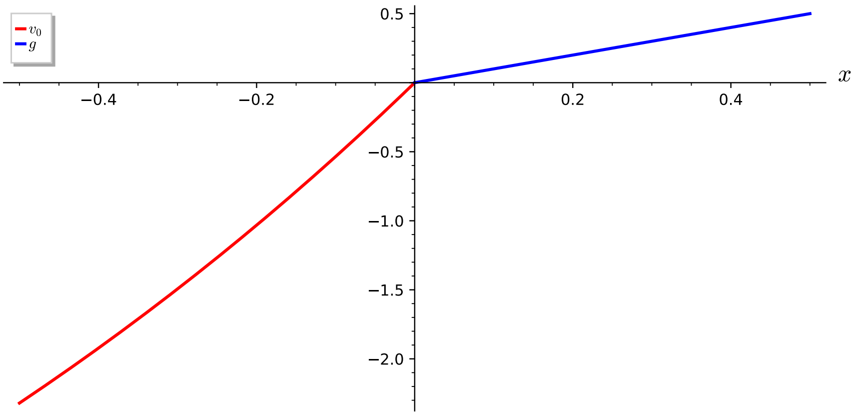

Figure 4.6: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.5$ and $p = -15$. Here $a^* = 0$.

According to case (i) in Theorem 4.13, $D = [0, \infty)$ here and Figure 4.6 shows that smooth pasting breaks down as a consequence of the discontinuity in g. Also note from Figure 4.5 that J is still monotone but no longer has a root. $\triangleleft$

4.6.2 When $\delta - (r + \lambda)/\rho > 0$.

In this subsection, we set $\delta = 0.6$ so that $\delta - (r + \lambda)/\rho > 0$ holds. There are four possible scenarios in this case w.r.t different choice of p. We have solved two of them while we do not yet know what happens for $p \in (\widehat{p}_0, -\delta/\lambda]$. We expect when $p \in (\widehat{p}_0, -\delta/\lambda]$ that we get to see optimal stopping regions at least as intricate as $[a^*, b^*] \cup [c^*, \infty)$ and we will present several examples to sparkle ideas.

Example 4.17. When $p = -1 \in (-\delta/\lambda, 0)$.

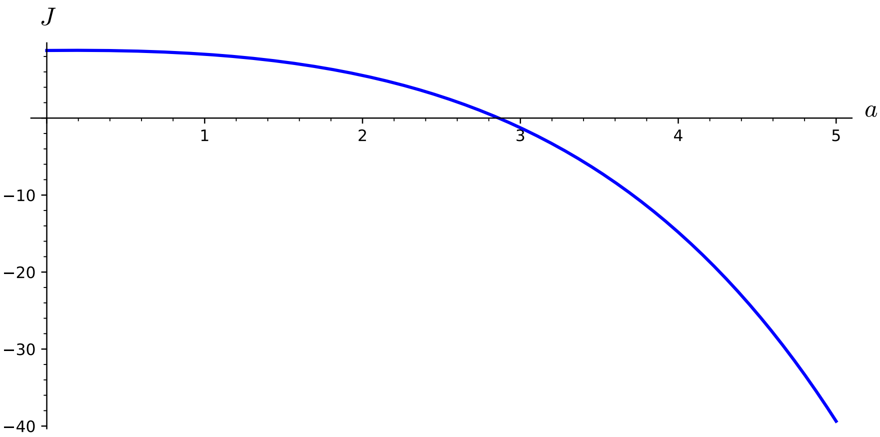

Figure 4.7: A plot of J when $\delta = 0.6$ and $p = -1$.

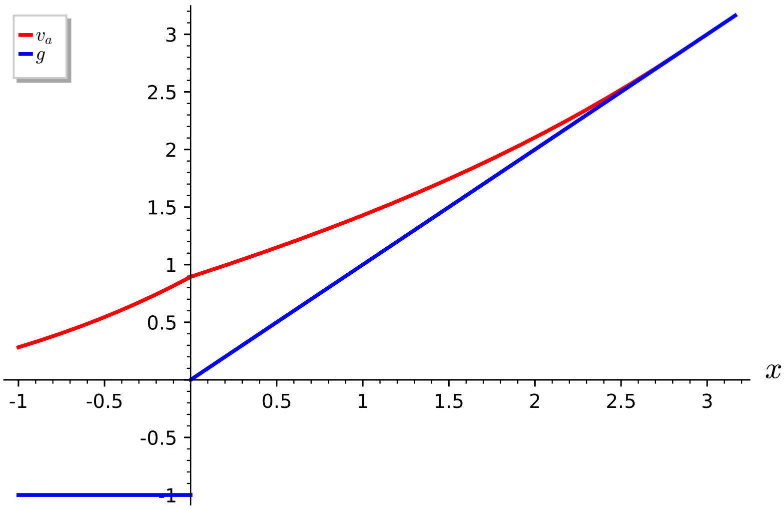

Figure 4.8: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -1$. Here $a^* = 2.863$.

It is nice to observe that Figure 4.7 and Figure 4.8 look quite similar to those in Example 4.14 and Example 4.15. In particular, Figure 4.7 shows that J is monotone with a unique root again. $\triangleleft$

When $p \in (\widehat{p}, -\delta/\lambda]$. We will give two examples with p close to $\widehat{p}$ and $-\delta/\lambda$ respectively.

Example 4.18. When $p = -1.52 \in (\widehat{p}, -\delta/\lambda]$ and close to $-\delta/\lambda$.

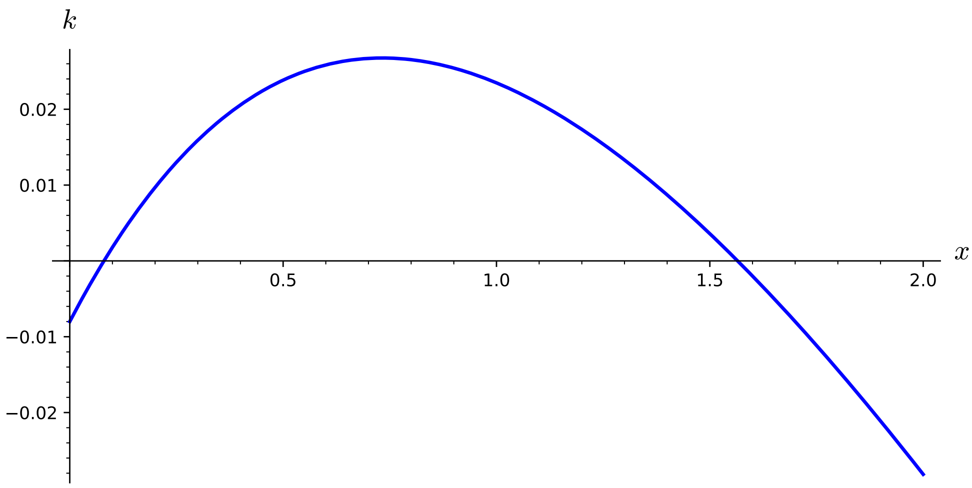

Figure 4.9: A plot of k when $\delta = 0.6$ and $p = -1.52$.

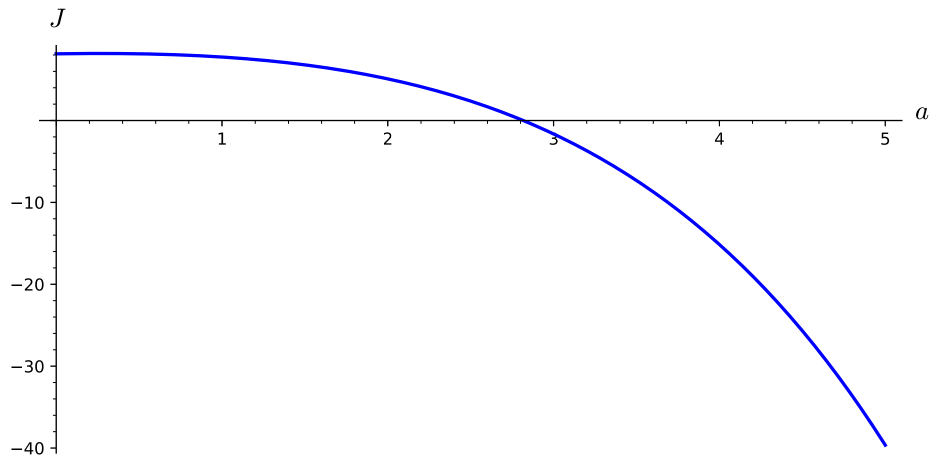

Figure 4.10: A plot of J when $\delta = 0.6$ and $p = -1.52$.

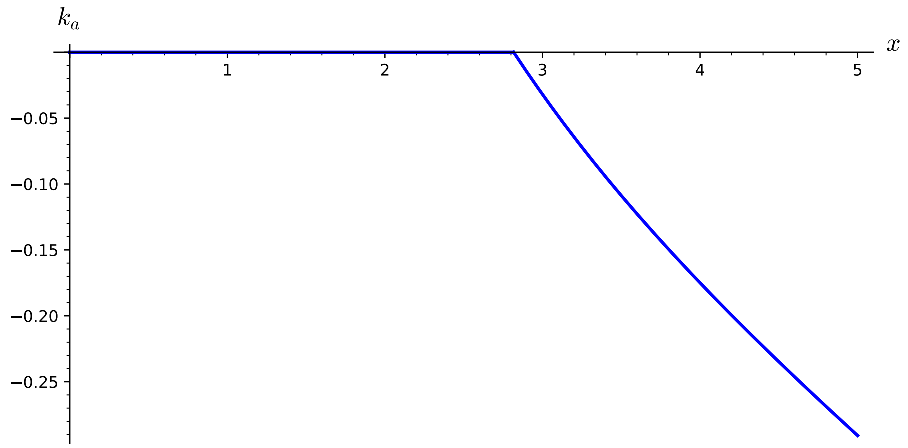

Figure 4.11: A plot of k_{a^*} when $\delta = 0.6$ and $p = -1.52$.

As can be seen from Figure 4.9 and Figure 4.10, J has a unique root which is greater than the largest root of k. Then by define a^* equal to the unique root of J, we make the plot for the drift function k_{a^*} of $Z^{(a^*)}$ as shown in Figure 4.11, which holds that $k_{a^*} = 0$ on $(-\infty, a^*]$ and $k_{a^*} < 0$ on (a^*, ∞). Based on this these observations, we also make the plot for v_{a^*} in Figure 4.12 below.

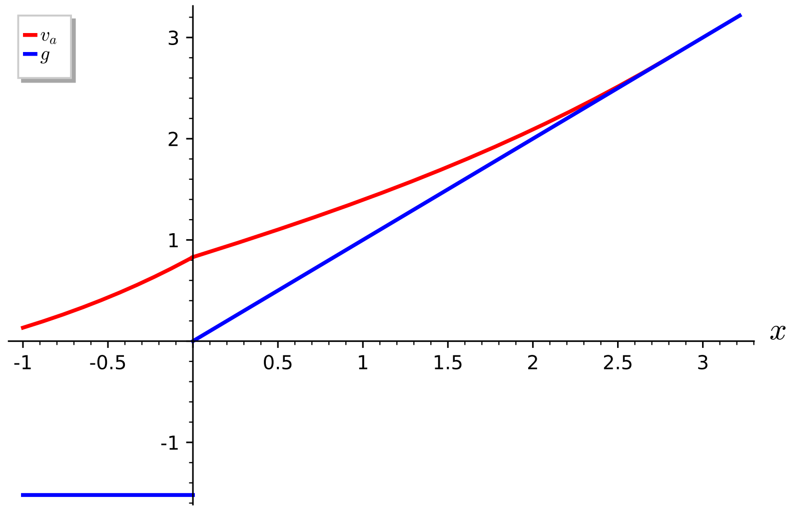

Figure 4.12: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -1.52$. Here $a^* = 2.8168$.

Figure 4.12 looks quite similar to Figure 4.2, Figure 4.4 and Figure 4.8 where an up-crossing strategy is optimal with a^* defined as the largest root of J. Also, it can be seen that v_{a^*} is not differentiable at zero.

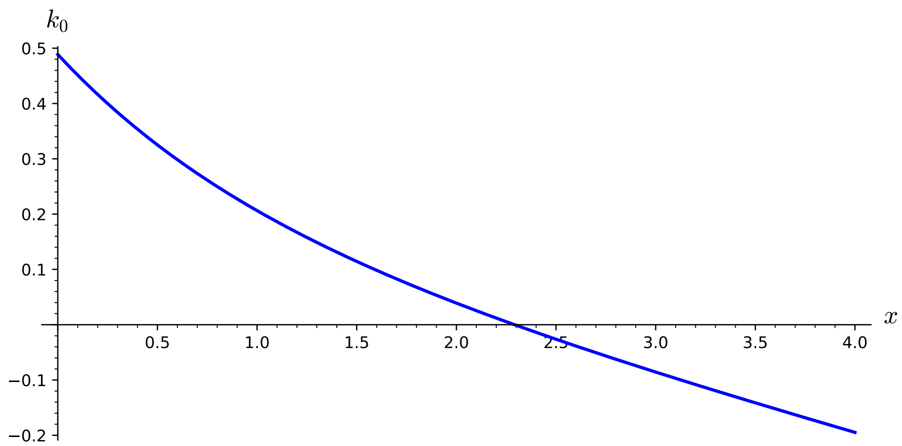

Figure 4.13: A plot of k_0 when $\delta = 0.6$ and $p = -1.52$.

We also make a plot for k_0 and as shown in Figure 4.13, k_0 is positive on a certain interval, which indicates that $a = 0$ is not optimal.

Then, according to step 6) in Theorem 4.12, when choosing $a^* = 2.8168$, Figure 4.12 shows that the smooth pasting holds at a^* and $v_{a^*} \geq g$. Also, it can be seen from Figure 4.11 that supermartingale property for process $Z^{(a^*)}$ holds. Hence, it seems that $D = [a^*, \infty)$ with $a^* = 2.8168$ is still optimal even though we have not been able to prove that yet. $\triangleleft$

Example 4.19. When $p = -1.67 \in (\widehat{p}, -\delta/\lambda]$ and close to $\widehat{p}$.

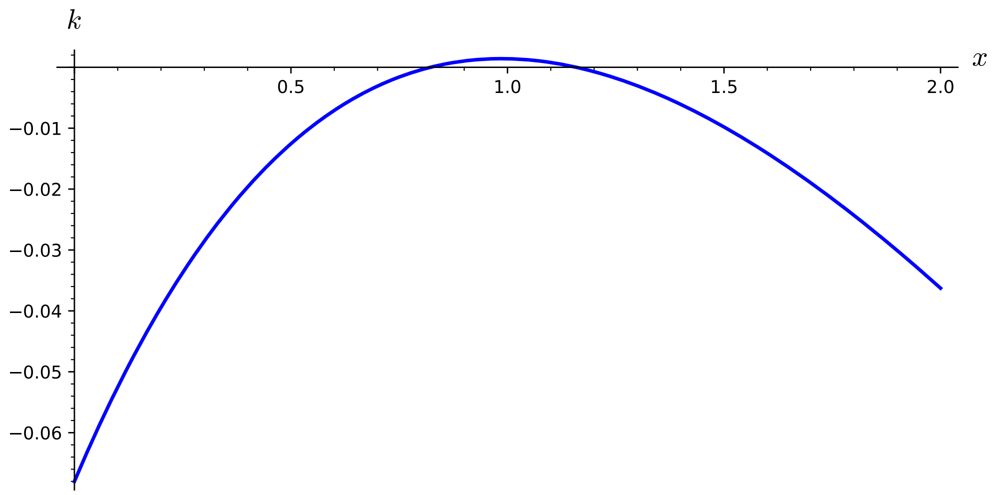

Figure 4.14: A plot of k when $\delta = 0.6$ and $p = -1.67$.

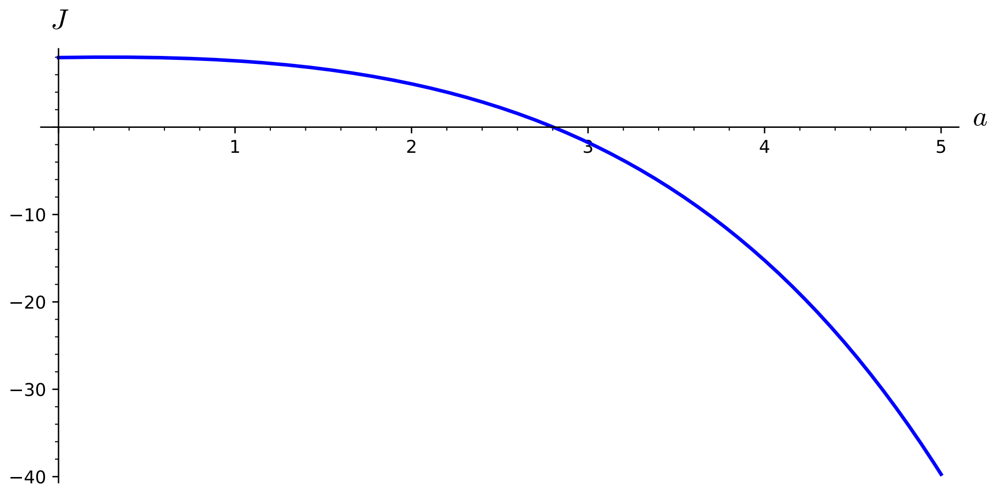

Figure 4.15: A plot of J when $\delta = 0.6$ and $p = -1.67$.

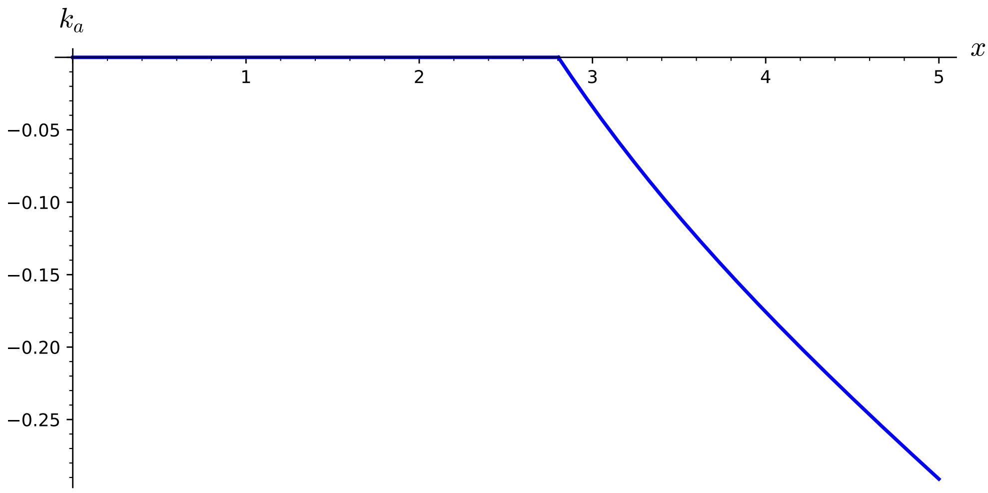

Figure 4.16: A plot of k_{a^*} when $\delta = 0.6$ and $p = -1.67$.

Similar to those in Example 4.18, J has a unique root a^* greater than the largest root of k and k_{a^*} indicates that $Z^{(a^*)}$ is a supermartingale.

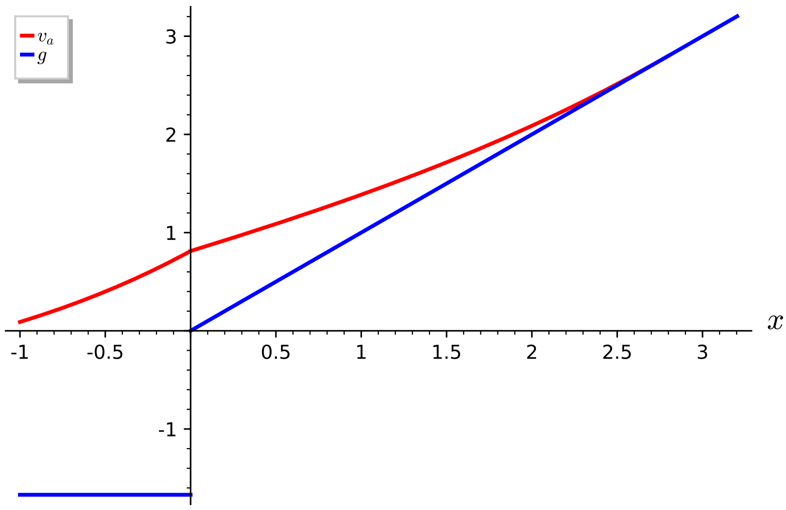

Figure 4.17: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -1.67$. Here $a^* = 2.803$.

We again make plots for v_{a^*} and again, it looks almost the same as Figure 4.2, Figure 4.4, Figure 4.8 and Figure 4.12.

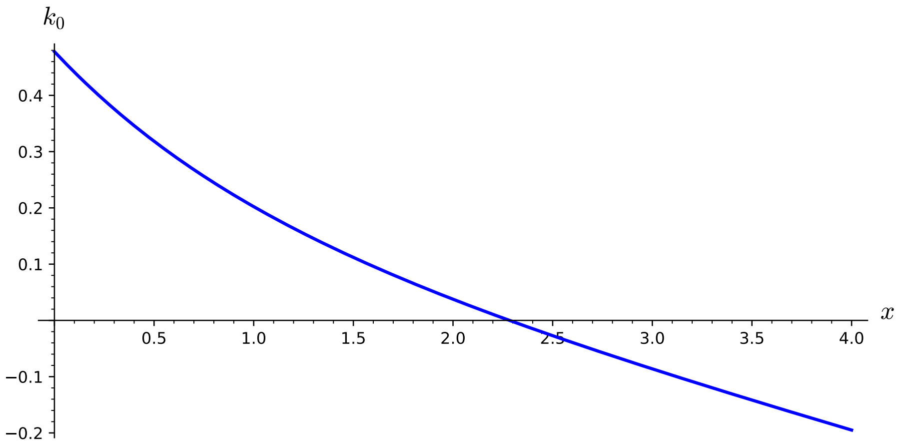

Figure 4.18: A plot of k_0 when $\delta = 0.6$ and $p = -1.67$.

Further, if letting $a = 0$, k_0 is positive on a certain interval $[0, x_0)$, which means that $Z^{(0)}$ is not a supermartingale, and hence an up-crossing strategy with threshold equal to zero is not optimal.

Now similar to Example 4.18, by setting $a^* = 2.803$, we can observe that $Z^{(a^*)}$ is a supermartingale in Figure 4.16 and $v_{a^*} \geq g$ in Figure 4.17, which again according to step 6), indicates that $D = [a^*, \infty)$ with $a^* = 2.803$ remains to be optimal although we have not been able to verify that yet. $\triangleleft$

We also give two examples when $p \in (\widehat{p}_0, \widehat{p}]$ with p close to $\widehat{p}$ from left and $\widehat{p}_0$ from right respectively.

Example 4.20. When $p = -1.7 \in (\widehat{p}_0, \widehat{p}]$ and close to $\widehat{p}$.

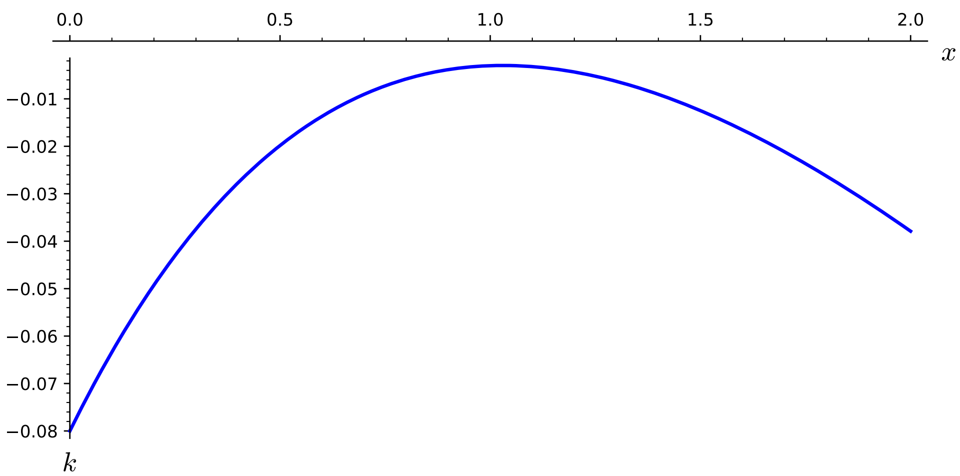

Figure 4.19: A plot of k when $\delta = 0.6$ and $p = -1.7$.

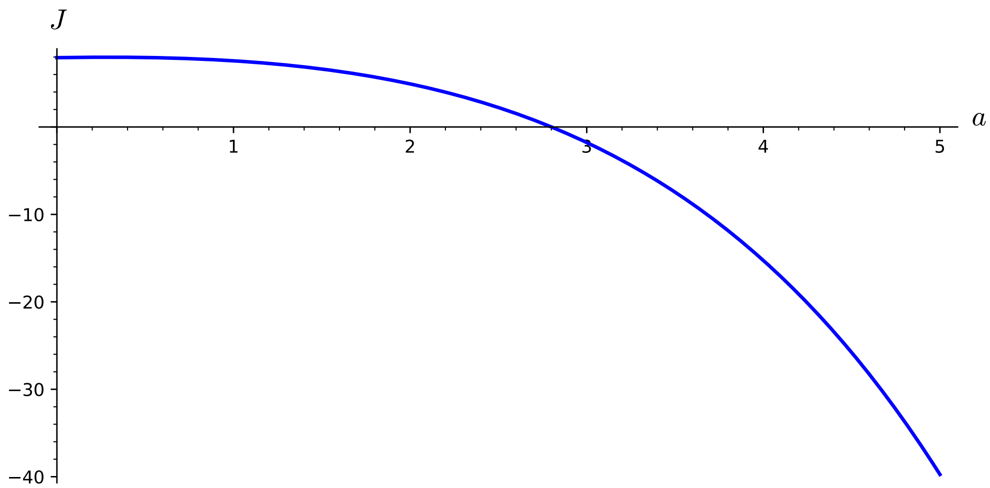

Figure 4.20: A plot of J when $\delta = 0.6$ and $p = -1.7$.

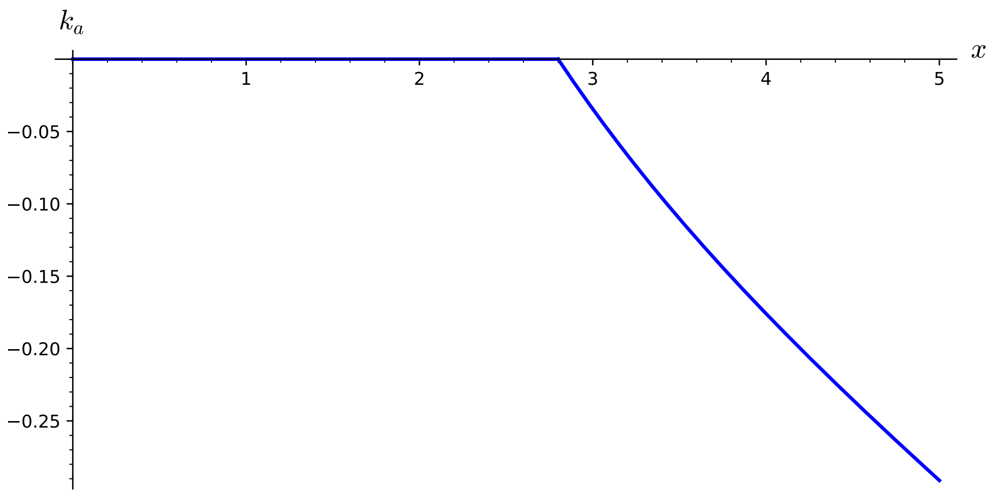

Figure 4.21: A plot of k_{a^*} when $\delta = 0.6$ and $p = -1.7$.

One interesting observation here is that k has no root on $[0, \infty)$ while J has a unique root. By setting a^* equal to the unique root of J, we can see from the shape of k_{a^*} in Figure 4.21. that the supermartingale property of $Z^{(a*)}$ holds.

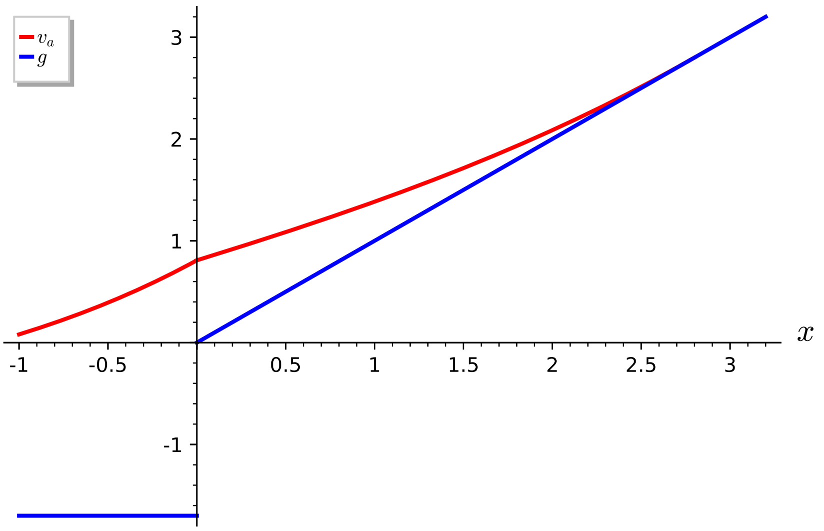

Figure 4.22: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -1.7$. Here $a^* = 2.800$.

Also, v_{a^*} behaves similar to those before with an upcrossing strategy to be optimal and it is also not differentiable at 0.

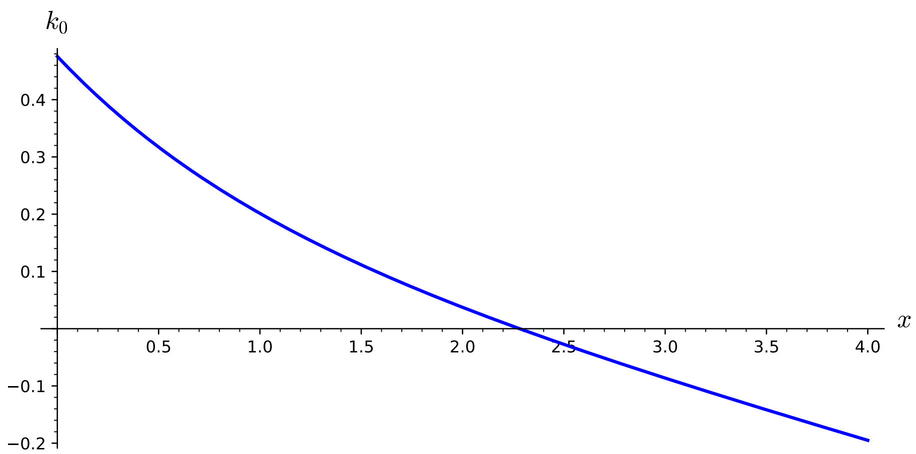

Figure 4.23: A plot of k_0 when $\delta = 0.6$ and $p = -1.7$.

Figure 4.23 again implies that $a = 0$ is not optimal. For the same reasons as discussed in Example 4.18 and Example 4.19, all these plots in Example 4.20 indicate that $D = [a^*, \infty)$ for some $a^* > 0$ would be optimal still though this has not yet been solved. $\triangleleft$

Example 4.21. When $p = -9 \in (\widehat{p}_0, \widehat{p}]$ and close to $\widehat{p}_0$. This is somehow the most interesting example we look at.

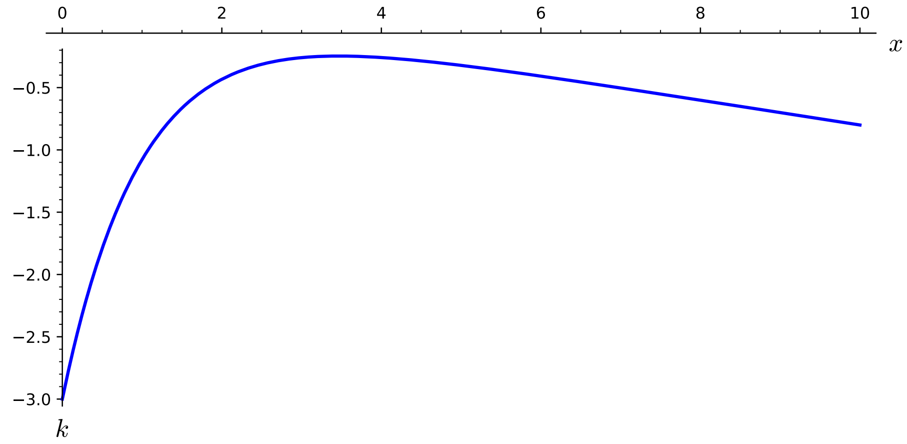

Figure 4.24: A plot of k when $\delta = 0.6$ and $p = -9$.

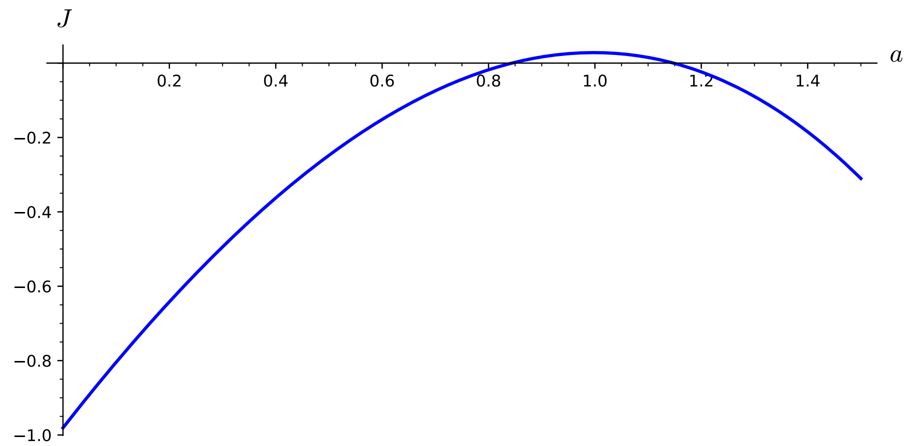

Figure 4.25: A plot of J when $\delta = 0.6$ and $p = -9$.

As we can see from Figure 4.24 and Figure 4.25, both k and J are not monotone and k has no root while J has two roots on $(0, \infty)$.

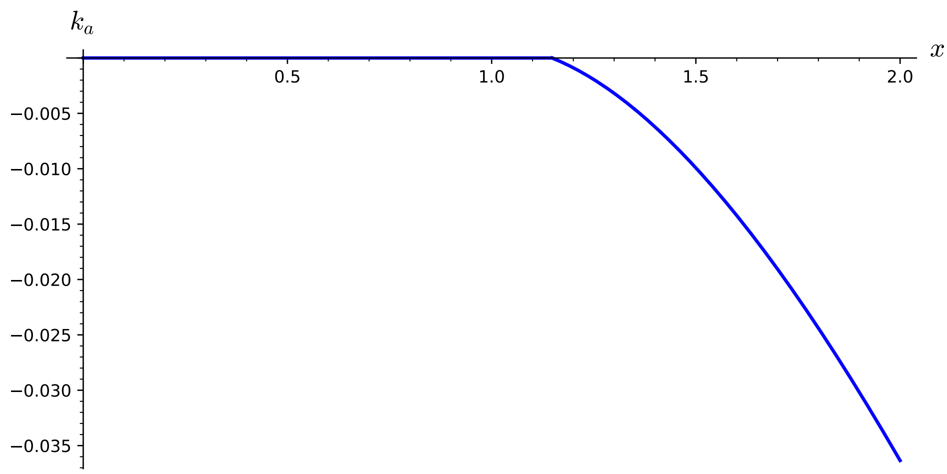

Figure 4.26: A plot of k_{a^*} when $\delta = 0.6$ and $p = -9$.

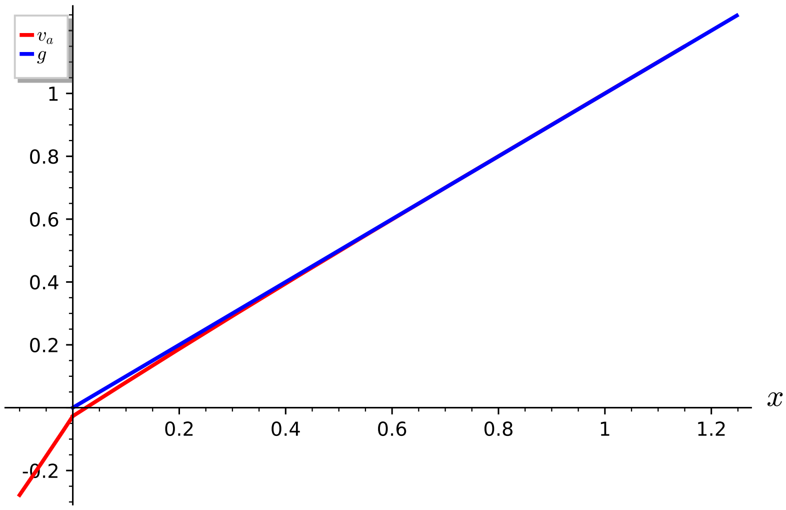

Figure 4.27: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -9$. Here $a^* = 1.148$.

By setting a^* equal to the largest root of J, we make the plots for k_{a^*} and v_{a^*} and from Figure 4.27, one can observe that v_{a^*} is below g at first and then goes above. To see this more clearly, we make a plot for $v_{a^*} - g$ as follows.

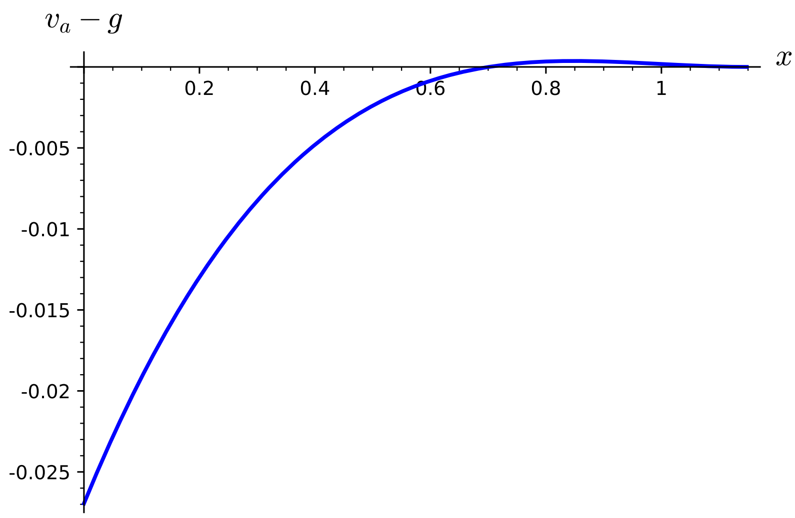

Figure 4.28: A plot of $v_{a^*} - g$. Here $a^* = 1.148$.

Figure 4.28 shows that v_{a^*} is less than g on $[0, x_0)$ with x_0 around 0.7. However, from the definition of v, we have that $v \geq g$ and hence v_{a^*} with $a^* = 1.148$ is not equal to v, that is, $[1.148, \infty)$ cannot be the optimal stopping region D.

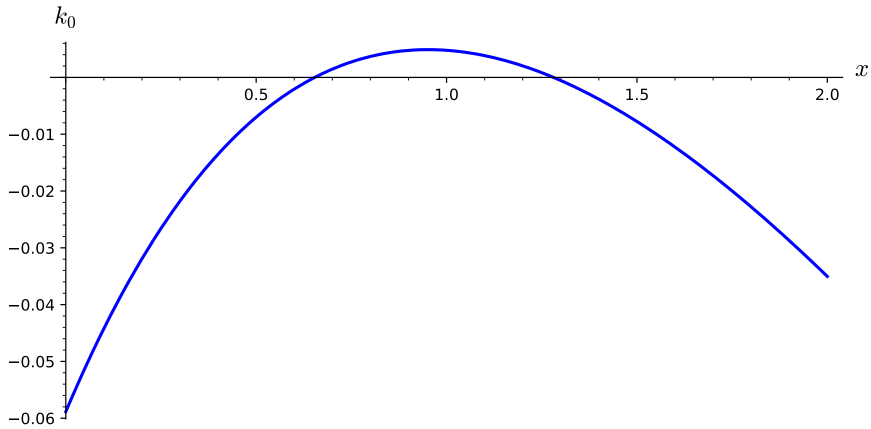

Figure 4.29: A plot of k_0 when $\delta = 0.6$ and $p = -9$.

We also make a plot for k_0 in Figure 4.29. By [38, Theorem 2.2], if $[0, \infty)$ were the optimal stopping region, then process $Z^{(0)}$ would be a supermartingale, which equivalently means that $k_0 \leq 0$ would hold everywhere. However, it can be seen from Figure 4.29 that $k_0 > 0$ on a certain interval, from which it follows that $Z^{(0)}$ is not a supermartingale and hence $[0, \infty)$ is not optimal.

So far we have shown that $[0, \infty)$ and $[1.148, \infty)$ are not optimal. Below we make analysis of possible stopping strategies.

First note that D cannot be empty due to the same reason as shown in step 1) from the proof of Theorem 4.12. Also, D cannot be of the form $[a^*, b^*]$ with $0 \leq a^* < b^* < \infty$, as for x large enough, $[a^*, b^*]$ is so far away that due to the discounting this situation is very similar to an empty stopping region and hence is not optimal.

Next, (a_0, a^*) is not in D where $a_0 < a^* = 1.148$ denote the two roots observed in Figure 4.25. This is because that for $a \in (a_0, a_1)$, we have $v_a(a) = a$ while $v_a'(a-) < 1$ due to $J(a) > 0$ as shown in Figure 4.25. It then follows that $v(x) \geq v_a(x) > x = g(x)$ for x close enough to a, i.e. $x \notin D$.

In fact, D cannot be of the form $[a, \infty)$ for $a \geq 0$. We have already seen that $D \neq [0, \infty)$, so below we consider $a > 0$. According to step 2) and 6) from the proof of Theorem 4.12, if $D = [a, \infty)$, then a must be the root of function J, i.e., a equals either a_0 or a^*. But we have shown above that (a_0, a^*) is not in D and $D \neq [1.148, \infty)$, it then follows that an up-crossing strategy is not optimal in this example.

Accordingly, we think that a stopping region of the form $[a^*, b^*] \cup [c^*, \infty)$ for some $0 \leq a^* \leq b^* \leq a_0$ and $c^* \geq a_*$ may be optimal here. Such guess may make sense in practice as well. When p becomes more negative, the insurer would choose to stop when U is close to zero from above to avoid killing and the following penalty; while when U is comparatively far away from negative half, it is also optimal to stop so that the profit can be taken.

In sum, combining the results shown in Example 4.20 and Example 4.21 together where $p \in (\widehat{p}_0, \widehat{p}]$, we expect that the optimal stopping region is either of the form $[a^*, \infty)$ as implied by Example 4.20, or of the form $[a^*, b^*] \cup [c^*, \infty)$ as suggested by Example 4.21. $\triangleleft$

Example 4.22. When $p = -12 \in (-\infty, \widehat{p}_0]$ and close to $\widehat{p}$.

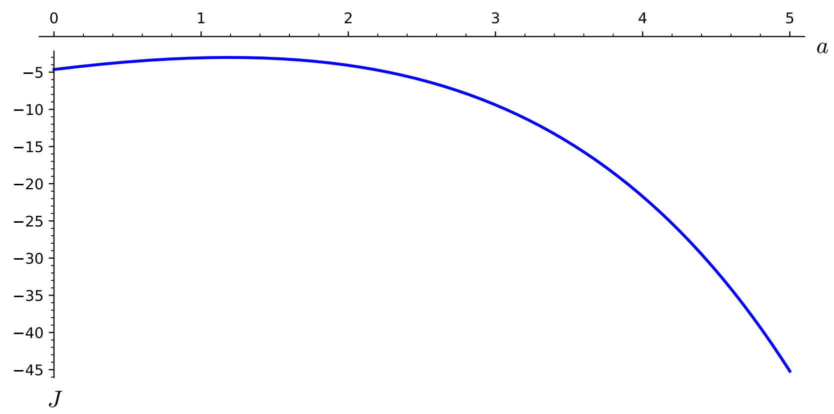

Figure 4.30: A plot of J when $\delta = 0.6$ and $p = -12$.

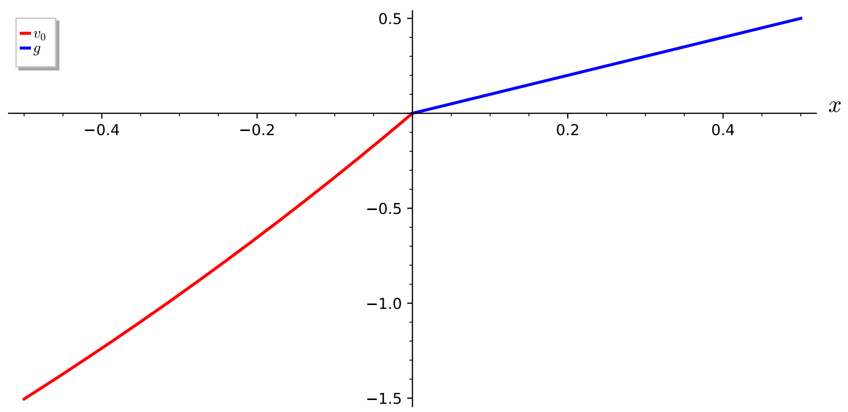

Figure 4.31: A plot of $v = v_{a^*}$ in red and g in blue when $\delta = 0.6$ and $p = -12$. Here $a^* = 0$.

Again, by case (i) in Theorem 4.13, we have $D = [0, \infty)$ and v_0 is similar to 4.16, v_0 is not C^1 across $x = 0$. While comparing Figure 4.30 with Figure 4.5, we can notice that J is no longer monontone here. $\triangleleft$

4.6.3 Other discussions

As shown in Theorem 4.12 and Theorem 4.13, an up-crossing strategy is optimal in certain cases. However, due to the existence of the penalty in payoff function g and an Omega clock in the model, traditional up-crossing strategies may not always be the optimal choice for some set of model parameters, e.g. see Example 4.21.

Consider penalty parameter p, we can immediately know that when p decreases, the optimal value function v must decrease as well, uniformly in x, and hence the optimal stopping region can only grow as p decreases. This can be seen for instance, from Example 4.14 and Example 4.15.

Consider the killing parameter q, if we let $q \to \infty$, the bankruptcy constructed by the Omega clock would be reduced to classical ruin, i.e. when the first time U drops below 0, ruin occurs and the insurer is forced to stop and left with the fine $p < 0$. Also note that $\widehat{p}$ given by (4.57) is independent of q while $\widehat{p}_0 \to \widehat{p}$ as $q \to \infty$. Hence, based on our existing results, we expect that a stopping region of the form $D = [a^*, \infty)$ with $a^* \geq 0$ would be optimal for our optimal stopping problem if the Omega clock is replaced by classic ruin.

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