OPTIMAL CONTROL OF TAXATION FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

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Dalal Ghanim Fahad Al Ghanim
School of Natural Science
Department of Mathematics

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In the context of loss-carry-forward taxation on the capital of an insurance company, we introduce two tax processes, latent and natural tax processes and show they are equivalent. This equivalence relation enables us to deal straightforward with the existence and uniqueness of the natural tax process, which is defined via an integral equation, and allows us to translate results from one model to the other. We clarify by our results the existing literature on tax processes. Using our equivalence relation, we derive an explicit expression for the expected deficit at ruin and the maximum surplus prior to ruin for the natural tax process when ruin happens before it reaches some positive level. We explain the relation of this expression with the draw-down literature. We introduce and solve two optimal control tax problems for a spectrally negative Lévy risk process. The first one aims to find the maximum tax value function and the tax strategy that achieves this. We prove a value function is the optimal value by putting it through a verification lemma. We find that, when the Lévy measure has a log-convex tail, the optimal tax strategy is a piecewise constant natural tax strategy. We show, on a special case, that our solution agrees with the solution of an optimal tax control problem considered in a previous literature. In the second optimal control tax problem, we add the bail-out concept to the model such that ruin is not allowed. An optimal strategy is defined as a tax and bail-out admissible strategy that maximises the net profit of taxation. In order to find the optimal tax value in this model, we introduce a new approach to find unknown fluctuation identities. Our work shows that the function representing the net present value of tax can be uniquely characterised by a PDE and a set of boundary conditions, and we use this to derive an explicit formula for this function. We verify that, on special cases, our results agree with existing results in the literature. We find, under no condition on the Lévy measure, that the optimal strategy is a piecewise constant tax rate function and a bail-out process which allows the capital to be injected back to zero whenever it becomes strictly negative. We introduce a natural tax model with bail-outs when ruin is allowed if the deficit at ruin exceeds some pre-specified level. We derive a new fluctuation identities for the Lévy process reflected at its infimum. We use these identities and our new approach, to find the net profit of taxation in this model. We do this under an assumption on the Lévy process, that it has positive Gaussian coefficient in the unbounded variation case.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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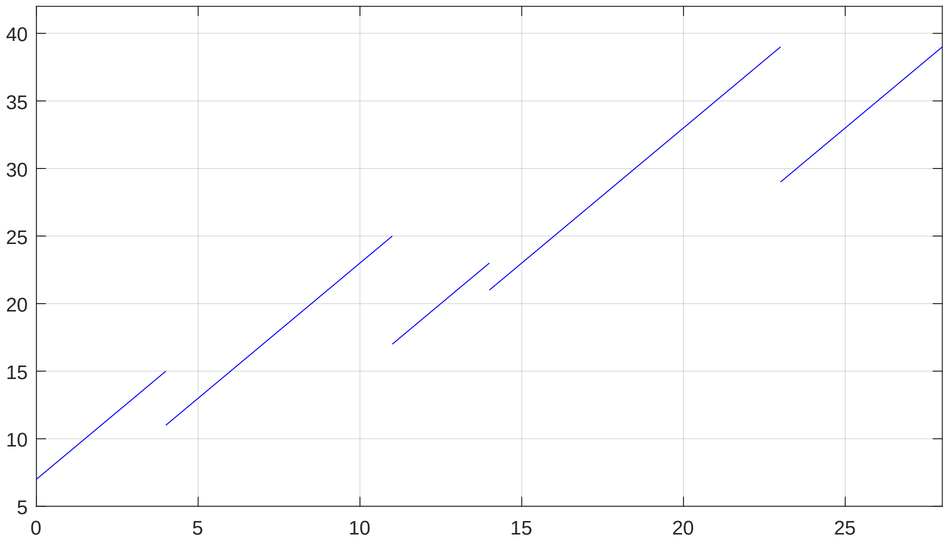
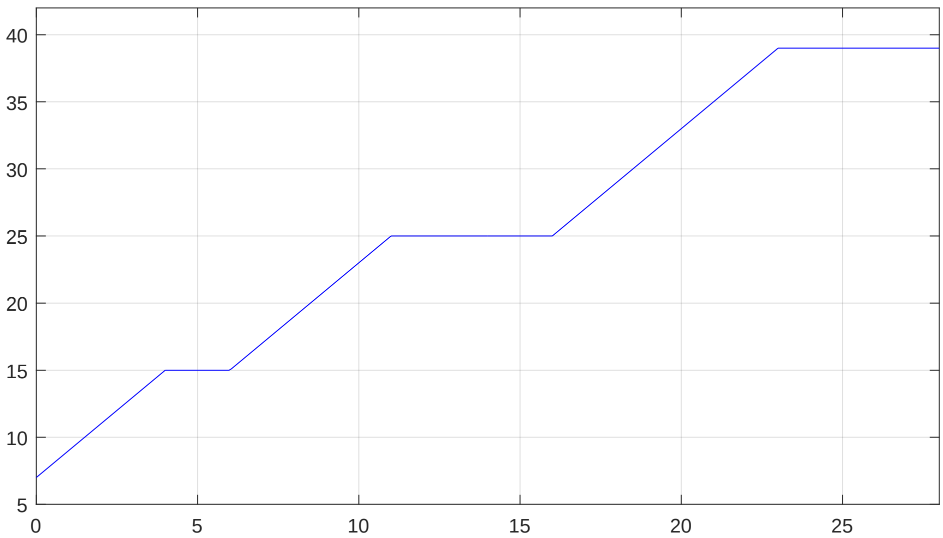
Chapter 1

Introduction

An insurance company usually monitors the evolution of the wealth, or surplus, that it has over the time in order to manage its businesses in the market. This allows the company to realise if it is gaining/losing money, or at risk of being in debt or bankrupt at some point of time. For this reason, the change of the wealth in time is modelled by the so called *risk process*. It refers to a collection of real valued random variables indexed by a subset of the real line, where randomness comes from the possible random events (or situations) that the company can face, and the indexed set interpreted as time. If we denote this risk process by $X = (X_t)_{t\geq 0}$, then X_t represents the capital of the company at time t. From the paths of X, one can easily understand if the company gains (when paths are increasing over time), or loses (when paths are decreasing). For instance, as an insurance company receives claims, and hence loses for covering these claims, then the paths of X are decreasing by downward jumps that represents the amounts of these claims. Denote the running maximum of X by $\overline{X}_t = \sup_{0\leq s\leq t} X_s$. As an example for X and $\overline{X}$, see figures 1.1a and 1.1b, respectively. The model X can be modified in order to cover any desirable features. For example, when an insurance company needs to pay out dividends of its surplus to its beneficiaries. De Finetti in [20] introduced this dividend model, and argued that, for a fixed level $b > 0$, any excess in capital of the barrier b is paid out to shareholders. That is, dividends at time t are given by $(b \vee \overline{X}_t) - b$, where we use the notation $b \vee a = \max\{b, a\}$. Note that, the process X reflected at its supremum with initial value say, b, is given by $(b \vee \overline{X}_t) - X_t$. Therefore, we can say that paying out dividends according to this strategy is the same as reflecting the paths of X at the barrier b. The resulting paths



(a) Path of a risk process X with $X_0 = 7$.



(b) The supremum process $\overline{X}$.

Figure 1.1: Plot of a risk process X and its supremum $\overline{X}$

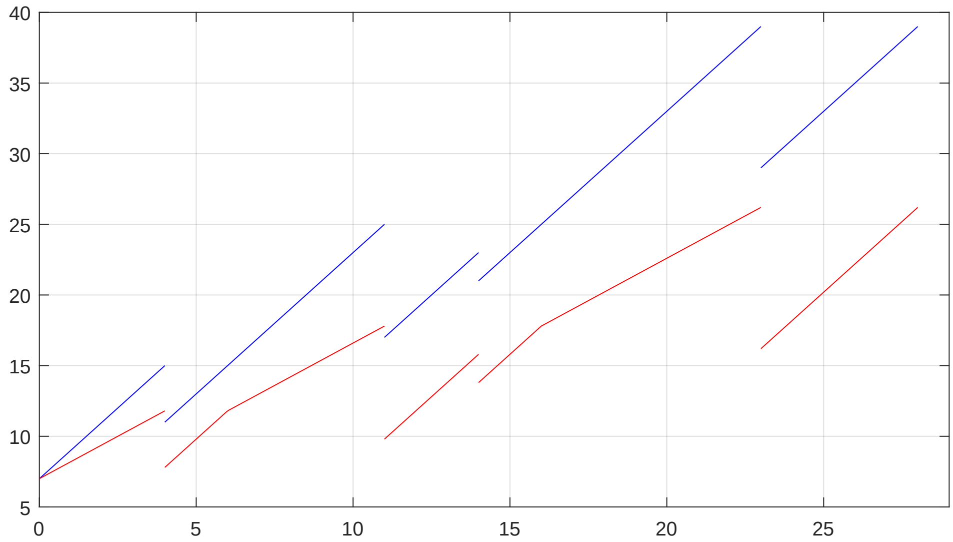


Figure 1.2: Paths of a risk process X (blue lines) and the associated tax process U^γ (red lines) with $X_0 = 7$ and $\gamma = 0.4$.

of X, which describes the reserve of the company after paying dividends, is given by

$$U_t := X_t - ((b \vee \overline{X}_t) - b) = X_t + b - (b \vee \overline{X}_t). \tag{1.1}$$

Further, X can be modified to model the case where dividends are paid out at a given fixed rate, say $\delta > 0$, whenever the capital is above a pre-specified level, say $b > 0$. This can be modelled by refracting the paths of X at the given level with a certain angle. This means that, a linear drift at rate δ is subtracted from the paths of X whenever it passes above level b. This modification is given in [31]. There is a class of modified risk processes between the reflected and refracted processes, which we study in this thesis and it is called *tax processes*. We give an example of a tax process, with constant tax rate, in order to simplify the idea of a tax process, and note that, our main tax model in this thesis is more general than this example. For a constant tax rate $\gamma \in (0,1)$, we introduce the tax process $U^\gamma := (U_t^\gamma)_{t\geq 0}$ with

$$U_t^\gamma = X_t - \gamma \left(\overline{X}_t - X_0\right), \qquad t \geq 0. \tag{1.2}$$

Figure 1.2 shows how the process X is transformed to U^γ by the subtraction of taxes, which can be seen as a partial reflection since $\gamma < 1$. That is, taxation with rate $\gamma < 1$ is similar to paying out dividends at a weaker rate than reflection. Suppose that $X_0 = b$, and γ is not constant, in such a way that before X reaching level b, $\gamma = 0$, and $\gamma = 1$ whenever X is above or equal to b, then, the two processes (1.1) and (1.2) are equal.

Looking at graph 1.2, it can be noticed that whenever the surplus process reaches a new maximum, partial reflection occurs. The times of these partial reflections can be understood to correspond to tax payments which happens only whenever the insurance company is in a profitable situation. This regime is called in the literature *loss-carry-forward taxation*. This phrase is explained as follows. Suppose that tax is due on an income of a company, but this income is negative. In this case, the company is allowed to carry this loss forward, and when it makes profit again, that previous loss can be used to offset the current profit. This procedure is called a loss-carry-forward taxation because it carries forward a tax loss at some time into a future time. A great deal of literature exists in the study of loss-carry-forward tax process, and was introduced in [2]. While γ is constant in some of these studies, like in [6], some authors extend γ to be a function that depends on the surplus process X, such as [33]. In the latter article, an unusual property of the process U^γ is, the tax rate function γ is a taxation at time t that depends on $\overline{X}_t$, the running maximum of X, and not on the running maximum of the process U^γ itself, $\overline{U}^\gamma = \sup_{s\leq t} U^\gamma_s$. That is, as explained in [1], the amount of tax payments the company makes at time t is not determined by the amount of capital the company has at that time but it depends on a latent capital level, namely $\overline{X}_t$, which is the amount of capital that the company would have at time t, if no taxes were paid out at all. For that reason, in this thesis, we define a new tax process $V^\delta = (V^\delta_t)_{t\geq 0}$, where δ is a tax rate function that depends on $\overline{V}^\delta_t = \sup_{s\leq t} V^\delta_s$ such that $\delta < 1$. We call V^δ a *natural tax process* or a tax process with *natural tax rate* δ, and U^γ a *latent tax process* or the tax process with *latent tax rate* γ (see [1]). Whereas latent and natural tax processes look quite different when considering their definitions, it appears that these two classes of tax processes are essentially equivalent. This observation has not been noticed in the literature before. In fact, this equivalence relation entitles us to deal in a straightforward way with the existence and uniqueness of the natural tax process, and again no one has studied this point before.

It is known that a risk process X has the Markov property when the future states of the process depends only upon the present state, not on the past states that preceded it. When X is modelled by a Markov process, we have the advantage that the two-dimensional process $(V^\delta, \overline{V}^\delta)$ is Markov. This advantage and the equivalence relation that we found between the two tax processes U^γ and V^δ enables us, for the first time in literature, to easily translate results derived for the latent tax process into results on the natural tax process, or vice versa, and present them in two dimensional expressions.

A risk process X belongs to a large class of random functions which is called stochastic processes. The literature is rich in various categories of stochastic processes based on their properties, and *Lévy processes* are one of them. In particular, in recent actuarial literature, see for example [29], the surplus of an insurance company is described as spectrally negative Lévy process. It is a Lévy process with downward jumps only, which correspond to the claims that a company receives. This feature makes it a suitable model for the capital process of insurance companies. Furthermore, many fluctuation identities in terms of a class of functions known as scale functions for spectrally negative Lévy process are available explicitly in literature, and this helps us to find many expressions of interest. As an example, in this thesis, for a natural tax process V^δ driven by spectrally negative Lévy process X, we derived the expected accumulated discounted tax payments in terms of scale functions for X. This expression is counted until ruin time (i.e when the company runs out of businesses or go bankrupt and mathematically defined as the first time that the tax process go below the zero level), and which is called the tax value function. Also, we established an analytic expression for the so-called overshoot identity, the expectation concerning the overshoot of the tax process over a fixed level. We derived some useful applications of this identity, such as the expected deficit at ruin and the expected aggregate surplus prior to ruin, before reaching some positive level. Moreover, an expression for the two sided exit problem is found, which is an identity concerning the exit of the tax process from a half-line or a strip. This latter identity gives a company an indication about the probability of ruin. For a spectrally negative Lévy process X, some research like [60] and [37] studied problems related to a first downward crossing time of the process X at a level that depends on $\overline{X}$, the so called *draw-down time.* Authors in [65, Section 4.1] explained how draw-down problems in models without tax is related to loss-carry-forward tax ruin problems. We point out here that, [64] found the overshoot identities of the latent tax process U^γ for a constant tax γ, by using this relation with the draw-down literature. In this thesis, we show that these overshoot identities found in [64] can be generalised for the latent tax process U^γ with γ a function of $\overline{X}$. We present these results for the natural tax process V^δ using our equivalence relation that we proved between the two tax processes.

As this thesis title indicates, we deal with optimal control of taxation for spectrally negative Lévy process. In general, an optimal control problem can be described briefly as follows. For a risk process X, we choose a control π that belongs to some pre-specified set of admissible controls depending on the problem that we study. The dynamics of X are changed by this control π, the controlled process is denoted by U^π, e.g U^γ in (1.2) and a value function v^π is assigned to each control. The value function v^π usually represents a cost or a reward corresponding to the control π, and defined as an expectation of a random variable that depends on the control π and the controlled process U^π. The optimal control problem is characterised as, finding the optimal value function, denoted by v^*, which is defined as the supremum (or infimum) of all value functions among all possible controls, and finding the optimal control π^* that achieves this value. In the optimal control problem that we study in this thesis, the tax payments takes the role of the control π. Naturally, a government wants to maximise the expected tax revenue and hence wants to know what is the tax strategy which produces this. We study in this work this type of optimal control problem, which was solved first in [57], where the control is the tax function γ that depends on $\overline{X}$ and the controlled process is U^γ. In this thesis, we generalise the optimal control problem they considered by defining a more general class of controls. Furthermore, by using our equivalence relation that we proved between U^γ and V^δ, we show that the solution of our problem is a natural tax process which agrees with the solution derived in [57] on a special case. Our work have lots of insight from the optimal control problem studied in many cases for dividends, such as [43, 45]. The methodology we use, which is the verification lemma for optimal control problem, and the condition we need to solve the problem, which is a condition on the Lévy measure of the spectrally negative Lévy process X, are similar to the ones considered for dividends.

It is normal to look at the optimal control problem for taxation when the government is bailing-out the company. In this thesis, we also study the tax process V^δ in the case of adding bail-outs. We mean by the word 'bail-out', when the government gives money from the tax fund they gain to the bankrupt institution, and in our work it is an insurance company, in order for this institution to survive and continue its businesses.

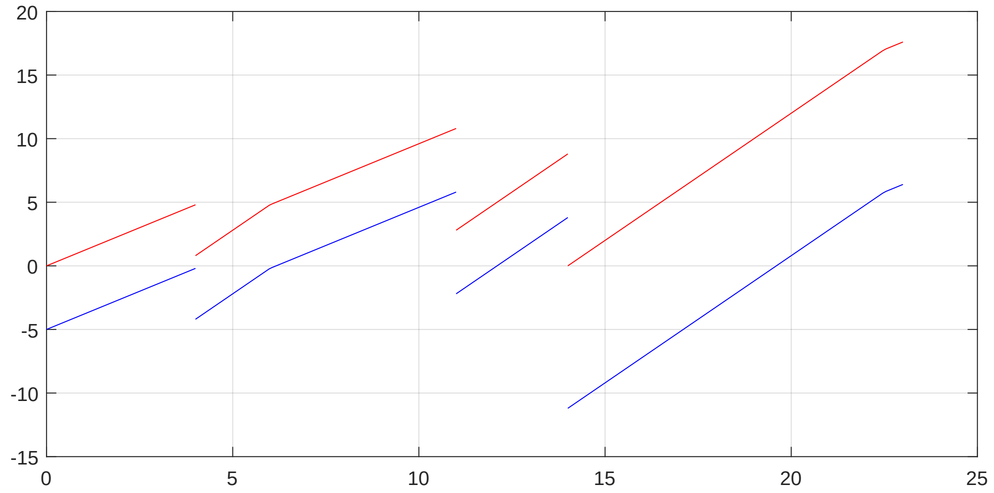


Figure 1.3: Paths of a natural tax process V^δ, with $\delta = 0.4$, before (blue lines) and after (red lines) adding the bail-out.

We denote by K^δ the bail-out process, which represents the capital injections made by the government whenever the tax process V^δ becomes strictly negative. Figure 1.3 shows an example of the process V^δ before and after adding the bail-out. We study two types of bailing-out. The first type, is when the bail-out is allowed forever. That is, the government always support the company by injecting the capital, whenever the aggregate process becomes below zero. We denote this tax process by $V^{\delta,\infty}$, and we call it *a natural tax process with forced bail-out*. We derive the net present value of profit that the government can gain. The tax value function is denoted by v_{tax}, and we denote by v_{inj} the total amount of bail-out injected by the government. Therefore, the net present value of profit is given by $v^{\delta,\infty} = v_{tax} - \eta\, v_{inj}$, where $\eta \geq 1$ represents the rate of loss the government can have when bailing-out, and called a *bail-out cost factor*. Authors in [3] and [4] found v_{tax}, v_{inj} and hence $v^{\delta,\infty}$, respectively, when the tax rate function is constant, while in our work, we generalise this expression for general natural tax rate function.

We point out that, the method we follow in finding each value function is created from the normal approach in literature of solving an optimal control problem by using a verification lemma. Our results prove that, if a function f of two variables satisfies some PDE and some boundary conditions, then it is the required value function; it turns out those conditions uniquely specify f, and we use them to derive an explicit formula for f. The first step in this method is that, we state and prove a lemma which includes the conditions and the PDE that a function should satisfy in order to be the required value function. Then, assuming that a function f satisfies these conditions and by using the Markov property of the process $(V, \overline{V})$, we derive a first order ODE for this f. We solve this ODE and find a candidate value function, f. At last, we verify that f satisfies all the conditions mentioned in the lemma given at the first step. We find that our methodology is flexible, as by changing some boundary conditions, we can find a different desirable value function. While we use our new approach to find the value functions, we needed to derive a two dimensional formula for some function satisfying some regularity conditions, which is called for the one dimensional case a Meyer-Itô formula.

In this thesis, for the first time in literature, we introduced an optimal control problem for the process $V^{\delta,\infty}$, to find the maximum tax net profit value and the optimal tax and bail-out strategy which produces this value. The second type of bailing-out, is when the bail-out is allowed to some pre-specified limited level. That is, the government is bailing-out the company only if the ruined process does not exceed a level $c < 0$, otherwise the government chooses to stop the bail-out process. We call the tax process in this type *a natural tax process with limited bail-out*. We derive, in a similar way to the forced bail-out case, the net present value of tax profit that the government make and we denote it by $v^{\delta,-c}$. Moreover, we find that $v^{\delta,\infty} = \lim_{c\downarrow-\infty} v^{\delta,-c}$.

This thesis is organised as follows. In Chapter 2, we present some background on general Lévy processes, then especially on spectrally negative Lévy process and their scale functions.

In Chapter 3, we clarify the equivalence relation between the two tax processes that we defined above, a latent tax process U^{γ} and the corresponding natural tax process V^{δ}. This is proved in Theorem 3.2.3, which allows us to use the corresponding results given in [33], for the latent tax processes, in deriving results for the natural tax process. We illustrate in the examples section, an existing natural tax process and a relation of our work with an existing literature.

In Chapter 4, we state and prove Theorem 4.1.2, which gives an Itô expansion for a two dimensional function defined on some specific space. This theorem will be used to prove many results all over the next chapters. Moreover, it allows us to establish, for the process $(V^{\delta}, \overline{V}^{\delta})$, an analytic expression of its overshoot over a certain level in terms of some given operator and the scale function. This work is in Theorem 4.1.4, in which we obtain an expression for the expectation concerning the overshoot of V^δ over some level c, and the maximum $\overline{V}^\delta$ prior to crossing that level, when crossing level c happens before reaching some positive level a. In addition, we use Theorem 3.2.3 to establish an explicit expression for that expectation in the case $c = 0$. Then, we provide some useful applications of the derived formula. In Section 4.3, we introduce our new approach to obtain useful identities of the natural tax process which will be implemented to find the net present tax value function in Chapter 6 and 7.

In Chapter 5, we introduce and solve a more general control problem than the one studied in [57]. A verification lemma is proved and used to show that a piecewise constant natural tax rate is an optimal strategy under suitable assumption imposed on the Lévy measure. This optimal solution agrees with the one found in [57] on a special case. Remarkably, we found out that Wang and Hu obtained a natural tax process as the optimal solution to the problem of controlling a latent tax process.

In Chapter 6, we define a natural tax process together with the effect of the minimal capital injections required to keep the surplus non-negative, where ruin is not allowed. An algorithm is given in Section 6.2 to explain the construction of this tax process and a proposition is proved to show its existence. The net present value of tax in this case is derived, by our new approach given in Section 4.3, for a general natural tax rate function. We verified at the end that our result agrees with [3] and [4] in the constant tax case. For the first time in literature, an optimal control problem of the natural taxation with forced bail-out case is introduced and solved, under no conditions on the Lévy measure.

In Chapter 7, we study the same natural tax process with bail-out introduced in Chapter 6 but in the case where ruin and injections are both allowed after ruin. The surplus is injected back to zero whenever it becomes negative as long as the deficit is above or equal some parameter $c < 0$. While we use in the previous chapter some results of the reflected Lévy process given in [10], we derived in Section 7.2 our own identities, such as the two sided-exit problem and the expected accumulated discounted amount of capital injections. The net present value of tax payments in this case is obtained in Section 7.3.

In Chapter 8, we give briefly, the progress in the study of loss-carry-forward tax in general. We state in summary, the existing literature and the corresponding problems they addressed.

Chapter 2

Preliminaries

The aim of this chapter is to provide a short review of definitions and results on Lévy processes. We also place emphasis on spectrally negative Lévy processes and scale functions that will be used in this research.

2.1 Lévy processes

We first give the definition of a Lévy process as in [30, Definition 1.1].

Definition 2.1.1 (Lévy process). A process $X = \{X_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Lévy process if it possesses the following properties:

i) The paths of X are $\mathbb{P}$-almost surely right-continuous with left limits.

ii) $\mathbb{P}(X_0 = 0) = 1$.

iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

It is clear from part iv) that all Lévy processes satisfy the Markov property, which is for each $B \in \mathcal{B}(\mathbb{R})$ and s, $t \geq 0$,

$$\mathbb{P}(X_{t+s} \in B|\mathcal{F}_t) = \mathbb{P}(X_{t+s} \in B|\sigma(X_t)),$$

where $\sigma(X_t)$ is the smallest σ-algebra such that X_t is measurable. The strong Markov property for Lévy processes is given by the next result.

Theorem 2.1.2 ([30, Theorem 3.1]) *Suppose that τ is a stopping time. Define on $\{\tau < \infty\}$ the process $\tilde{X} = \{\tilde{X}_t : t \geq 0\}$ where*

$$\tilde{X}_t = X_{\tau+t} - X_\tau,\ t \geq 0.$$

Then, on the event $\{\tau < \infty\}$, the process $\tilde{X}$ is independent of $\mathcal{F}_\tau$, has the same law as X and hence in particular is a Lévy process.

Let Π be a Borel measure concentrated on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty.$$

According to Theorems 1.3 and 1.6 in [30], there exist parameters (a, σ^2, Π) with $a \in \mathbb{R}$, $\sigma \geq 0$ called the Gaussian coefficient, and a measure Π, that identify the distribution of X_t via its characteristic function

$$\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)},$$

where

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x).$$

It is known that every Lévy process is a semi-martingale, see Theorem 9 in [52]. The next result gives a condition that clarifies when a Lévy process X is of bounded variation.

Lemma 2.1.3 ([30, Lemma 2.12]) *A Lévy process with triplet (a, σ^2, Π) has paths of bounded variation if and only if*

$$\sigma = 0 \ \textit{and} \ \int_{\mathbb{R}} (1 \wedge |x|)\Pi(dx) < \infty.$$

A Lévy process is càdlàg, that is, right-continuous process with left limits. Therefore, the only type of discontinuities it can have is jump discontinuities. Let the jump of a Lévy process X at time t be defined as

$$\Delta X_t = X_t - X_{t-}.$$

In order to explain the jump structure of a Lévy process X, and also excursions of X, we need to understand the meaning of a Poisson random measure and a Poisson point process.

Definition 2.1.4 ([30, Definition 2.3]) Let $(S, \mathcal{S}, \eta)$ be an arbitrary sigma-finite measure space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $N\colon \Omega \times \mathcal{S} \to \{0, 1, 2, ...\} \cup \{\infty\}$ in such a way that the family $\{N(., A)\colon A \in \mathcal{S}\}$ are random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then N is called a Poisson random measure on $(S, \mathcal{S}, \eta)$ (or Poisson random measure on S with intensity η) if

(i) for mutually disjoint $A_1, ..., A_n$ in $\mathcal{S}$, the variables $N(A_1), ..., N(A_n)$ are independent,

(ii) for each $A \in \mathcal{S}, N(A)$ is Poisson distributed with parameter $\eta(A)$, where we allow $0 \leq \eta(A) \leq \infty$,

(iii) $N(.)$ is a measure $\mathbb{P}$ almost surely.

The supporting random set of points of N on $(S, \mathcal{S}, \eta)$ is called Poisson point process on S with intensity η.

Let $A \in \mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$ and define

$$N(A) = card \{(t, \Lambda) \in A, t > 0\},$$

where *card* means the cardinality of a set, and which can be written as,

$$N(A) = \sum_{0 \leq s \leq t} \mathbf{1}_\Lambda(\Delta X_s).$$

Then, by Lemma 2.2 in [30], $N\colon \mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R} \setminus \{0\}) \to \{0, 1, 2, ...\} \cup \{\infty\}$ is a Poisson random measure with intensity $\mathrm{d}t \times \Pi(\mathrm{d}x)$ that describes the jumps of a Lévy process X.

The next result is a combination of Theorem 4.4 and Corollary 4.6 in [30]. It generalizes Theorem 2.7 in [30] and gives the compensation formula for a Lévy process X.

Theorem 2.1.5 *Suppose $\phi\colon [0, \infty) \times \mathbb{R} \times \Omega \to [0, \infty)$ is a random time-space function such that*

(i) $\phi(t, x)[\omega]$ is measurable,

(ii) for each $t \geq 0$, $\phi(t, x)[\omega]$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$-measurable,

(iii) for each $x \in \mathbb{R}$, with probability one, $\{\phi(t,x) \colon t \geq 0\}$ is a left continuous process.

Then, for all $t \geq 0$,

$$\mathbb{E}\left(\int_{[0,t]}\int_{\mathbb{R}} \phi(s,x)N(\mathrm{d}s \times \mathrm{d}x)\right) = \mathbb{E}\left(\int_{[0,t]}\int_{\mathbb{R}} \phi(s,x)\Pi(\mathrm{d}x)\mathrm{d}s\right) \tag{2.1}$$

with the understanding that the right-hand side is infinite if and only if the left-hand side is. Moreover, if for all $t \geq 0$,

$$\mathbb{E}\left(\int_{[0,t]}\int_{\mathbb{R}} \phi(s,x)\Pi(\mathrm{d}x)\mathrm{d}s\right) < \infty,$$

then we have that

$$M_t := \int_{[0,t]}\int_{\mathbb{R}} \phi(s,x)N(\mathrm{d}s \times \mathrm{d}x) - \int_{[0,t]}\int_{\mathbb{R}} \phi(s,x)\Pi(\mathrm{d}x)\mathrm{d}s, \ t \geq 0, \tag{2.2}$$

is a martingale.

According to Theorem 42 in [52], a Lévy process X with characteristic triple (a, σ^2, Π) has a decomposition

$$\begin{aligned} X_t = \sigma B_t + at &+ \left(\int_{[0,t]}\int_{\{|x|<1\}} xN(\mathrm{d}s \times \mathrm{d}x) - t\int_{\{|x|<1\}} x\Pi(\mathrm{d}x)\right) \\ &+ \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq 1\}}, \end{aligned} \tag{2.3}$$

where B is a Brownian motion, N is a Poisson random measure with intensity $\mathrm{d}t \times \Pi(\mathrm{d}x)$, such that for any $A \in \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$, $N(A)$ is a Poisson process independent of B.

A spectrally negative Lévy process is a Lévy process without upward jumps. We give an example of this type of processes, which is explained in details in Section 1.3.1 [30].

Example 2.1.6 Cramér-Lundberg process. Suppose that the capital of an insurance company is modelled as X at time t

$$X_t = X_0 + c\,t - \sum_{i=1}^{N_t} \xi_i,$$

where X_0 denotes the initial surplus, $c > 0$ represents the premium rate, $(N_t)_{t\geq 0}$ is a Poisson process with rate λ, and the sequence $(\xi_i)_{i\geq 1}$ consists of positive random variables which represent the claims. They are independent and identically distributed with a common distribution F, and $(N_t)_{t\geq 0}$, $(\xi_i)_{i\geq 1}$ are independent. The summation itself is called a compound Poisson process. The process X is a spectrally negative Lévy process that has triplet $(c - \lambda\int_0^1 x\,F(dx), 0, \lambda\,F(-dx)\mathbf{1}_{\{x<0\}})$.

For the rest of this chapter, the process X will only be considered as a spectrally negative Lévy process. We also exclude the cases of X having monotone paths. We define the Lévy measure of X, $\nu(\mathrm{d}x) := \Pi(-\mathrm{d}x)$, as a measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty.$$

We define the Laplace exponent as follows;

$$\psi(\theta) := -\Psi(-i\theta).$$

That is, the Lévy triplet of X is given by (a, σ^2, ν), and for $\theta \geq 0$,

$$\psi(\theta) = \frac{1}{t}\log\mathbb{E}\left(e^{\theta X_t}\right) = -a\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \left(1 - e^{-\theta x} - \theta x\mathbf{1}_{\{0<x<1\}}\right)\nu(\mathrm{d}x).$$

One of the important properties of this Laplace exponent is that it is infinitely differentiable and strictly convex on $(0, \infty)$. The right inverse of ψ is defined as

$$\Phi(q) = \sup\{\theta \geq 0 \colon \psi(\theta) = q\}, \tag{2.4}$$

for $q \geq 0$. Also, $\psi'(0+) = \mathbb{E}[X_1] \in [-\infty, \infty)$ and the asymptotic behaviour of X can be recognized by $\psi'(0+)$ as in [30, Theorem 7.2]. So, $\psi'(0+) > 0$ if and only if X drifts to infinity, $\psi'(0+) < 0$ if and only if X drifts to minus infinity, and $\psi'(0+) = 0$ if and only if X oscillates. As given in [30, Theorem 3.6], the Laplace exponent $\psi(\theta)$ is finite if and only if

$$\int_{|x|\geq 1} e^{-\theta x}\nu(\mathrm{d}x) < \infty.$$

Under this assumption, one can show that

$$\mathcal{E}_t(\theta) = e^{\theta X_t - \psi(\theta)t}, \qquad t \geq 0,$$

is a unit mean martingale with respect to $\mathbb{F}$. Then we can define a change of measure as

$$\left.\frac{\mathrm{d}\mathbb{P}_t^\theta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\theta), \qquad 0 \leq t < \infty, \qquad \theta \geq 0,$$

which means that for $A \in \mathcal{F}_t$

$$\mathbb{P}_t^{\theta}(A) = \mathbb{E}\left(\mathbf{1}_A \mathcal{E}_t(\theta)\right), \qquad 0 \leq t < \infty, \qquad \theta \geq 0,$$

which is called the Esscher transform. It is well known that $(X_s, \mathbb{P}_s^{\theta})_{0 \leq s \leq t}$ is also a spectrally negative Lévy process, as shown in [30, p.83]. If we perform the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_t^{\Phi(q)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\Phi(q)),$$

to the spectrally negative Lévy process $(X, \mathbb{P})$, then the Laplace exponent of $(X, \mathbb{P}^{\Phi(q)})$ is given by $\psi_{\Phi(q)}(\theta) = \psi(\theta + \Phi(q)) - q$. Note that, by the strict convexity of ψ and for $q > 0$, $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$. This implies that $(X, \mathbb{P}^{\Phi(q)})$ drifts always to infinity for $q > 0$.

For an $\mathbb{F}$-stopping time τ, as $t \wedge \tau \leq t$, then by Doob's optional sampling theorem, [52, Theorem 16, p.9], for $A \in \mathcal{F}_{t\wedge\tau}$,

$$\mathbb{P}_t^{\theta}(A) = \mathbb{E}\left[\mathbf{1}_A e^{\theta X_t - \psi(\theta) t}\right] = \mathbb{E}\left[\mathbf{1}_A e^{\theta X_{t\wedge\tau} - \psi(\theta)(t\wedge\tau)}\right]. \tag{2.5}$$

2.2 Scale functions

In this section, we define the q-scale functions of X and state some of their properties and significant results. The existence of these scale functions is proven in [30, Theorem 8.1], which is presented in the following theorem.

Theorem 2.2.1 *There exist a family of functions $W^{(q)} : \mathbb{R} \to [0, \infty)$ and*

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)\,\mathrm{d}y, \ \textit{for } x \in \mathbb{R},$$

defined for $q \geq 0$, such that the following hold:

i) For any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is characterised on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^{\infty} e^{-\beta x} W^{(q)}(x)\,dx = \frac{1}{\Psi(\beta) - q} \quad \textit{for } \beta > \Phi(q).$$

ii) Let $\rho_0^- = \inf\{t \geq 0 : X_t < 0\}$, then for any $x \in \mathbb{R}$ and $q \geq 0$

$$\mathbb{E}_x[e^{-q\rho_0^-}\mathbf{1}_{(\rho_0^- < \infty)}] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

where we understand $\frac{q}{\Phi(q)}$ *in the limiting sense for* $q = 0$*, so that*

$$\mathbb{P}_x\left(\rho_0^- < \infty\right) = \begin{cases} 1 - \psi'(0+)\, W(x) & \text{if } \psi'(0+) \geq 0 \\ 1 & \text{if } \psi'(0+) < 0. \end{cases} \tag{2.6}$$

iii) For any $x \leq a$ *and* $q \geq 0$*, let* $\rho_a^+ = \inf\{t \geq 0 : X_t > a\}$*, then*

$$\mathbb{E}_x[e^{-q\rho_a^+}\mathbf{1}_{(\rho_0^- > \rho_a^+)}] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.7}$$

and

$$\mathbb{E}_x[e^{-q\rho_0^-}\mathbf{1}_{(\rho_0^- < \rho_a^+)}] = Z^{(q)}(x) - Z^{(q)}(a)\,\frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.8}$$

where $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$.

If we let $\overline{W}^{(q)}(y) = \int_0^y W^{(q)}(z)\,\mathrm{d}z$, then the relation between $W^{(q)}$ and $\overline{W}^{(q)}$ is given by

Lemma 2.2.2 ***[10, Lemma 1]*** *For* $y \in [0, a]$ *and* $a > 0$*, it holds that*

$$\frac{\overline{W}^{(q)}(y)}{\overline{W}^{(q)}(a)} \leq \frac{W^{(q)}(y)}{W^{(q)}(a)}.$$

For $q = 0$, we shall write W instead of $W^{(0)}$ and we call it the scale function instead of the 0-scale function. It is known that, as shown in [30, Section 8.2], for $q \geq 0$,

$$W^{(q)}(x) = e^{\Phi(q)x}\, W_{\Phi(q)}(x), \tag{2.9}$$

where it is clear that $W_{\Phi(q)}$ is the scale function for the process $(X, \mathbb{P}^{\Phi(q)})$. By relation (2.9), it is proven in [44] that $W^{(q)}$ is log-concave on $(0, \infty)$ for all $q \geq 0$. That is, for $q \geq 0$, $\log(W^{(q)}(x))$ is concave on $(0, \infty)$. It has been shown in [28, Lemma 2.3], that for any $q \geq 0$, $W^{(q)}$ is absolutely continuous with respect to the Lebesgue measure and strictly increasing. We denote by $W^{(q)\prime}$ the associated density. Moreover, we have the following result.

Lemma 2.2.3 ***[28, Lemma 2.4]*** *For each* $q \geq 0$*, the scale function* $W^{(q)}$ *is continuously differentiable if and only if at least one of the following three criteria holds,*

(i) $\sigma \neq 0$

(ii) $\int_{(0,1)} x\,\nu(dx) = \infty$

(iii) The tail of the Lévy measure, $\nu(x,\infty) = \int_x^\infty \nu(dr)$, is continuous.

And in the unbounded variation case, we have the next result.

Theorem 2.2.4 ***[16, Theorem 1]*** *Suppose that X has a Gaussian component. For each fixed $q \geq 0$, the function $W^{(q)}$ is twice continuously differentiable on $(0,\infty)$.*

By the help of (2.9), the asymptotic behaviour of the scale function at infinity is given through the following result.

Lemma 2.2.5 ***[19, Lemma 3.3]*** *For $q \geq 0$, we have*

$$\lim_{x\uparrow\infty} e^{-\Phi(q)x} W^{(q)}(x) = \frac{1}{\psi'(\Phi(q))},$$

and

$$\lim_{x\uparrow\infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)}, \tag{2.10}$$

where the right hand side above is understood in the limiting sense $\lim_{q\downarrow 0} \frac{q}{\Phi(q)} = 0 \vee (\frac{1}{\psi'(0+)})$ when $q = 0$.

Therefore, by the above lemma and as given in [35], when $q > 0$, the function $W^{(q)}(x)$ behaves like the exponential function $e^{\Phi(q)x}$ for large x. Thus, for $q > 0$, $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$.

The behaviour of $W^{(q)}$ in the neighbourhood of the origin is given in the next result.

Lemma 2.2.6 ([35, Lemma 1]) *As $x \downarrow 0$, the value of the scale function $W^{(q)}(x)$ and its right derivative are determined for every $q \geq 0$ as follows:*

$$W^{(q)}(0+) = \begin{cases} \dfrac{1}{d} & \text{when } \ \sigma = 0 \ \text{ and } \ \int_0^1 x\nu(\mathrm{d}x) < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

$$W^{(q)\prime}(0+) = \begin{cases} \dfrac{2}{\sigma^2} & \text{when } \ \sigma > 0, \\ \dfrac{\nu(0,\infty)+q}{d^2} & \text{when } \ \sigma = 0 \ \text{ and } \ \nu(0,\infty) < \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{2.12}$$

where $d = a + \int_0^1 x\nu(\mathrm{d}x) > 0$ stands for the drift of X when it is of bounded variation.

Let $Y_t := X_t - I_t$, where $I_t = \inf_{0\le s\le t}(X_s \wedge 0)$. We call Y, the reflected process at its infimum. The next result gives the analytical expression of the Laplace transform of the entrance time of the reflected process Y into (a,∞) with $a > 0$.

Lemma 2.2.7 ***[50, Proposition 2]*** *For $a > 0$, define $T_a^+ := \inf\{t \ge 0 : Y_t > a\}$. Let $x \in [0,a]$, and $q \ge 0$. Then we have*

$$\mathbb{E}_x\left[e^{-qT_a^+}\right] = \frac{Z^{(q)}(x)}{Z^{(q)}(a)}.$$

Recall that $\rho_a^+ = \inf\{t \ge 0 : X_t > a\}$ and $\rho_0^- = \inf\{t \ge 0 : X_t < 0\}$. Through the proof of Lemma 2.2.7 in [50], it is shown that

$$\{e^{-q(t\wedge\rho_a^+\wedge\rho_0^-)}\, W^{(q)}(X_{t\wedge\rho_a^+\wedge\rho_0^-}),\ t \ge 0\}, \tag{2.13}$$

and

$$\{e^{-q(t\wedge\rho_a^+\wedge\rho_0^-)}\, Z^{(q)}(X_{t\wedge\rho_a^+\wedge\rho_0^-}),\ t \ge 0\} \tag{2.14}$$

are martingales. Moreover, we have the following result which is proved in [10, Proposition 2].

Lemma 2.2.8 *Define*

$$\overline{Z}^{(q)}(y) = \int_0^y Z^{(q)}(z)\,\mathrm{d}z = y + q\int_0^y\int_0^z W^{(q)}(w)\,\mathrm{d}w\,\mathrm{d}z,\ \ y \in \mathbb{R}. \tag{2.15}$$

If $\psi'(0+) > -\infty$, then

$$\left\{e^{-q(t\wedge\rho_0^-)}\left[\overline{Z}^{(q)}(X_{t\wedge\rho_0^-}) + \frac{\psi'(0+)}{q}\right],\ t \ge 0\right\} \tag{2.16}$$

is a martingale.

A key object which plays a central role in scale functions theory, is the excursion measure, which helps to prove many fluctuation identities and results, within the context of spectrally negative Lévy process. For example, the proof of Theorem 3.2.9 in Chapter 3 below shows how we use excursion measure to find the net present value of taxation. The basic idea of excursion theory for a Lévy process is to describe the successive sections of its trajectory, which make up excursions from its previous maximum. This is explained in defining the excursion space below. In order to give this definition, we need to introduce a new time-scale which locates the times at which a Lévy process X creates new maxima, which we call a local time at the maximum of X, and denote it by L.

Definition 2.2.9 ([30, Definition 6.1]) A continuous, non-decreasing, $[0,\infty)$-valued, $\mathbb{F}$-adapted process, $L = \{L_t : t \geq 0\}$, is called a local time at the maximum if the following hold.

(i) The support of the Stieltjes measure $\mathrm{d}L$ is the closure of the random set of times $\{t \geq 0 : \overline{X}_t = X_t\}$.

(ii) For every $\mathbb{F}$-stopping time T such that $\overline{X}_T = X_T$ on $\{T < \infty\}$ almost surely, the shifted process

$$\{L_{T+t} - L_T : t \geq 0\}$$

is independent of $\mathcal{F}_T$ on $\{T < \infty\}$ and has the same law as L under $\mathbb{P}$.

Remark 1 Let $\overline{X} = \sup_{0\leq s\leq t} X_s$, then $L = \overline{X}$ satisfies Definition 2.2.9.

The inverse local time, $L^{-1} := \{L_t^{-1} : t \geq 0\}$, is defined in [30, Section 6.2] as

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty \\ \infty & \text{otherwise,} \end{cases} \tag{2.17}$$

where in this definition t is a local time and L_t^{-1} is the real time at which X reaches a new maxima.

The next result is the change of variables formula for Stieltjes integrals.

Lemma 2.2.10 ([54, Proposition 4.9]) *Let A be an increasing, possibly infinite, right-continuous function and for $t \geq 0$,*

$$C_t = \inf\{s > 0 : A_s > t\}$$

is the right-continuous inverse of A. If f is a positive Borel function on $[0,\infty)$, then

$$\int_0^\infty f(u)\,\mathrm{d}A(u) = \int_0^{A_\infty} f(C_s)\,\mathrm{d}s.$$

Note that, we can use Lemma 2.2.10 with $A = L$, the local time of the process X at the maximum.

We are now able to give the definition of an excursion space, which is based on Section 2.3 in [19]. The space of excursions, $\mathcal{E}$, is the space of real valued, right-continuous paths with left limits, and which are killed at the first hitting time of

$(-\infty, 0]$. These paths are the excursions of X from its running supremum and denoted by ε_t, where

$$\varepsilon_t = \{\varepsilon_t(s) := X_{L_t^{-1}} - X_{L_{t^-}^{-1}+s} \colon 0 < s \leq L_t^{-1} - L_{t^-}^{-1}\},$$

whenever $L_t^{-1} - L_{t^-}^{-1} > 0$. The height of each excursion ε is given by $\bar{\varepsilon} = \sup_{s\geq 0} \varepsilon(s)$. According to Definition 2.1.4, the process $\{(t, \varepsilon_t(s)) \colon t \geq 0, 0 < s \leq L_t^{-1} - L_{t^-}^{-1}\}$ is a Poisson point process on $[0, \infty) \times \mathcal{E}$ with intensity $\mathrm{d}t \times \mathrm{d}n$, where n is the excursion measure defined on the space $(\mathcal{E}, \mathcal{F})$, with $\mathcal{F} = \sigma(\mathcal{E}_t(s))$. The n- measurable functional, $\bar{\varepsilon}$, has a very useful formula related with the scale function, given in [30, Chapter 8], which is

$$n(\bar{\varepsilon} > x) = \frac{W'(x)}{W(x)}, \tag{2.18}$$

such that x is not a point of discontinuity in W'.

Explicit forms of scale functions are found in many examples in the literature such as [26]. Also, [30, Chapter 9] explains a method for generating many examples of spectrally negative Lévy process such that one can compute their associated scale functions explicitly. In [28, pp.157-181], several numerical methods with examples are explained to compute scale functions for a general spectrally negative Lévy process.

Chapter 3

Tax processes

This chapter is taken from our published article [1] with some modification on the assumptions that gives a slight generalisation on the results.

3.1 Introduction

We start this chapter by rigorously defining the type of tax processes that we are interested in this thesis. First, we make some assumptions on X. We redefine $\overline{X}$ as

$$\overline{X}_t := \overline{X}_0 \vee \sup_{0 \le s \le t} X_s, \quad t \ge 0,$$

where we assume that $X_0 = x$ and $\overline{X}_0 = \bar{x}$ such that $x \le \bar{x}$. We assume that X is a stochastic process where $\overline{X}$ is a continuous process. An example of such a process, is a stochastic process with càdlàg paths and without upward jumps. Also, X is associated with probabilities

$$\mathbb{P}_{x,\bar{x}}\left[\cdot\right] = \mathbb{P}\left[\cdot | X_0 = x, \overline{X}_0 = \bar{x}\right],$$

and

$$\mathbb{P}_x\left[\cdot\right] = \mathbb{P}_{x,x}\left[\cdot\right].$$

In the loss-carry-forward taxation regime for the risk process X, we define the tax process $U^\gamma := (U_t^\gamma)_{t \ge 0}$ by

$$U_t^\gamma = X_t - \int_{0+}^t \gamma(\overline{X}_s)\, \mathrm{d}\overline{X}_s, \qquad t \ge 0, \tag{3.1}$$

where $\gamma\colon [\bar{x}, \infty) \to [0, 1)$ is a measurable function. This tax process was introduced first in [33] for spectrally negative Lévy process X. Note that, the notation $\int_{0+}^t = \int_{(0,t]}$

means the integral over $(0,t]$. Due to the assumptions on X, every path $t \mapsto \overline{X}_t$ is continuous, therefore, the integral in (3.1) is a well-defined Lebesgue-Stieltjes integral. We call U^γ a *latent tax process* or the tax process with *latent tax rate* γ (see [1]). We explain this latent tax process in the following way. For a small $h > 0$, in the time interval $[t, t+h]$, a fraction $\gamma(\overline{X}_t)$ of the increment $\overline{X}_{t+h} - \overline{X}_t$ represents the tax payment. These tax payments are made only whenever X reaches a new maximum (which is whenever U^γ reaches a new maximum as proved in Lemma 3.2.1), which is why we can see the taxation structure in (3.1) to be of the loss-carry-forward type. Since $\gamma < 1$, this can be seen as partial reflection. Note that, the case $\gamma = \mathbf{1}_{[b,\infty)}$ corresponds to fully reflecting the path at the barrier b.

The motivation of this part of research started by [33], [53] and [57]. This was while trying to understand the reason that the two tax value functions [57, Equation 5.7] and [53, Equation 14] are different, although the two articles use the same latent tax process U^γ. We noticed an unnatural property of the process U^γ, which is the taxation at time t depends on $\overline{X}_t$, the running maximum of X, and not on the running maximum of the process U^γ itself, $\overline{U}^\gamma = \sup_{s \le t} U^\gamma_s$. Moreover, when X is modelled by a Markov process, the process $(U^\gamma, \overline{U}^\gamma)$ is not Markovian, and in order to have the Markov property, one needs to consider the three-dimensional process $(U^\gamma, \overline{U}^\gamma, \overline{X})$. In this chapter, we introduce the tax process $V^\delta = (V^\delta_t)_{t \ge 0}$, which satisfies the equation

$$V^\delta_t = X_t - \int_{0+}^{t} \delta(\overline{V}^\delta_s)\, \mathrm{d}\overline{X}_s, \tag{3.2}$$

where $\overline{V}^\delta_t = \sup_{s \le t} V^\delta_s$ and $\delta \colon [\bar{x}, \infty) \to [0,1)$ is a measurable function. As given in [1], we call V^δ a *natural tax process* or a tax process with *natural tax rate* δ. It can be seen that (3.2) is an integral equation, and hence, it is not immediately clear whether a process V^δ exists and if so if it is uniquely defined. We will give, in this chapter, a simple condition for existence and uniqueness of V^δ. Remarkably, if we assume that X is a Markov process and existence and uniqueness of the natural tax process V^δ holds, then we have the advantage that the two-dimensional process $(V^\delta, \overline{V}^\delta)$ is Markovian. We realised that authors in [7] looked at tax processes with a natural tax rate in the case where X is a Cramér-Lundberg risk process while they were studying the ruin probability, though they did not provide a definition of the tax process in terms of an integral equation and did not discuss existence and uniqueness of such a process. In the setting where X is a Cramér-Lundberg risk process, [68] and [18] considered a more general class of natural tax processes than ours in which the associated premium rate is allowed to be surplus-dependent. Although [68, Section 1] and [18, Equation (1.2)] contain the definition (3.2) for the natural tax process, with $\delta = 0$ and the function $c(\cdot)$ being constant, respectively, neither paper mentions the question of existence and uniqueness of the tax process.

Note that, most of the context of this chapter is copied from [1] verbatim except the point that we present the results here for any x and $\bar{x}$ such that $x \leq \bar{x}$, whereas the results in our article only represents the case $x = \bar{x}$. Moreover, Theorem 3.2.9 and Remark 5 are added to the thesis, while they are not written in the article. In Section 3.2, Theorem 3.2.3 clarifies the equivalence relation between the two tax processes (3.1) and (3.2). This relation enables us to translate results derived for U^γ into results on V^δ and vice versa, and by using this feature, some Corollaries of Theorem 3.2.3 are proved. We give two examples for applying our result in Section 3.3.

Before we go to the next section, we emphasise that our assumptions allow for a large class of stochastic processes for X that includes, amongst others, spectrally negative Lévy processes, spectrally negative Markov additive processes (see [8]), diffusion processes (see [36]) and fractional Brownian motion. However, from a practical modelling point of view, (3.1) and (3.2) might not, in all cases, be the right way to define a taxed process. For instance, when one considers a Cramér-Lundberg risk process where the company earns interest on its capital as well as pays tax according to a loss-carry-forward scheme, then one should not work with a process of the form (3.1) or (3.2), but instead define the tax process differently, as in [68]. Our definitions (3.1) and (3.2) are practically suitable for modelling tax processes when the underlying risk process without tax X has a spatial homogeneity property, which is the case for, for instance, spectrally negative Lévy processes or spectrally negative Markov additive processes.

3.2 The equivalence of two tax processes

In this section, we give the main result of the chapter, which is Theorem 3.2.3. This theorem gives the equivalence relation between U^γ and V^δ defined in (3.1) and (3.2) respectively. This equivalence allows us to deal in a rather straightforward way with the existence and uniqueness of the natural tax process, which is something that has not been dealt with before. In order to present our results, we will need to consider the following ordinary differential equation, for a given measurable function $\delta\colon [\bar{x},\infty) \to [0,1)$:

$$\begin{aligned} \frac{\mathrm{d}y^{\delta}_{\bar{x}}(t)}{\mathrm{d}t} &= 1-\delta\left(y^{\delta}_{\bar{x}}(t)\right), \qquad t\geq 0, \\ y^{\delta}_{\bar{x}}(0) &= \bar{x}. \end{aligned} \tag{3.4}$$

We say that $y^{\delta}_{\bar{x}}\colon [0,\infty) \to \mathbb{R}$ is a *solution* of this ODE if it is an absolutely continuous function and satisfies (3.4) for almost every t. The next results are applicable for all $\bar{x} \geq x$.

Before we state and prove Theorem 3.2.3, we need to present first the following lemmas. We first start with a lemma generalising a result from [33].

Lemma 3.2.1 *Let $H = (H_t)_{t\geq 0}$ be a stochastic process for which every path is measurable as a function of time and such that $H_t < 1$ for every $t \geq 0$. Define*

$$Y_t = X_t - \int_{0+}^{t} H_s \,\mathrm{d}\overline{X}_s, \qquad t\geq 0.$$

Then,

$$\overline{Y}_t = \overline{X}_t - \int_{0+}^{t} H_s \,\mathrm{d}\overline{X}_s,$$

where $\overline{Y}_t = \sup_{s\leq t} Y_s$. Moreover, $\{t\geq 0 : Y_t = \overline{Y}_t\} = \{t \geq 0 : X_t = \overline{X}_t\}$.

Proof Since $H_t < 1$ for all $t \geq 0$, the proof in [33, Lemma 2.1] works without alteration.□

Lemma 3.2.2 *Let $\delta\colon [\bar{x},\infty) \to [0,1)$ be a measurable function and assume that there exists a unique solution $y^{\delta}_{\bar{x}}$ of (3.4). Define $\gamma^{\delta}_{\bar{x}}\colon [\bar{x},\infty) \to [0,1)$ by $\gamma^{\delta}_{\bar{x}}(s) = \delta\left(y^{\delta}_{\bar{x}}(s-\bar{x})\right)$. If there exists a solution $V^{\delta} = (V^{\delta}_t)_{t\geq 0}$ to the integral equation*

$$V^{\delta}_t = X_t - \int_{0+}^{t} \delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r, \qquad t\geq 0, \tag{3.5}$$

then $\overline{V}^{\delta}_t = y^{\delta}_{\bar{x}}(\overline{X}_t - \bar{x})$ and hence V^{δ} is a latent tax process with latent tax rate given by $\gamma^{\delta}_{\bar{x}}$.

PROOF Suppose that V^δ solves (3.5). By Lemma 3.2.1,

$$\overline{V}^\delta_t = \overline{X}_t - \int_{0+}^t \delta(\overline{V}^\delta_r)\,\mathrm{d}\overline{X}_r, \qquad t \geq 0. \tag{3.6}$$

We define $L_t = \overline{X}_t - \bar{x}$ and we let L_a^{-1} be its right-inverse, i.e.

$$L_a^{-1} := \begin{cases} \inf\{t > 0 : L_t > a\} = \inf\{t > 0 : \overline{X}_t > a + \bar{x}\}, & \text{if } 0 \leq a < L_\infty, \\ \infty, & \text{if } a \geq L_\infty. \end{cases}$$

As $t \mapsto \overline{X}_t$ is continuous, then this implies

$$\overline{X}_{L_a^{-1}} = \bar{x} + (a \wedge L_\infty). \tag{3.7}$$

Using respectively (3.6) for $t = L_a^{-1} = L^{-1}_{a\wedge L_\infty}$, (3.7) and the change of variables formula, Lemma 2.2.10, with $r = L_b^{-1}$, we have for $a \geq 0$,

$$\begin{aligned} \overline{V}^\delta_{L^{-1}_{a\wedge L_\infty}} &= \overline{X}_{L^{-1}_{a\wedge L_\infty}} - \int_{0+}^{L^{-1}_{a\wedge L_\infty}} \delta(\overline{V}^\delta_r)\,\mathrm{d}\overline{X}_r \\ &= \bar{x} + (a \wedge L_\infty) - \int_{0+}^\infty 1_{\{r \leq L^{-1}_{a\wedge L_\infty}\}} \delta(\overline{V}^\delta_r)\,\mathrm{d}\overline{X}_r \\ &= \bar{x} + (a \wedge L_\infty) - \int_0^\infty 1_{\{0 < L_b^{-1} \leq L^{-1}_{a\wedge L_\infty}\}} \delta(\overline{V}^\delta_{L_b^{-1}})\,\mathrm{d}b \\ &= \bar{x} + \int_0^{a\wedge L_\infty} \left(1 - \delta\left(\overline{V}^\delta_{L_b^{-1}}\right)\right)\,\mathrm{d}b, \end{aligned}$$

where for the last equality we used that L_b^{-1} is strictly increasing on $[0, L_\infty]$, which follows because $t \mapsto \overline{X}_t$ is continuous. By the hypothesis that (3.4) has a unique solution $y^\delta_{\bar{x}}$, we deduce,

$$\overline{V}^\delta_{L^{-1}_{a\wedge L_\infty}} = y^\delta_{\bar{x}}(a \wedge L_\infty) = y^\delta_{\bar{x}}\left(\overline{X}_{L^{-1}_{a\wedge L_\infty}} - \bar{x}\right), \qquad a \geq 0, \tag{3.8}$$

where the last equality follows by (3.7). As $t \mapsto \overline{X}_t$ is continuous, $\overline{X}_{L^{-1}_{L_t}} = \overline{X}_t$ for all $t \geq 0$, which implies via (3.6) that $\overline{V}^\delta_{L^{-1}_{L_t}} = \overline{V}^\delta_t$ for all $t \geq 0$. So by invoking (3.8) for $a = L_t$, we conclude that $\overline{V}^\delta_t = y^\delta_{\bar{x}}(\overline{X}_t - \bar{x})$ for all $t \geq 0$.□

Now, we are ready to state and prove the main result in this chapter.

Theorem 3.2.3 *Recall that $\overline{X}_0 = \bar{x}$.*

(i) Let U^γ be the tax process with latent tax rate γ, where $\gamma\colon [\bar{x}, \infty) \to [0, 1)$ is a measurable function. Define $\bar{\gamma}_{\bar{x}}\colon [\bar{x}, \infty) \to \mathbb{R}$ by

$$\bar{\gamma}_{\bar{x}}(s) = \bar{x} + \int_{\bar{x}}^s (1 - \gamma(y))\,\mathrm{d}y, \qquad s \geq \bar{x}, \tag{3.9}$$

and consider its inverse $\bar{\gamma}_{\bar{x}}^{-1} \colon [\bar{x}, \infty] \to [\bar{x}, \infty]$, *with the convention that* $\bar{\gamma}_{\bar{x}}^{-1}(s) = \infty$ *when* $s \geq \bar{\gamma}_{\bar{x}}(\infty)$. *Define* $\delta_{\bar{x}}^{\gamma} \colon [\bar{x}, \bar{\gamma}_{\bar{x}}(\infty)) \to [0,1)$ *by* $\delta_{\bar{x}}^{\gamma}(s) = \gamma(\bar{\gamma}_{\bar{x}}^{-1}(s))$. *Then,*

$$\overline{U}_t^{\gamma} = \bar{\gamma}_{\bar{x}}(\overline{X}_t), \qquad t \geq 0, \tag{3.10}$$

and U^{γ} *is a natural tax process with natural tax rate* $\delta_{\bar{x}}^{\gamma}$.

(ii) Let $\delta \colon [\bar{x}, \infty) \to [0,1)$ *be a measurable function and assume that there exists a unique solution* $y_{\bar{x}}^{\delta}(t)$ *of* (3.4). *Define* $\gamma_{\bar{x}}^{\delta} \colon [\bar{x}, \infty) \to [0,1)$ *by* $\gamma_{\bar{x}}^{\delta}(s) = \delta\left(y_{\bar{x}}^{\delta}(s - \bar{x})\right)$. *Then, the integral equation* (3.2) *defining the natural tax process has a unique solution* $V^{\delta} = (V_t^{\delta})_{t \geq 0}$. *Moreover,*

$$\overline{V}_t^{\delta} = y_{\bar{x}}^{\delta}(\overline{X}_t - \bar{x}), \qquad t \geq 0, \tag{3.11}$$

and so the solution V^{δ} *to* (3.2) *is a latent tax process with latent tax rate given by* $\gamma_{\bar{x}}^{\delta}$.

Proof (i) Fix $t \geq 0$. By Lemma 3.2.1, we have

$$\overline{U}_t^{\gamma} = \overline{X}_t - \int_{0+}^{t} \gamma(\overline{X}_r) \, \mathrm{d}\overline{X}_r.$$

By applying the change of variable $y = \overline{X}_r$, we obtain

$$\overline{U}_t^{\gamma} = \overline{X}_t - \int_{\bar{x}}^{\overline{X}_t} \gamma(y) \, \mathrm{d}y = \bar{\gamma}_{\bar{x}}(\overline{X}_t),$$

where we recall that $\bar{\gamma}_{\bar{x}}(s) = \bar{x} + \int_{\bar{x}}^{s} (1 - \gamma(y)) \, \mathrm{d}y$. Hence, $\bar{\gamma}_{\bar{x}}^{-1}(\overline{U}_t^{\gamma}) = \overline{X}_t$, and so $\gamma(\overline{X}_t) = \gamma(\bar{\gamma}_{\bar{x}}^{-1}(\overline{U}_t^{\gamma})) = \delta_{\bar{x}}^{\gamma}(\overline{U}_t^{\gamma})$. It follows that U^{γ} is a natural tax process with natural tax rate $\delta_{\bar{x}}^{\gamma}$.

(ii) The uniqueness of a solution to (3.2), and the equality (3.11), follow directly from Lemma 3.2.2. So it remains to prove the existence of a solution to (3.2). By the hypothesis there exists a unique solution $y_{\bar{x}}^{\delta}$ to (3.4). With $\gamma_{\bar{x}}^{\delta}(z) = \delta\left(y_{\bar{x}}^{\delta}(z - \bar{x})\right)$, we define $\bar{\delta} \colon [\bar{x}, \infty) \to [0,1)$ by

$$\bar{\delta}(z) = \gamma_{\bar{x}}^{\delta}\big((\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(z)\big) = \delta\left(y_{\bar{x}}^{\delta}\left(\left(\bar{\gamma}_{\bar{x}}^{\delta}\right)^{-1}(z) - \bar{x}\right)\right),$$

where $(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}$ is the inverse function of

$$\bar{\gamma}_{\bar{x}}^{\delta}(z) = \bar{x} + \int_{\bar{x}}^{z} (1 - \gamma_{\bar{x}}^{\delta}(y)) \, \mathrm{d}y. \tag{3.12}$$

By part (i), the tax process with latent tax rate $\gamma^{\delta}_{\bar{x}}$ is a natural tax process with natural tax rate $\bar{\delta}$. Thus, it remains to show that $\bar{\delta}(z) = \delta(z)$ for $z \geq \bar{x}$.

Note that, $\bar{\gamma}^{\delta}_{\bar{x}}$ is an absolutely continuous function and hence $(\bar{\gamma}^{\delta}_{\bar{x}})'$ exists almost everywhere. By (3.4) we have that, for z such that $(\bar{\gamma}^{\delta}_{\bar{x}})'(z)$ exists,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left(y^{\delta}_{\bar{x}}((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z) - \bar{x})\right) &= \left[1 - \delta\left(y^{\delta}_{\bar{x}}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z) - \bar{x}\right)\right)\right]\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right)\\ &= \left[1 - \gamma^{\delta}_{\bar{x}}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right)\right]\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right).\end{aligned}$$

Since by the inverse function theorem [51, Theorem 31.1],

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right) = \frac{1}{(\bar{\gamma}^{\delta}_{\bar{x}})'\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right)} = \frac{1}{1 - \gamma^{\delta}_{\bar{x}}\left((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z)\right)},$$

we see that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(y^{\delta}_{\bar{x}}((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z) - \bar{x})\right) = 1 \qquad \text{a.e.},$$

and therefore, by the absolute continuity, for some constant c, we have that

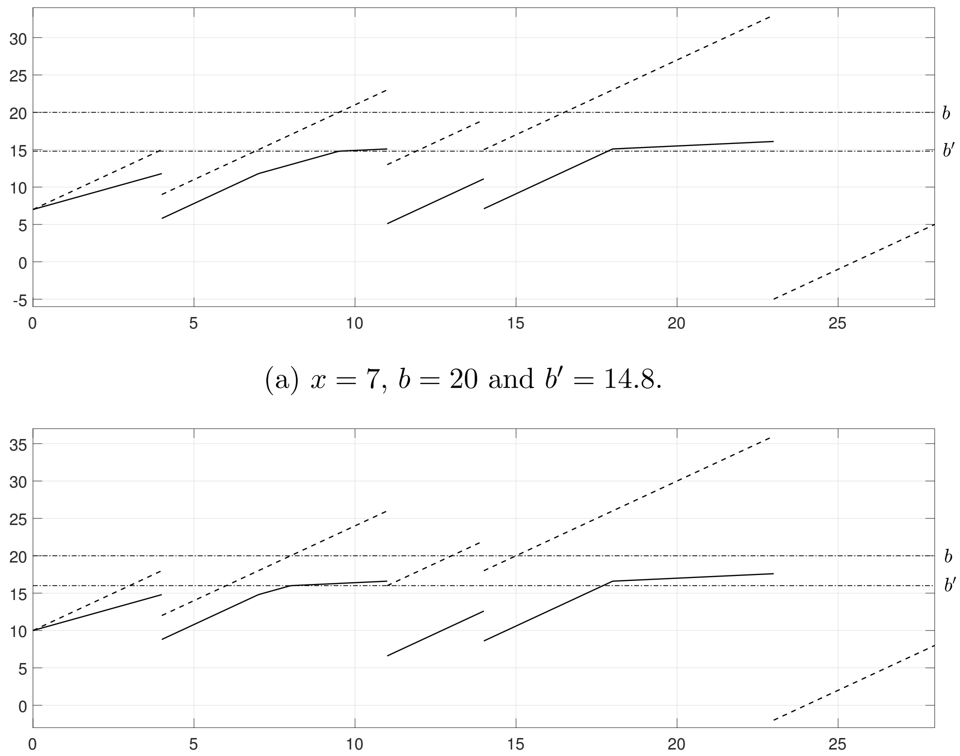
$$y^{\delta}_{\bar{x}}((\bar{\gamma}^{\delta}_{\bar{x}})^{-1}(z) - \bar{x}) = z + c, \qquad z \geq \bar{x}.$$

Since $(\bar{\gamma}^{\delta}_{\bar{x}}(\bar{x}))^{-1} = \bar{x} = y^{\delta}_{\bar{x}}(0)$, we get that $c = 0$. We conclude that $\bar{\delta}(z) = \delta(z)$ for $z \geq \bar{x}$, and this completes the proof.$\square$

This theorem states that a sufficient condition for existence and uniqueness of solutions to (3.2) can be given in terms of a simple ODE. From the proofs given above, it is not difficult to see that the existence and uniqueness of the ODE (3.4) is also a necessary condition for existence and uniqueness of a solution to (3.2). Theorem 3.2.3 also gives a precise relationship between the two types of tax processes. In particular, every latent tax process is a natural tax process, though the corresponding latent and natural tax rates may differ. Conversely, every well-defined natural tax process is also a latent tax process. The next example illustrates this equivalence for piecewise constant tax rates.

Example 3.2.4 Let $X_0 = \overline{X}_0 = x$. Define the piecewise constant function f^b by

$$f^b(z) = \begin{cases}\alpha, & z \leq b,\\ \beta, & z > b,\end{cases} \tag{3.13}$$



(a) $x = 7$, $b = 20$ and $b' = 14.8$.

(b) $x = 10$, $b = 20$ and $b' = 16$.

Figure 3.1: Plots of the risk process X (dashed line) and the associated latent tax process U^{f^b} or equivalently natural tax process $V^{f^{b'}}$ (solid line), where f^b is the piecewise constant function defined by (3.13) with $\alpha = 0.4$ and $\beta = 0.9$. The dashed-dot lines mark the values of b and b'.

where $b > x = X_0$ and $0 \leq \alpha \leq \beta < 1$. Note that, the ODE (3.4) with $\delta = f^b$ has a unique solution, see e.g. Example 3.3.1. It is clear that the tax process with latent tax rate f^b differs from the tax process with natural tax rate f^b, unless $\alpha = \beta$ or $\alpha = 0$. However, from Theorem 3.2.3 we deduce that the tax process with latent tax rate f^b is equal to the tax process with natural tax rate $f^{b'}$ for

$$b' = (1 - \alpha)b + \alpha x.$$

Note that b' depends on the starting point x, unless $\alpha = 0$. Figure 3.1 contains two plots in which an example of X and the corresponding tax process U^{f^b}, or equivalently $V^{f^{b'}}$, are drawn. From this figure, we see that indeed the first time X reaches the level b, is equal to the first time the tax process reaches the level b'.

We give the following remark about the natural tax process V^δ.

Remark 2 (i) **Markov property.** If X is a Markov process, then it follows from the integral equation (3.2) for the natural tax process V^δ that the process $(V^\delta, \overline{V}^\delta)$ is Markov. One might expect that the equivalence between the two types of tax processes should imply the same for $(U^\gamma, \overline{U}^\gamma)$ where U^γ is an arbitrary latent tax process, since we know by Theorem 3.2.3i that U^γ is also a natural tax process. However, the corresponding natural tax rate is $\delta^\gamma_{\bar{x}} = \delta^\gamma_{\overline{X}_0}$, which depends upon the initial value of $\overline{X}$. Looked at another way, although one can recover $\overline{X}$ from the formula $\overline{X}_t = \bar{\gamma}_{\bar{x}}^{-1}(\overline{U}^\gamma_t)$, this too depends on knowledge of the initial value $\overline{X}_0$. For this reason, we do not obtain the Markov property for $(U^\gamma, \overline{U}^\gamma)$ in general.

(ii) **An alternative definition of the natural tax process.** It would also appear to be reasonable to define a natural type of tax process by $W^\kappa = (W^\kappa_t)_{t\geq 0}$ with

$$W^\kappa_t = X_t - \int_{0+}^t \kappa(\overline{W}^\kappa_s)\,\mathrm{d}\overline{W}^\kappa_s,$$

where $\kappa\colon [\bar{x}, \infty) \to [0,\infty)$. However, due to Lemma 3.2.1, the process W^κ is actually equivalent to the natural tax process V^δ, where $\kappa = \frac{\delta}{1-\delta}$. We believe that describing this process in terms of the natural tax rate δ is more customary, as it defines the tax rate as a proportion of the increments of capital prior to taxation, rather than after taxation.

Next, we present and prove our results for the natural tax process V^δ associated with the spectrally negative Lévy process X. Before that, we recall first the results for the latent tax process U^γ as given in [33]. In these results, for $a > 0$, the first passage times are defined by $\sigma^+_a := \inf\{t > 0 : U^\gamma_t > a\}$ and $\sigma^-_0 := \inf\{t > 0 : U^\gamma_t < 0\}$. Note also in these results, that $X_0 = \overline{X}_0 = x$.

Theorem 3.2.5 ***[33, Theorem 1.1](Two sided exit problem)*** *For any $0 \leq x \leq a$ and $q \geq 0$, we have*

$$\mathbb{E}_x\left[e^{-q\sigma^+_a}\mathbf{1}_{\{\sigma^+_a < \sigma^-_0\}}\right] = \exp\left\{-\int_x^a \frac{W^{(q)\prime}(y)}{W^{(q)}(y)(1-\gamma(\bar{\gamma}_x^{-1}(y)))}\,\mathrm{d}y\right\}.$$

Theorem 3.2.6 ***[33, Theorem 1.2](Net present value of tax paid until ruin)*** *For any $x \geq 0$ and $q \geq 0$, we have*

$$\mathbb{E}_x\left[\int_{0+}^{\sigma^-_0} e^{-qr}\gamma(\overline{X}_r)\mathrm{d}\overline{X}_r\right] = \int_x^\infty \exp\left\{-\int_x^t \frac{W^{(q)\prime}(\bar{\gamma}_x(s))}{W^{(q)}(\bar{\gamma}_x(s))}\,\mathrm{d}s\right\}\gamma(t)\,\mathrm{d}t.$$

The next two results are corollaries to Theorem 3.2.3.

Corollary 3.2.7 *Let X be a spectrally negative Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{x,\bar{x}})$ such that $\mathbb{P}_{x,\bar{x}}(X_0 = x, \overline{X}_0 = \bar{x}) = 1$. Let $\delta\colon [\bar{x}, \infty) \to [0,1)$ be a measurable function such that there exists a unique solution $y^{\delta}_{\bar{x}}$ to (3.4). Let V^{δ} be the tax process with natural rate δ associated with the spectrally negative Lévy process X. Then, for any $0 \le x \le \bar{x} < y^{\delta}_{\bar{x}}(\infty)$ and $q \ge 0$, we have*

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right] \\
&\qquad = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{y^{\delta}_{\bar{x}}(\infty)} \exp\left\{-\int_{\bar{x}}^{y}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\,\mathrm{d}r\right\}\frac{\delta(y)}{(1-\delta(y))}\,\mathrm{d}y. \qquad (3.14)
\end{aligned}$$

Proof The taxation starts when the process V^{δ} reaches $\bar{x}$. So, using the strong Markov property of the process $(V^{\delta}, \overline{V}^{\delta})$, we have the following

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\,\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_0^-\}}\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[\mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\,\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_0^-\}}|\mathcal{F}_{\tau_{\bar{x}}^+}\right]\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_0^-\}}\,e^{-q\tau_{\bar{x}}^+}\,\mathbb{E}_{x,\bar{x}}\left[e^{q\tau_{\bar{x}}^+}\int_{\tau_{\bar{x}}^+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r|\mathcal{F}_{\tau_{\bar{x}}^+}\right]\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_{\bar{x}}^+}\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_0^-\}}\,\mathbb{E}_{V^{\delta}_{\tau_{\bar{x}}^+},\overline{V}^{\delta}_{\tau_{\bar{x}}^+}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right]\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_{\bar{x}}^+}\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_0^-\}}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right] \\
&= \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right],
\end{aligned}$$

where in the last equality, since the process V^{δ} before reaching $\bar{x}$ is just the process X, we can use the third part of Theorem 2.2.1. Then, since by part ii in Theorem 3.2.3, V^{δ} is a latent tax process with a latent tax rate given by $\gamma^{\delta}_{\bar{x}}$, where for $z \ge \bar{x}$, $\gamma^{\delta}_{\bar{x}}(z) := \delta\left(y^{\delta}_{\bar{x}}(z-\bar{x})\right)$, we can apply Theorem 3.2.6 and get

$$\begin{aligned}
\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right] &= \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\sigma_0^-} e^{-qr}\,\gamma^{\delta}_{\bar{x}}(\overline{X}_r)\,\mathrm{d}\overline{X}_r\right] \\
&= \int_{\bar{x}}^{\infty}\exp\left\{-\int_{\bar{x}}^{t}\frac{W^{(q)\prime}(\bar{\gamma}^{\delta}_{\bar{x}}(s))}{W^{(q)}(\bar{\gamma}^{\delta}_{\bar{x}}(s))}\,\mathrm{d}s\right\}\gamma^{\delta}_{\bar{x}}(t)\,\mathrm{d}t. \qquad (3.15)
\end{aligned}$$

Use the change of variables $r = \bar{\gamma}_{\bar{x}}^{\delta}(s)$ and then $t = (\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(y)$, then (3.15) equals

$$\int_{\bar{x}}^{\bar{\gamma}_{\bar{x}}^{\delta}(\infty)} \exp\left\{-\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\gamma_{\bar{x}}^{\delta}(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(r))}\,\mathrm{d}r\right\} \frac{\gamma_{\bar{x}}^{\delta}(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(y)}{(1-\gamma_{\bar{x}}^{\delta}(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(y))}\,\mathrm{d}y.$$

From the proof of Theorem 3.2.3, part ii, we proved that $\gamma_{\bar{x}}^{\delta}(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(s) = \delta(s)$ for all $s \geq \bar{x}$. Also, since $\bar{\gamma}_{\bar{x}}^{\delta}(\infty) = y_{\bar{x}}^{\delta}(\infty)$, therefore, we get the required result.□

Corollary 3.2.8 *Suppose that we have the assumptions given in Corollary 3.2.7. Then, for $q \geq 0$ and $0 \leq x \leq \bar{x} < a < y_{\bar{x}}^{\delta}(\infty)$, we have*

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\exp\left\{-\int_{\bar{x}}^{a} \frac{W^{(q)\prime}(y)}{W^{(q)}(y)(1-\delta(y))}\,\mathrm{d}y\right\}, \tag{3.16}$$

where $W^{(q)\prime}$ denotes a density of $W^{(q)}$ on $(0,\infty)$. On the other hand, if $a \geq y_{\bar{x}}^{\delta}(\infty)$, then $\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = 0$.

Proof From (3.11) we see that $\tau_a^+ = \infty$ when $a \geq y_{\bar{x}}(\infty)$. Hence, we can assume without loss of generality that $a < y_{\bar{x}}^{\delta}(\infty)$. By part ii of Theorem 3.2.3, we know that V^{δ} is a latent tax process with latent tax rate $\gamma_{\bar{x}}^{\delta}$. Hence, we can use Theorem 3.2.5 to conclude that,

$$\begin{aligned}
&\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] \\
&= \mathbb{E}_{\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] \\
&= \exp\left\{-\int_{\bar{x}}^{a} \frac{W^{(q)\prime}(y)}{W^{(q)}(y)\left(1-\gamma_{\bar{x}}^{\delta}\left((\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(y)\right)\right)}\,\mathrm{d}y\right\},
\end{aligned}$$

where $(\bar{\gamma}_{\bar{x}}^{\delta})^{-1}$ is the inverse of the function $\bar{\gamma}_{\bar{x}}^{\delta}$ given by (3.12). Note that in [33], the additional assumption $\int_0^{\infty}(1-\gamma_{\bar{x}}^{\delta}(z))\,\mathrm{d}z = \infty$ is made on the latent tax rate, but from the proof of Theorem 1.1 in [33] it is clear that this assumption is unnecessary when $a < y_{\bar{x}}^{\delta}(\infty)$. In the proof of Theorem 3.2.3ii we showed that $\gamma_{\bar{x}}^{\delta}\left((\bar{\gamma}_{\bar{x}}^{\delta})^{-1}(y)\right) = \delta(y)$ for all $y \geq \bar{x}$, so we get

$$\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \exp\left\{-\int_{\bar{x}}^{a} \frac{W^{(q)\prime}(y)}{W^{(q)}(y)\,(1-\delta(y))}\,\mathrm{d}y\right\}.$$

In order to finish the proof, we use the strong Markov property of $(V^{\delta}, \overline{V}^{\delta})$ similarly as in the proof of Corollary 3.2.7.□

The next theorem gives the net present value of tax payments when initial values x and $\bar{x}$ belong to $[0,a]$ for some $0 < a < y_{\bar{x}}^{\delta}(\infty)$. To prove this result, we adapt the proof of [33, Theorem 1.2] which uses excursion theory. We will need this result in Chapter 6.

Theorem 3.2.9 *Let X be a spectrally negative Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{x,\bar{x}})$ such that $\mathbb{P}_{x,\bar{x}}(X_0 = x, \overline{X}_0 = \bar{x}) = 1$. Let $\delta\colon [\bar{x}, \infty) \to [0, 1)$ be a measurable function such that there exists a unique solution $y^{\delta}_{\bar{x}}$ to (3.4). Let V^{δ} be the tax process with natural rate δ associated with the spectrally negative Lévy process X. Then, for $q \geq 0$ and any $0 \leq x \leq \bar{x} \leq a < y^{\delta}_{\bar{x}}(\infty)$, where $a > 0$, we have*

$$\mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^- \wedge \tau_a^+} e^{-qr}\,\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right]$$
$$= \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\} \frac{\delta(y)}{(1-\delta(y))}\,\mathrm{d}y. \tag{3.17}$$

Proof First, we show that

$$\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_0^- \wedge \tau_a^+} e^{-qr}\delta(\overline{V}^{\delta}_r)\,\mathrm{d}\overline{X}_r\right] = \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\} \frac{\delta(y)}{(1-\delta(y))}\,\mathrm{d}y.$$

Let $L_t = \overline{X}_t - \bar{x}$ be the local time at the maximum for the process X under $\mathbb{P}_{\bar{x},\bar{x}}$. For simplicity of notations, let $T = \tau_0^- \wedge \tau_a^+$ and $G_t = L^{-1}_{L_t-}$. Then, since the process $\overline{X}$ does not increase on the time interval (G_T, T), we have

$$\begin{aligned}
\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{T} e^{-qr}\,\delta(\bar{V}_r)\,\mathrm{d}\bar{X}_r\right] &= \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{G_T} e^{-qr}\,\delta(\bar{V}_r)\,\mathrm{d}\bar{X}_r\right]\\
&= \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\infty} \mathbf{1}_{\{r<G_T\}}\, e^{-qr}\,\delta(\bar{V}_r)\,\mathrm{d}L_r\right]\\
&= \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0}^{L_\infty} \mathbf{1}_{\{L_t^{-1}<G_T\}}\, e^{-qL_t^{-1}}\,\delta(\overline{V}_{L_t^{-1}})\,\mathrm{d}t\right],
\end{aligned} \tag{3.18}$$

where we use Lemma 2.2.10 in the last equality. Since $t < L_\infty$ inside the integral (3.18), then by definition (2.17) and by (3.11), $\overline{V}_{L_t^{-1}} = y^{\delta}_{\bar{x}}(t)$. Also, since L_t^{-1} is strictly increasing on $[0, L_\infty]$ and by Fubini's theorem, (3.18) equals

$$\begin{aligned}
&\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0}^{L_\infty} \mathbf{1}_{\{t<L_T\}}\, e^{-qL_t^{-1}}\,\delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t\right]\\
&= \int_{0}^{L_\infty} \mathbb{E}_{\bar{x},\bar{x}}\left[e^{-qL_t^{-1}}\,\mathbf{1}_{\{t<L_T\}}\right]\delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t\\
&= \int_{0}^{L_\infty \wedge (y^{\delta}_{\bar{x}})^{-1}(a)} \mathbb{E}_{\bar{x},\bar{x}}\left[e^{-qL_t^{-1}}\,\mathbf{1}_{\{t<L_{\tau_0^-}\}}\right]\delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t,
\end{aligned} \tag{3.19}$$

where we get the last equality since we know that $\mathbf{1}_{\{t<L_T\}} = \mathbf{1}_{\{t<L_{\tau_0^-}\}}\mathbf{1}_{\{t<L_{\tau_a^+}\}}$, and also, from (3.11), $L_{\tau_a^+} = \overline{X}_{\tau_a^+} - \bar{x} = (y^{\delta}_{\bar{x}})^{-1}(a)$. Moreover, since $\overline{X}$ is increasing, then $(y^{\delta}_{\bar{x}})^{-1}(a) = \overline{X}_{\tau_a^+} - \bar{x} \leq \overline{X}_\infty - \bar{x} = L_\infty$. Therefore, $L_\infty \wedge (y^{\delta}_{\bar{x}})^{-1}(a) = (y^{\delta}_{\bar{x}})^{-1}(a)$. Let

ε be the excursion of the process V^δ from its running maximum and $\bar{\varepsilon}_t = \sup_{0\le s\le t}\varepsilon_s$ and

$$A = \{(s,\bar{\varepsilon}_s) : 0 \le s \le t, \bar{\varepsilon}_s > y^\delta_{\bar{x}}(s)\}.$$

Then A is a Poisson point process with parameter

$$\eta(A) = \int \mathbf{1}_A \,\mathrm{d}s\, n(\mathrm{d}\varepsilon) = \int_0^t n(\bar{\varepsilon}_s > y^\delta_{\bar{x}}(s))\,\mathrm{d}s.$$

Next, let $\tau = L_t^{-1}$ and we perform the change of measure at $\tau \wedge m = L_t^{-1} \wedge m$, where m is a finite deterministic time and so that according to (2.5), we have for any $B \in \mathcal{F}_{\tau\wedge m}$,

$$\mathbb{P}^{\Phi(q)}_{m,\bar{x},\bar{x}}(B) = \mathbb{E}_{\bar{x},\bar{x}}\left[\mathbf{1}_B\, e^{\Phi(q)(X_{\tau\wedge m}-\bar{x})-q(\tau\wedge m)}\right]. \tag{3.20}$$

Since $\{t < L_{\tau_0^-}\}$ and $\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0 \le s \le t\}$ represents the same event for any $t \ge 0$, then by (3.20) we have that

$$\begin{aligned}
&\mathbb{P}^{\Phi(q)}_{m,\bar{x},\bar{x}}\left(\mathbf{1}_{\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0\le s\le t\wedge L_m\}}\right)\\
&\quad= \mathbb{E}_{\bar{x},\bar{x}}\left[\mathbf{1}_{\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0\le s\le t\wedge L_m\}} e^{\Phi(q)(X_{\tau\wedge m}-\bar{x})-q(\tau\wedge m)}\right].
\end{aligned} \tag{3.21}$$

Since $X_{L_t^{-1}} = \overline{X}_{L_t^{-1}}$, then $t = \overline{X}_{L_t^{-1}} - \bar{x} = X_{L_t^{-1}} - \bar{x}$. Then (3.19) becomes

$$\begin{aligned}
&\int_0^{(y^\delta_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t}\,\mathbb{E}_{\bar{x},\bar{x}}\left[\mathbf{1}_{\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0\le s\le t\}} e^{-q\tau}\, e^{\Phi(q)(X_\tau-\bar{x})}\right]\delta(y^\delta_{\bar{x}}(t))\,\mathrm{d}t\\
&= \int_0^{(y^\delta_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t}\\
&\qquad \times \lim_{m\to\infty}\mathbb{E}_{\bar{x},\bar{x}}\left[\mathbf{1}_{\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0\le s\le t\wedge L_m\}} e^{\Phi(q)(X_{\tau\wedge m}-\bar{x})-q(\tau\wedge m)}\right]\delta(y^\delta_{\bar{x}}(t))\,\mathrm{d}t,
\end{aligned} \tag{3.22}$$

where we use the dominated convergence theorem in the last equality since we have that

$$|X_{\tau\wedge m}| \le |\overline{X}_{\tau\wedge m}| \le (y^\delta_{\bar{x}})^{-1}(a).$$

Therefore, by (3.21), (3.22) equals

$$\begin{aligned}
&\int_0^{(y^\delta_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t}\lim_{m\to\infty}\mathbb{P}^{\Phi(q)}_{m,\bar{x},\bar{x}}\left(\mathbf{1}_{\{\bar{\varepsilon}_s \le y^\delta_{\bar{x}}(s) \;\text{ for all }\; 0\le s\le t\wedge L_m\}}\right)\delta(y^\delta_{\bar{x}}(t))\,\mathrm{d}t\\
&= \int_0^{(y^\delta_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t}\lim_{m\to\infty}\exp\left\{-\int_0^{t\wedge L_m} n_{\Phi(q)}\left(\bar{\varepsilon}_s > y^\delta_{\bar{x}}(s)\right)\mathrm{d}s\right\}\delta(y^\delta_{\bar{x}}(t))\,\mathrm{d}t\\
&= \int_0^{(y^\delta_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t}\exp\left\{-\int_0^t \frac{W'_{\Phi(q)}(y^\delta_{\bar{x}}(s))}{W_{\Phi(q)}(y^\delta_{\bar{x}}(s))}\mathrm{d}s\right\}\delta(y^\delta_{\bar{x}}(t))\,\mathrm{d}t,
\end{aligned} \tag{3.23}$$

where we used (2.18) in the last equality. Recall now that

$$\frac{W'_{\Phi(q)}(y^{\delta}_{\bar{x}}(s))}{W_{\Phi(q)}(y^{\delta}_{\bar{x}}(s))} = \frac{W^{(q)\prime}(y^{\delta}_{\bar{x}}(s))}{W^{(q)}(y^{\delta}_{\bar{x}}(s))} - \Phi(q).$$

Thus, (3.23) becomes

$$\begin{aligned}
&\int_0^{(y^{\delta}_{\bar{x}})^{-1}(a)} e^{-\Phi(q)t} \exp\left\{-\int_0^t \left[\frac{W^{(q)\prime}(y^{\delta}_{\bar{x}}(s))}{W^{(q)}(y^{\delta}_{\bar{x}}(s))} - \Phi(q)\right] \mathrm{d}s\right\} \delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t \\
&= \int_0^{(y^{\delta}_{\bar{x}})^{-1}(a)} \exp\left\{-\int_0^t \frac{W^{(q)\prime}(y^{\delta}_{\bar{x}}(s))}{W^{(q)}(y^{\delta}_{\bar{x}}(s))} \mathrm{d}s\right\} \delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t \\
&= \int_0^{(y^{\delta}_{\bar{x}})^{-1}(a)} \exp\left\{-\int_{\bar{x}}^{y^{\delta}_{\bar{x}}(t)} \frac{W^{(q)\prime}(z)}{W^{(q)}(z)(1-\delta(z))} \mathrm{d}z\right\} \delta(y^{\delta}_{\bar{x}}(t))\,\mathrm{d}t \\
&= \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(z)}{W^{(q)}(z)(1-\delta(z))} \mathrm{d}z\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y,
\end{aligned}$$

where for the last two equalities, we use the change of variables, $z = y^{\delta}_{\bar{x}}(s)$ and then $y = y^{\delta}_{\bar{x}}(t)$. In order to get (3.17), we use the strong Markov property of $(V^{\delta}, \overline{V}^{\delta})$.□

Remark 3 If we let $a \uparrow y^{\delta}_{\bar{x}}(\infty)$ in Theorem 3.2.9, we have an alternative way of proving the analytical expression of the net present value given in (3.14).

Remark 4 Note that, for $x = \bar{x} \geq 0$, if we apply Corollary 3.2.7 for a constant tax rate $\gamma < 1$, we get the expected discounted tax value function

$$\begin{aligned}
v_{\gamma,q}(x) &:= \mathbb{E}_{x,x}\left[\int_{0+}^{\tau_0^-} e^{-qr}\gamma\,\mathrm{d}\overline{X}_r\right] \\
&= \frac{\gamma}{1-\gamma}\int_x^{\infty}\left(\frac{W^{(q)}(x)}{W^{(q)}(s)}\right)^{1/(1-\gamma)} \mathrm{d}s,
\end{aligned}$$

which coincides with [6, Equation (3.2)].

3.3 Examples

In this section, we present two examples. The first one demonstrates existence of the tax process V^{δ} with progressive natural tax rate δ. The second example proves the tax identity for the process V^{δ}.

Example 3.3.1 When the tax rate increases with the amount of capital one has, the taxation regime is typically called progressive. We will show that, when δ is an increasing (in the weak sense) measurable function $\delta\colon [\bar{x}, \infty) \to [0, 1)$, then the ODE

(3.4) has a unique solution, which implies the existence and uniqueness of the natural tax process with tax rate δ.

For existence, since δ is an increasing function, we have that

$$g(z) := \frac{1}{1-\delta(z)}, \qquad z \geq \bar{x},$$

is a strictly positive, increasing measurable function, and hence integrable, so

$$G(y) := \int_{\bar{x}}^{y} g(z)\,\mathrm{d}z, \qquad y \geq \bar{x},$$

is absolutely continuous. Moreover, since G is continuous and strictly increasing, G^{-1} exists and, as $G' > 0$ a.e., G^{-1} is absolutely continuous [15, Vol. I, p. 389]. Thus, $(G^{-1})'(t)$ exists for almost every t, and it follows that a solution to (3.4) is given by $y_{\bar{x}}(t) = G^{-1}(t)$. This is because, by the inverse function theorem [51, Theorem 31.1], it holds that

$$\frac{\mathrm{d}G^{-1}(t)}{\mathrm{d}t} = \frac{1}{g(G^{-1}(t))} = 1 - \delta(G^{-1}(t)), \qquad \text{for a.e. } t > 0,$$

and since $G(\bar{x}) = 0$, we have $G^{-1}(0) = \bar{x}$.

For uniqueness, since δ is increasing, the right hand side of (3.4) is decreasing. This guarantees uniqueness, as can be proved using, for instance, [27, Theorem 1.3.8].

Example 3.3.2 Assume we are in the setting of Corollary 3.2.8 where $X_0 = \overline{X}_0 = x$. We are interested here in the tax identity: a relationship between the survival probability of the natural tax process V^δ and the one of the risk process with out tax X. To this end, let

$$\phi_\delta(x) = \mathbb{P}_x\left(\inf_{t\geq 0} V_t^\delta \geq 0\right) \quad \text{and} \quad \phi_0(x) = \mathbb{P}_x\left(\inf_{t\geq 0} X_t \geq 0\right)$$

be the survival probability in the risk model with and without taxation, respectively.

If $y_x^\delta(\infty) < \infty$, the process V^δ cannot exceed the level $y_x^\delta(\infty)$. Since from every starting level (and thus in particular from $y_x^\delta(\infty)$), there is a strictly positive probability of V^δ going below zero, a standard renewal argument shows that the survival probability $\phi_\delta(x)$ is zero in this case.

On the other hand, if $y_x(\infty) = \infty$, then we can apply Corollary 3.2.8 to get a relation between the two survival probabilities. Namely, by letting $q \to 0$ and $a \to \infty$

in (3.16) and using the well-known expression for $\phi_0(x)$ when $\psi'(0+) > 0$ as in (2.6), we have that

$$\phi_\delta(x) = \exp\left\{-\int_x^\infty \frac{W'(y)}{W(y)(1-\delta(y))}\mathrm{d}y\right\} = \exp\left\{-\int_x^\infty \frac{\mathrm{d}\ln(\phi_0(y))}{\mathrm{d}y}\cdot\frac{1}{(1-\delta(y))}\mathrm{d}y\right\}.$$

This agrees with [7, Proposition 3.1] for the special case where X is a Cramér-Lundberg risk process, which confirms that in [7] natural tax processes are considered.

Remark 5 We explain the renewal argument used in Example 3.3.2 in the case $y_x^\delta(\infty) < \infty$. Suppose that, on the same probability space, we take two paths, one of them is the process X with dividends such that the dividend barrier is $y_{\bar{x}}^\delta(\infty)$, and the other path is the process V^δ. This implies that the V^δ path will lie below the X with dividends path. Denote by $Q_{x,y_{\bar{x}}^\delta(\infty)}$ the probability of the process X with dividends with dividend barrier $y_{\bar{x}}^\delta(\infty)$ and starting point x, and by $P_{x,\bar{x}}$ the probability of the process V^δ which starts at x and begins tax when it reaches level $\bar{x}$. Then, clearly we have the relation between the two paths for the ruin probability as follows:

$$P_{x,x}(\text{ruin occurs}) \geq P_{x,\bar{x}}(\text{ruin occurs}) \geq Q_{x,y_{\bar{x}}^\delta(\infty)}(\text{ruin occurs}).$$

So, we only need to prove that $Q_{x,y_{\bar{x}}^\delta(\infty)}(\text{ruin occurs}) = 1$. The path starts from x and if we consider the upcrossing times of x, then the periods between these upcrossing times are finite. Now define N by saying that $N = n$ if and only if ruin occurs between $(n-1)$st and nth upcrossings, or $N = \infty$ if no ruin occurs. We know that, before reaching level $y_{\bar{x}}^\delta(\infty)$, the process is the Lévy processes X, so the path sections between the $(i-1)$st and ith upcrossing times are identically independent distributed and each has positive probability of having ruin occur. So N is geometric random variable and hence finite a.s.

Chapter 4

Identities for natural tax processes

The expectation concerning the overshoot of a risk process X over some level c is a well studied object in the actuarial literature. For instance, in [41], for a spectrally negative Lévy process X, an analytic expression has been derived for the following expectation

$$\mathbb{E}_x\left[e^{-q\rho_c^-} f(X_{\rho_c^-})\ \mathbf{1}_{\{\rho_c^- < \rho_a^+\}}\right], \tag{4.1}$$

where $-\infty < c < a < \infty, q \geq 0, x \in (c, a]$, $\rho_c^- := \{t \geq 0 : X_t < c\}$, $\rho_a^+ := \{t \geq 0 : X_t > a\}$ and $f \colon (-\infty, c] \to \mathbb{R}$ is a function satisfying some regularity conditions. In many applications of spectrally negative Lévy process, (4.1) appears, such as solving exit problems for refracted Lévy processes in [31]. At some point in the work of Chapter 6, we needed to have expression (4.1) for the natural tax process, V^δ, and its maximum, $\overline{V}^\delta$, in order to find the net present value of taxation. In this chapter, we derive this expression by using a similar technique to the proof of [41, Theorem 2]. The key difficulty in our proof compared to the one in [41, Theorem 2], was the need to have a two dimensional (extant second derivative) Meyer-Itô formula on a function that satisfies some specific regularity conditions. In Section 4.1, we define the space of such functions and derive the required Itô expansion. Then, we state and prove the main result for this chapter, Theorem 4.1.4, in which we obtain an expression for the expectation concerning the overshoot of V^δ over some level c, and the maximum $\overline{V}^\delta$ prior to crossing that level, when crossing level c happens before reaching some positive level. Then, we use Theorem 3.2.3, to deduce an explicit expression for that expectation in the case $c = 0$. In Section 4.2, we give some applications of Theorem 4.1.4. Further, we explain the relation of some of these applications with the draw-down literature and a recent article [64]. Section 4.3 is devoted for explaining our new approach to find fluctuation identities of interest for natural tax process.

4.1 Deficit at and maximum surplus prior to ruin

Recall the first passage times

$$\tau_c^- = \inf\{t \geq 0 : V_t^\delta < c\} \text{ and } \tau_a^+ = \inf\{t \geq 0 : V_t^\delta > a\},$$

where $c, a \in \mathbb{R}$. In this section, we derive the following expectation

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}\ f(V^\delta_{\tau_c^-},\ \overline{V}^\delta_{\tau_c^-})\ \mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right], \tag{4.2}$$

where τ_c^- is the time of ruin, $-V^\delta_{\tau_c^-}$ is the deficit at ruin, and $\overline{V}^\delta_{\tau_c^-}$ is the maximum surplus prior to ruin, before the surplus reaching some level $a > 0$. First, we give a space definition, which will be used in most of the next upcoming statements of our results.

Recall that, we say a function $f \in \mathcal{C}^1[b, a]$ when f and its first derivative are continuous on (b, a), right-continuous at b and left-continuous at a. Also, we recall that L^1 is the space of measurable functions whose absolute value is Lebesgue integrable.

Definition 4.1.1 For $b \leq a < \infty$ and $c \leq d < \infty$, suppose that Y and Z are stochastic processes, where Z is a continuous process of bounded variation, such that $Y_0 \in [b, a]$ and $Z_0 \in [c, d]$. Let ν be the Lévy measure of X. Let $\mathcal{S}_{[b,a]\times[c,d]}$ be the function space consisting of finite sums of measurable functions $f : D_f \to \mathbb{R}$, where $[b, a] \times [c, d] \subseteq D_f$, and of the form $f(y, z) = g(y)\,h(z)$ such that the following holds,

(I) (a) If Y is of unbounded variation, $g \in \mathcal{C}^1[b, a]$ with the derivative being absolutely continuous on $[b, a]$ and having a density which is in L^1 on $[b, a]$.

(b) If Y is of bounded variation, g is absolutely continuous on $[b, a]$ with a locally bounded density on $[b, a]$.

(II) There exists $\lambda > 0$ such that $s \mapsto \int_\lambda^\infty g(s - \theta)\ \nu(\mathrm{d}\theta)$ is bounded on (b, a).

(III) h is absolutely continuous on $[c, d]$ with a locally bounded density on $[c, d]$.

In order to prove most of our results in this section and the upcoming chapters, we need a two dimensional (extant second derivative) Meyer-Itô formula on a function $f \in \mathcal{S}_{\mathbb{R}\times\mathbb{R}}$. For that reason, we state and prove the next theorem which gives an Itô expansion for this specific f.

Theorem 4.1.2 *Suppose that Y and Z are semi-martingales, where Z is a continuous process of bounded variation. Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be in $\mathcal{S}_{\mathbb{R}\times\mathbb{R}}$. Then,*

$$\begin{aligned}
&e^{-qt} f(Y_t, Z_t) - f(Y_0, Z_0) \\
&= \int_{0+}^{t} -qe^{-qs} f(Y_{s-}, Z_s)\,\mathrm{d}s + \int_{0+}^{t} e^{-qs}\frac{\partial f}{\partial y}(Y_{s-}, Z_s)\,\mathrm{d}Y_s \\
&\quad + \int_{0+}^{t} e^{-qs}\frac{\partial f}{\partial z}(Y_{s-}, Z_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_{0}^{t} e^{-qs}\frac{\partial^2 f}{\partial y^2}(Y_{s-}, Z_s)\,\mathrm{d}\,[Y,Y]^c_s \\
&\quad + \sum_{0<s\le t} e^{-qs}\big(\Delta f(Y_s, Z_s) - \frac{\partial f}{\partial y}(Y_{s-}, Z_s)\Delta Y_s\big),
\end{aligned} \tag{4.3}$$

where $\Delta Y_s = Y_s - Y_{s^-}$, $\Delta f(Y_s, Z_s) = f(Y_s, Z_s) - f(Y_{s^-}, Z_s)$ and $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial^2 f}{\partial y^2}$ exist as Radon-Nikodým derivatives.

Proof Since $f \in \mathcal{S}_{\mathbb{R}\times\mathbb{R}}$, we can assume that f is of the form $f(y,z) = g(y)\,h(z)$, otherwise the proof follows by linearity. Suppose first that Y is of unbounded variation, by the assumptions on g, we can use the extant second derivative Meyer-Itô formula [52, Theorem 71] to have the expansion,

$$\begin{aligned}
g(Y_t) - g(Y_0) &= \int_{0+}^{t} g'(Y_{s-})\,\mathrm{d}Y_s + \frac{1}{2}\int_{0}^{t} g''(Y_{s-})\,\mathrm{d}\,[Y,Y]^c_s \\
&\quad + \sum_{0<s\le t}\left[g(Y_s) - g(Y_{s-}) - g'(Y_{s-})\Delta Y_s\right].
\end{aligned} \tag{4.4}$$

Also, since Z is a continuous semi-martingale, then by the assumptions on h, we can use [24, (2.5)] to have the expansion

$$h(Z_t) = h(Z_0) + \int_{0+}^{t} h'(Z_s)\;\mathrm{d}Z_s, \tag{4.5}$$

where the local time term in the expansion, as in [24, (2.5)], vanishes because Z is of bounded variation (see Corollary [52, p.230]). Now, as $g(Y)$ and $h(Z)$ are semi-martingales, we can apply the integration by parts formula [52, p. 68] twice, and get for $t \geq 0$,

$$
\begin{aligned}
e^{-qt} f(Y_t, Z_t) - f(Y_0, Z_0) &= e^{-qt} g(Y_t)\, h(Z_t) - f(Y_0, Z_0) \\
&= \int_{0^+}^{t} e^{-qs} g(Y_{s^-})\, \mathrm{d}h(Z_s) + \int_{0^+}^{t} h(Z_s)\, \mathrm{d}[e^{-qs} g(Y_s)]_s \\
&\quad + [e^{-qt} g(Y_t), h(Z_t)]_t. \\
&= \int_{0^+}^{t} e^{-qs} g(Y_{s^-})\ \mathrm{d}h(Z_s) + \int_{0^+}^{t} -q e^{-qs} g(Y_{s^-}) h(Z_s)\ \mathrm{d}s \\
&\quad + \int_{0^+}^{t} e^{-qs} h(Z_s)\ \mathrm{d}g(Y_s), \qquad (4.6)
\end{aligned}
$$

where we used that $[e^{-qt} g(Y_t), h(Z_t)]_t = 0$ as $h(Z)$ is a continuous process of bounded variation. Substitute (4.5) and (4.4) in (4.6) to get expansion (4.3).

Now, suppose that Y is of bounded variation process, then by the assumptions on g, we can use [52, Theorem 78] to get the expansion,

$$
g(Y_t) - g(Y_0) = \int_{0^+}^{t} g'(Y_{s-})\, \mathrm{d}Y_s + \sum_{0<s\leq t} \left[g(Y_s) - g(Y_{s-}) - g'(Y_{s-}) \Delta Y_s \right]. \qquad (4.7)
$$

Similarly, substitute (4.5) and (4.7) in (4.6), to have expansion (4.3).□

Remark 6 (i) Condition II in Definition 4.1.1 is not required for Theorem 4.1.2, but it is needed to define the operator $\mathcal{A}$, which will be defined next and used during this thesis.

(ii) Although in the bounded variation case, the second derivative is not well defined, the expansion (4.3) does not change. This is because in that case, we will have σ from the quadratic variation part $[Y, Y]$, and $\sigma = 0$ in that case. Hence, the term which involves the second derivative vanishes.

For $f \in \mathcal{S}_{[b,a]\times[c,d]}$, $y \in [b, a]$ and $z \in [c, d]$, and with the triple (μ, σ^2, ν) that we introduced in Section 2.1, we define the operators $\mathcal{A}$ and Γ^δ by,

$$
\begin{aligned}
\mathcal{A} f(y, z) &= \mu \frac{\partial f}{\partial y}(y, z) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2}(y, z) \\
&\quad + \int_{0^+}^{\infty} \left[f(y - \theta, z) - f(y, z) + \theta \frac{\partial f}{\partial y}(y, z) \mathbf{1}_{\{0<\theta\leq 1\}} \right] \nu(\mathrm{d}\theta), \qquad (4.8)
\end{aligned}
$$

and

$$
\Gamma^\delta f(y, z) = \frac{\partial}{\partial y} f(y, z)\ \delta(z) - \frac{\partial}{\partial z} f(y, z)(1 - \delta(z)). \qquad (4.9)
$$

Note that, we will also use the operator $\mathcal{A}$ for a one variable function $f \in \mathcal{S}_{[b,a]}$, where $\mathcal{S}_{[b,a]}$ is a one dimensional functional space defined as in Definition 4.1.1 with $f = g$,

$$\begin{aligned}\mathcal{A}f(y) &= \mu f'(y) + \frac{\sigma^2}{2} f''(y) \\ &\quad + \int_{0+}^{\infty} \left[f(y-\theta) - f(y) + \theta f'(y) \mathbf{1}_{\{0<\theta\leq 1\}} \right] \nu(\mathrm{d}\theta). \end{aligned} \tag{4.10}$$

Remark 7 Note that, since $f \in \mathcal{S}_{[b,a]\times[c,d]}$, then by [41, Lemma 4], the integral term in $\mathcal{A}f(y,z)$ is well-defined.

The next corollary finds a two dimensional Itô expansion for $f \in \mathcal{S}_{[b,a]\times[c,d]}$.

Corollary 4.1.3 *Suppose that Y and Z are semi-martingales, where Z is a continuous process of bounded variation. Let $Y_0 \in [b,a]$ and $Z_0 \in [c,d]$. Define the following first passage times:*
$\tau_b^- := \inf\{t \geq 0 : Y_t < b\}$, $\tau_a^+ := \inf\{t \geq 0 : Y_t > a\}$, $\kappa_c^- := \inf\{t \geq 0 : Z_t < c\}$, *and* $\kappa_d^+ := \inf\{t \geq 0 : Z_t > d\}$.

Suppose that $f \in \mathcal{S}_{[b,a]\times[c,d]}$ and let $T = \tau_b^- \wedge \tau_a^+ \wedge \kappa_c^- \wedge \kappa_d^+$. Then,

$$\begin{aligned} & e^{-q(t\wedge T)} f(Y_{t\wedge T}, Z_{t\wedge T}) - f(Y_0, Z_0) \\ &= \int_{0+}^{t\wedge T} -qe^{-qs} f(Y_{s-}, Z_s)\,\mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial y}(Y_{s-}, Z_s)\,\mathrm{d}Y_s \\ &\quad + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial z}(Y_{s-}, Z_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_{0}^{t\wedge T} e^{-qs} \frac{\partial^2 f}{\partial y^2}(Y_{s-}, Z_s)\,\mathrm{d}\,[Y,Y]_s^c \\ &\quad + \sum_{0<s\leq t\wedge T} e^{-qs} \big[\Delta f(Y_s, Z_s) - \frac{\partial f}{\partial y}(Y_{s-}, Z_s)\Delta Y_s\big], \end{aligned} \tag{4.11}$$

Proof Since $f \in \mathcal{S}_{[b,a]\times[c,d]}$, then $f = \sum_k g_k\; h_k$, where g_k and h_k for each k are satisfying the regularity conditions on closed bounded intervals as given in Definition 4.1.1. Therefore, each g_k and h_k can be extended to functions $\tilde{g}_k$ and $\tilde{h}_k$, respectively, for each k, that satisfy their regularity conditions on the whole real line. For example, if we define the following extensions, for each $k \geq 1$,

$$\tilde{g}_k(y) = \begin{cases} g_k(y) & \text{if } b \leq y \leq a, \\ g_k(a) + g_k'(a)(y-a) & \text{if } y > a, \\ g_k(b) + g_k'(b)(y-b) & \text{if } y < b, \end{cases}$$

and

$$\tilde{h}_k(z) = \begin{cases} h_k(z) & \text{if } c \leq z \leq d, \\ h_k(d) & \text{if } z > d, \\ h_k(c) & \text{or } z < c. \end{cases}$$

Then, we get a function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form $\tilde{f} = \sum_k \tilde{f}_k = \sum_k \tilde{g}_k \, \tilde{h}_k$ such that each $\tilde{f}_k$ satisfies the assumptions in Theorem 4.1.2 and we have that $\tilde{f}|_{[b,a]\times[c,d]} = f$. So, by applying Theorem 4.1.2 on each $\tilde{f}_k$ at $t \wedge T$, then by linearity we get the following expansion of $\tilde{f}$ at $t \wedge T$,

$$\begin{aligned} & e^{-q(t\wedge T)}\tilde{f}(Y_{t\wedge T}, Z_{t\wedge T}) - \tilde{f}(Y_0, Z_0) \\ &= \int_{0+}^{t\wedge T} -qe^{-qs}\tilde{f}(Y_{s-}, Z_s)\,\mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial \tilde{f}}{\partial y}(Y_{s-}, Z_s)\,\mathrm{d}Y_s \\ &\quad + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial \tilde{f}}{\partial z}(Y_{s-}, Z_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_{0}^{t\wedge T} e^{-qs}\frac{\partial^2 \tilde{f}}{\partial y^2}(Y_{s-}, Z_s)\,\mathrm{d}\,[Y,Y]^c_s \\ &\quad + \sum_{0<s\leq t\wedge T} e^{-qs}\big[\Delta\tilde{f}(Y_s, Z_s) - \frac{\partial \tilde{f}}{\partial y}(Y_{s-}, Z_s)\Delta Y_s\big]. \end{aligned}$$

As $Y_0 \in [b,a]$ and $Z_0 \in [c,d]$, $\tilde{f}(Y_0, Z_0) = f(Y_0, Z_0)$, and since $\tilde{f}|_{[b,a]\times[c,a]} = f$, we get that,

$$\begin{aligned} & e^{-q(t\wedge T)}\tilde{f}(Y_{t\wedge T}, Z_{t\wedge T}) - f(Y_0, Z_0) \\ &= \int_{0+}^{t\wedge T} -qe^{-qs}f(Y_{s-}, Z_s)\,\mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial y}(Y_{s-}, Z_s)\,\mathrm{d}Y_s \\ &\quad + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial z}(Y_{s-}, Z_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_{0}^{t\wedge T} e^{-qs}\frac{\partial^2 f}{\partial y^2}(Y_{s-}, Z_s)\,\mathrm{d}\,[Y,Y]^c_s \\ &\quad + \sum_{0<s<t\wedge T} e^{-qs}\left[f(Y_s, Z_s) - f(Y_{s-}, Z_s)\right] + e^{-q(t\wedge T)}\big[\tilde{f}(Y_{t\wedge T}, Z_{t\wedge T}) - \tilde{f}(Y_{(t\wedge T)-}, Z_{t\wedge T})\big] \\ &\quad + \sum_{0<s\leq t\wedge T} e^{-qs}\big[-\frac{\partial f}{\partial y}(Y_{s-}, Z_s)\Delta Y_s\big], \end{aligned}$$

Now $e^{-q(t\wedge T)}\tilde{f}(Y_{t\wedge T}, Z_{t\wedge T})$ is in both sides, so we can delete it. Use that

$$\tilde{f}(Y_{(t\wedge T)-}, Z_{t\wedge T}) = f(Y_{(t\wedge T)-}, Z_{t\wedge T}),$$

and add $e^{-q(t\wedge T)}f(Y_{t\wedge T}, Z_{t\wedge T})$ to both sides, then we have expansion (4.11).□

We are now ready to state and prove the main result in this chapter, which gives the expression for (4.2).

Theorem 4.1.4 *For $a > 0$, let $-\infty < c < a < \infty$ and $x,\ \bar{x} \in (c,a] \times (c,a]$ such that $x \leq \bar{x}$. Let $\delta\colon [\bar{x},\infty) \to [0,1)$ be a measurable function such that there exists a unique solution $y^\delta_{\bar{x}}$ to (3.4). Let $q > 0$ and V^δ be the tax process with natural rate δ associated with the spectrally negative Lévy process X. Let $f\colon (-\infty, c] \times [c,a] \to \mathbb{R}$ be a measurable, locally bounded function, and of the form $f(y,z) = g(y)\,h(z)$. Suppose that there exists $\lambda > a-c$ such that $s \mapsto \int_\lambda^\infty g(s-\theta)\,\nu(\mathrm{d}\theta)$ is bounded on (c,a), and h is an absolutely continuous with a bounded density on $[c,a]$. Let $\tilde{f}\colon (-\infty,a] \times [c,a] \to \mathbb{R}$ be an extension of f that lies in $\mathcal{S}_{[c,a]\times[c,a]}$ and of the form $\tilde{f}(y,z) = \tilde{g}(y)\,h(z)$, where $\tilde{g}$ is bounded on $(-\infty,a]$. Then,*

$$
\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-} f(V^\delta_{\tau_c^-},\ \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] \\
&= \tilde{f}(x,\bar{x}) + \int_c^{\bar{x}} (\mathcal{A}-q)\tilde{f}(y,\bar{x})\Big[\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)}(\bar{x}-y) - W^{(q)}(x-y)\Big]\mathrm{d}y \\
&\quad + \frac{\sigma^2}{2}\left[f(c,\bar{x}) - \tilde{f}(c+,\bar{x})\right]\left[W^{(q)\prime}(x-c) - \frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)\prime}(\bar{x}-c)\right] \\
&\quad + \frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)} \times \left\{\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-},\overline{V}^\delta_s)\,\mathrm{d}s\right]\right. \\
&\quad - \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\Gamma^\delta\tilde{f}(\overline{V}^\delta_s,\overline{V}^\delta_s)\,\mathrm{d}\overline{X}_s\right] - \tilde{f}(a,a)\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_c^-\}}\right] \\
&\quad \left. + \mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}\left[f(c,\overline{V}^\delta_{\tau_c^-}) - \tilde{f}(c+,\overline{V}^\delta_{\tau_c^-})\right]\mathbf{1}_{\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\}}\right]\right\}.
\end{aligned}
$$

Further, for the case $x = c$, if the paths of X are of bounded variation, then

$$
\begin{aligned}
&\mathbb{E}_{c,\bar{x}}\left[e^{-q\tau_c^-} f(V^\delta_{\tau_c^-},\ \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] \\
&= \tilde{f}(c+,\bar{x}) + \frac{W^{(q)}(0)}{W^{(q)}(\bar{x}-c)}\int_c^{\bar{x}} (\mathcal{A}-q)\tilde{f}(y,\bar{x})W^{(q)}(\bar{x}-y)\,\mathrm{d}y \\
&\quad + \frac{W^{(q)}(0)}{W^{(q)}(\bar{x}-c)} \times \left\{\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-},\overline{V}^\delta_s)\,\mathrm{d}s\right]\right. \\
&\quad \left. - \mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\Gamma^\delta\tilde{f}(\overline{V}^\delta_s,\overline{V}^\delta_s)\,\mathrm{d}\overline{X}_s\right] - \tilde{f}(a,a)\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_c^-\}}\right]\right\},
\end{aligned}
$$

whereas if the paths of X are of unbounded variation,

$$
\mathbb{E}_{c,\bar{x}}\left[e^{-q\tau_c^-} f(V^\delta_{\tau_c^-},\ \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] = f(c,\bar{x}). \tag{4.12}
$$

Proof Let $T = \tau_c^- \wedge \tau_a^+$. Note that, T equals the stopping time defined in Corollary 4.1.3. This is because, according to the definition of $\overline{V}^\delta$, we have $\tau_c^- \wedge \kappa_c^- = \tau_c^-$ and

$\tau_a^+ = \kappa_a^+$. Since $\overline{V}^\delta$ is a continuous process of bounded variation, and $\tilde{f} \in \mathcal{S}_{[c,a]\times[c,a]}$, then by Corollary 4.1.3, we have

$$
\begin{aligned}
& e^{-q(t\wedge T)}\tilde{f}(V^\delta_{t\wedge T}, \overline{V}^\delta_{t\wedge T}) - \tilde{f}(x,\bar{x}) \\
&= \int_{0+}^{t\wedge T} -qe^{-qs}\tilde{f}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}V^\delta_s \\
&\quad + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial \tilde{f}}{\partial z}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}\overline{V}^\delta_s + \frac{1}{2}\int_{0}^{t\wedge T} e^{-qs}\frac{\partial^2 \tilde{f}}{\partial y^2}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}\left[V^\delta, V^\delta\right]^c_s \\
&\quad + \sum_{0<s\le t} e^{-qs}\Big[\Delta\tilde{f}(V^\delta_s, \overline{V}^\delta_s) - \frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\Delta V^\delta_s\Big],
\end{aligned}
\tag{4.13}
$$

where we use the notations: $\Delta V^\delta_s = V^\delta_s - V^\delta_{s-}$ and for a stochastic process Z, $(Z_s)^c = Z_s - \sum_{0<u\le s}\Delta Z_u$. From the definition of the natural tax process V^δ, we have that

$$
\Delta\tilde{f}(V^\delta_{s-} + \Delta X_s, \overline{V}^\delta_s) = \Delta\tilde{f}(V^\delta_s, \overline{V}^\delta_s),
$$

and also by the decomposition of X in (2.3), for any $s > 0$

$$
\left[V^\delta, V^\delta\right]^c_s = [X, X]^c_s = \sigma^2\, s.
$$

Rearranging the terms in (4.13),

$$
\begin{aligned}
& e^{-q(t\wedge T)}\tilde{f}(V^\delta_{t\wedge T}, \overline{V}^\delta_{t\wedge T}) \\
&\quad = M_t + \tilde{f}(x,\bar{x}) + \int_{0+}^{t\wedge T} e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}s \\
&\qquad + \int_{0+}^{t\wedge T} e^{-qs}\Big[\frac{\partial \tilde{f}}{\partial z}(V^\delta_{s-}, \overline{V}^\delta_s)(1-\delta(\overline{V}^\delta_s)) - \frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\delta(\overline{V}^\delta_s)\Big]\mathrm{d}\overline{X}_s,
\end{aligned}
$$

where

$$
\begin{aligned}
M_t &= \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}\big[X_s - \mu s - \sum_{0<u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|>1\}}\big] \\
&\quad + \Bigg\{\sum_{0<s\le t\wedge T} e^{-qs}\big[\Delta\tilde{f}(V^\delta_{s-} + \Delta X_s, \overline{V}^\delta_s) - \frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\Delta X_s \mathbf{1}_{\{|\Delta X_s|\le 1\}}\big] \\
&\quad - \int_{0+}^{t\wedge T}\int_{0+}^{\infty} e^{-qs}\big[\tilde{f}(V^\delta_{s-} - \theta, \overline{V}^\delta_s) - \tilde{f}(V^\delta_{s-}, \overline{V}^\delta_s) + \theta\frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\mathbf{1}_{\{0<\theta\le 1\}}\big]\nu(\mathrm{d}\theta)\,\mathrm{d}s\Bigg\}
\end{aligned}
$$

is a zero mean martingale.

Take expectation, let $t \uparrow \infty$, and since $\tilde{g}$ is bounded on $(-\infty, a]$, we can use the dominated convergence theorem to get that

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q(\tau_c^- \wedge \tau_a^+)}\tilde{f}(V^\delta_{\tau_c^- \wedge \tau_a^+}, \overline{V}^\delta_{\tau_c^- \wedge \tau_a^+})\right] \\
&\quad = \tilde{f}(x,\bar{x}) + \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^- \wedge \tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}s\right] \\
&\qquad + \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^- \wedge \tau_a^+} e^{-qs}\big[\frac{\partial \tilde{f}}{\partial z}(V^\delta_{s-}, \overline{V}^\delta_s)(1-\delta(\overline{V}^\delta_s)) - \frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\delta(\overline{V}^\delta_s)\big]\mathrm{d}\overline{X}_s\right].
\end{aligned} \tag{4.14}$$

On the other hand, by the lack of upward jumps, we have

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q(\tau_c^- \wedge \tau_a^+)}\tilde{f}(V^\delta_{\tau_c^- \wedge \tau_a^+}, \overline{V}^\delta_{\tau_c^- \wedge \tau_a^+})\right] \\
&\quad = \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}\tilde{f}(V^\delta_{\tau_c^-}, \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] + \tilde{f}(a,a)\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_c^- > \tau_a^+\}}\right].
\end{aligned} \tag{4.15}$$

Put (4.15) in the left hand side of (4.14) to have that

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}\tilde{f}(V^\delta_{\tau_c^-}, \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] \\
&\quad = \tilde{f}(x,\bar{x}) + \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^- \wedge \tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-}, \overline{V}^\delta_s)\,\mathrm{d}s\right] \\
&\qquad + \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^- \wedge \tau_a^+} e^{-qs}\big[\frac{\partial \tilde{f}}{\partial z}(V^\delta_{s-}, \overline{V}^\delta_s)(1-\delta(\overline{V}^\delta_s)) - \frac{\partial \tilde{f}}{\partial y}(V^\delta_{s-}, \overline{V}^\delta_s)\delta(\overline{V}^\delta_s)\big]\mathrm{d}\overline{X}_s\right] \\
&\qquad - \tilde{f}(a,a)\,\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_c^- > \tau_a^+\}}\right].
\end{aligned} \tag{4.16}$$

Also,

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-}, \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right] \\
&\quad = \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_c^-}f(X_{\rho_c^-}, \bar{x})\mathbf{1}_{\{\rho_c^- < \rho_{\bar{x}}^+\}}\right] \\
&\qquad + \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_{\bar{x}}^+}\mathbf{1}_{\{\rho_{\bar{x}}^+ < \rho_c^-\}}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-}, \overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^- < \tau_a^+\}}\right],
\end{aligned} \tag{4.17}$$

where

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_{\bar{x}}^+}\mathbf{1}_{\{\rho_{\bar{x}}^+ < \rho_c^-\}}\right] = \frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}.$$

From [41, Equation 7], we know that the first term in (4.17) is given by

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_c^-}f(X_{\rho_c^-},\bar{x})\mathbf{1}_{\{\rho_c^-<\rho_{\bar{x}}^+\}}\right]\\
&\quad=\tilde{f}(x,\bar{x})-\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}\tilde{f}(\bar{x},\bar{x})\\
&\qquad+\int_c^{\bar{x}}(\mathcal{A}-q)\tilde{f}(y,\bar{x})\Big[\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)}(\bar{x}-y)-W^{(q)}(x-y)\Big]\mathrm{d}y\\
&\qquad+\frac{\sigma^2}{2}\left[f(c,\bar{x})-\tilde{f}(c+,\bar{x})\right]\left[W^{(q)\prime}(x-c)-\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)\prime}(\bar{x}-c)\right].
\end{aligned}$$

Now, we find the last expectation in (4.17),

$$\begin{aligned}
&\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^-<\tau_a^+\}}\right]\\
&=\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}<c,\ \tau_c^-<\tau_a^+\right\}}\right]+\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\right\}}\right]\\
&=\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}\tilde{f}(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}<c,\ \tau_c^-<\tau_a^+\right\}}\right]+\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(c,\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\right\}}\right]\\
&=\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}\tilde{f}(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^-<\tau_a^+\}}\right]+\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}f(c,\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\right\}}\right]\\
&\quad-\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}\tilde{f}(c+,\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\left\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\right\}}\right].
\end{aligned}\tag{4.18}$$

Then, substitute (4.16), with $x=\bar{x}$, in (4.18), and get the second term in (4.17).

Therefore, we get that

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}f(V^\delta_{\tau_c^-},\overline{V}^\delta_{\tau_c^-})\mathbf{1}_{\{\tau_c^-<\tau_a^+\}}\right]\\
&\quad=\tilde{f}(x,\bar{x})-\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}\tilde{f}(\bar{x},\bar{x})\\
&\qquad+\int_c^{\bar{x}}(\mathcal{A}-q)\tilde{f}(y,\bar{x})\Big[\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)}(\bar{x}-y)-W^{(q)}(x-y)\Big]\mathrm{d}y\\
&\qquad+\frac{\sigma^2}{2}\big[f(c,\bar{x})-\tilde{f}(c+,\bar{x})\big]\big[W^{(q)\prime}(x-c)-\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}W^{(q)\prime}(\bar{x}-c)\big]\\
&\qquad+\frac{W^{(q)}(x-c)}{W^{(q)}(\bar{x}-c)}\times\Bigg\{\tilde{f}(\bar{x},\bar{x})+\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+}e^{-qs}(\mathcal{A}-q)\tilde{f}(V^\delta_{s-},\overline{V}^\delta_s)\,\mathrm{d}s\right]\\
&\qquad-\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+}e^{-qs}\Gamma^\delta\tilde{f}(\overline{V}^\delta_s,\overline{V}^\delta_s)\,\mathrm{d}\overline{X}_s\right]-\tilde{f}(a,a)\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_c^->\tau_a^+\}}\right]\\
&\qquad+\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_c^-}\big[f(c,\overline{V}^\delta_{\tau_c^-})-\tilde{f}(c+,\overline{V}^\delta_{\tau_c^-})\big]\mathbf{1}_{\left\{V^\delta_{\tau_c^-}=c,\ \tau_c^-<\tau_a^+\right\}}\right]\Bigg\},
\end{aligned}$$

which gives the required statement.□

By the next lemma, we can get an explicit analytic expression for (4.2) when $c = 0$.

Lemma 4.1.5 *Given the assumptions in Theorem 4.1.4, if we let $c = 0$ and $x = \bar{x}$, then*

(I)

$$\mathbb{E}_{x,x}\left[\int_{0+}^{\tau_0^- \wedge \tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V_s^\delta, \overline{V}_s^\delta)\mathrm{d}s\right]$$
$$= \int_x^a \int_0^a (\mathcal{A}-q)\tilde{f}(y,z)\left\{\left[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right]\mathrm{d}y + W^{(q)}(0+)\delta_z(\mathrm{d}y)\right\} l_x(z)\,\mathrm{d}z,$$

where δ_z is the Dirac measure which assigns unit mass to the point z and $l_x(z)$ is given by

$$l_x(z) = \frac{1}{1-\delta(z)}\exp\left\{-\int_x^z \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}. \tag{4.19}$$

(II)

$$\mathbb{E}_{x,x}\left[\int_{0+}^{\tau_0^- \wedge \tau_a^+} e^{-qs}\Gamma^\delta \tilde{f}(\overline{V}_s^\delta, \overline{V}_s^\delta)\,\mathrm{d}\overline{X}_s\right] = \int_x^a l_x(z)\,\Gamma^\delta \tilde{f}(z,z)\,\mathrm{d}z.$$

Proof (I) Let

$$V_{0,a}(x,\mathrm{d}y,\mathrm{d}z) = \int_0^\infty e^{-qs}\mathbb{P}_{x,x}(V_s^\delta \in \mathrm{d}y,\ \overline{V}_s^\delta \in \mathrm{d}z,\ s < \tau_0^- \wedge \tau_a^+)\,\mathrm{d}s \tag{4.20}$$

be the q-potential measure of $(V^\delta, \overline{V}^\delta)$ killed on exiting $[0,a]$. Then,

$$\mathbb{E}_{x,x}\left[\int_{0+}^{\tau_0^- \wedge \tau_a^+} e^{-qs}(\mathcal{A}-q)\tilde{f}(V_s^\delta, \overline{V}_s^\delta)\,\mathrm{d}s\right] = \int_x^a \int_0^a (\mathcal{A}-q)\tilde{f}(y,z)V_{0,a}(x,\mathrm{d}y,\mathrm{d}z).$$

By Theorem 3.2.3, we can use [33, Theorem 1.3] to extract the following result for the natural tax process V^δ,

$$
\begin{aligned}
&\mathbb{E}_{x,x}\left[e^{-q\tau_0^-};\ -V^\delta_{\tau_0^-}\in \mathrm{d}Q,\ \overline{V}^\delta_{\tau_0^-}\in \mathrm{d}z\right]\\
&=\left\{\int_0^z \left[W^{(q)\prime}(z-m)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-m)\right]\nu(m+\mathrm{d}Q)\,\mathrm{d}m\right.\\
&\qquad\left.+W^{(q)}(0+)\,\nu(z+\mathrm{d}Q)\right\}l_x(z)\,\mathrm{d}z\\
&=\int_0^z \left[W^{(q)\prime}(z-m)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-m)\right]\nu(m+\mathrm{d}Q)\,\mathrm{d}m\,l_x(z)\,\mathrm{d}z\\
&\qquad+W^{(q)}(0+)\,\nu(z+\mathrm{d}Q)\,l_x(z)\,\mathrm{d}z\\
&=\int_0^z \left[W^{(q)\prime}(z-m)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-m)\right]\nu(m+\mathrm{d}Q)\,\mathrm{d}m\,l_x(z)\,\mathrm{d}z\\
&\qquad+W^{(q)}(0+)\left[\int_0^z \delta_z(\mathrm{d}m)\nu(m+\mathrm{d}Q)\right]l_x(z)\,\mathrm{d}z\\
&=\int_0^\infty \nu(m+\mathrm{d}Q)\Big\{\big[W^{(q)\prime}(z-m)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-m)\big]\,\mathrm{d}m\\
&\qquad+W^{(q)}(0+)\delta_z(\mathrm{d}m)\Big\}\,l_x(z)\,\mathrm{d}z. \qquad (4.21)
\end{aligned}
$$

On the one hand, we can use similar method to the proof of [33, Lemma 2.2] and get that

$$
\begin{aligned}
&\mathbb{E}_{x,x}\left[e^{-q\tau_0^-}f(-V^\delta_{\tau_0^-},\overline{V}^\delta_{\tau_0^-})\mathbf{1}_{\left\{\tau_0^-<\tau_a^+\right\}}\right]\\
&=\mathbb{E}_{x,x}\left[\sum_{0\le t<\infty}e^{-qt}f(-(V^\delta_{t-}+\Delta V^\delta_t),\overline{V}^\delta_t)\mathbf{1}_{\left\{\underline{V}^\delta_{t-}\ge 0,x\le\overline{V}^\delta_t\le a,V^\delta_{t-}+\Delta V^\delta_t<0\right\}}\right]\\
&=\mathbb{E}_{x,x}\left[\int_{-\infty}^0\int_0^\infty e^{-qt}\mathbf{1}_{\left\{t<\tau_0^-\right\}}\mathbf{1}_{\left\{x\le\overline{V}^\delta_t\le a\right\}}f(-(V^\delta_{t-}+\theta),\overline{V}^\delta_t)\mathbf{1}_{\left\{V^\delta_{t-}+\theta<0\right\}}N(\mathrm{d}t,\mathrm{d}\theta)\right]\\
&=\mathbb{E}_{x,x}\left[\int_0^\infty\int_0^\infty e^{-qt}\mathbf{1}_{\left\{t<\tau_0^-\right\}}f(\theta-V^\delta_{t-},\overline{V}^\delta_t)\mathbf{1}_{\left\{x\le\overline{V}^\delta_t\le a\right\}}\mathbf{1}_{\left\{V^\delta_{t-}-\theta<0\right\}}N(\mathrm{d}t,-\mathrm{d}\theta)\right]\\
&=\mathbb{E}_{x,x}\left[\int_0^\infty\int_0^\infty e^{-qt}\mathbf{1}_{\left\{t<\tau_0^-\right\}}f(\theta-V^\delta_{t-},\overline{V}^\delta_t)\mathbf{1}_{\left\{x\le\overline{V}^\delta_t\le a\right\}}\mathbf{1}_{\left\{\theta>V^\delta_{t-}\right\}}\nu(\mathrm{d}\theta)\,\mathrm{d}t\right]\\
&=\int_0^\infty e^{-qt}\mathrm{d}t\int_0^\infty\mathbb{E}_{x,x}\left[f(\theta-V^\delta_{t-},\overline{V}^\delta_t)\mathbf{1}_{\left\{t<\tau_0^-\right\}}\right]\mathbf{1}_{\left\{\theta>V^\delta_{t-}\right\}}\mathbf{1}_{\left\{x\le\overline{V}^\delta_t\le a\right\}}\nu(\mathrm{d}\theta)\\
&=\int_0^\infty e^{-qt}\mathrm{d}t\int_0^\infty\int_x^a\int_0^a f(\theta-y,z)\mathbb{P}_{x,x}(V^\delta_t\in\mathrm{d}y,\overline{V}^\delta_t\in\mathrm{d}z,t<\tau_0^-\wedge\tau_a^+)\mathbf{1}_{\{\theta>y\}}\nu(\mathrm{d}\theta)\\
&=\int_0^\infty\int_x^a\int_0^a f(\theta-y,z)\,V_{0,a}(x,\mathrm{d}y,\mathrm{d}z)\,\mathbf{1}_{\{\theta>y\}}\,\nu(\mathrm{d}\theta)\\
&=\int_y^\infty\int_x^a\int_0^a f(\theta-y,z)\,V_{0,a}(x,\mathrm{d}y,\mathrm{d}z)\,\nu(\mathrm{d}\theta)\\
&=\int_0^\infty\int_x^a\int_0^a f(Y,z)\,V_{0,a}(x,\mathrm{d}y,\mathrm{d}z)\,\nu(\mathrm{d}Y+y)\\
&=\int_x^a\int_0^a f(Y,z)\int_0^\infty\nu(\mathrm{d}Y+y)\,V_{0,a}(x,\mathrm{d}y,\mathrm{d}z),
\end{aligned}
\tag{4.22}
$$

where in the fourth equality we use (2.1), in the seventh equality we use (4.20), and in the last two equalities we use the change of variable $\theta-y=Y$, so that $\mathrm{d}\theta=\mathrm{d}Y+y$.

On the other hand, we can use (4.21) and get that

$$
\begin{aligned}
&\mathbb{E}_{x,x}\left[e^{-q\tau_0^-}f(-V^\delta_{\tau_0^-},\overline{V}^\delta_{\tau_0^-})\mathbf{1}_{\left\{\tau_0^-<\tau_a^+\right\}}\right]\\
&=\int_x^a\int_0^a f(Q,z)\mathbb{E}_{x,x}\left[e^{-q\tau_0^-};-V^\delta_{\tau_0^-}\in\mathrm{d}Q,\overline{V}^\delta_{\tau_0^-}\in\mathrm{d}z,\tau_0^-<\tau_a^+\right]\\
&=\int_x^a\int_0^a f(Q,z)\times\\
&\quad\int_0^\infty\nu(m+\mathrm{d}Q)\Big\{\left[W^{(q)\prime}(z-m)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-m)\right]\mathrm{d}m\\
&\qquad+W^{(q)}(0+)\delta_z(\mathrm{d}m)\Big\}\,l_x(z)\,\mathrm{d}z.
\end{aligned}
\tag{4.23}
$$

It is clear, by comparing (4.22) and (4.23), that

$$V_{0,a}(x, \mathrm{d}y, \mathrm{d}z)$$
$$= \left[\left\{W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right\}\mathrm{d}y + W^{(q)}(0+)\delta_z(\mathrm{d}y)\right] l_x(z)\,\mathrm{d}z.$$

and hence the statement is proved.

(II) We can prove this result with a similar argument to the proof of Theorem 3.2.9.

□

Corollary 4.1.6 *Given the assumptions in Theorem 4.1.4 with $c = 0$,*

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\, f(V^\delta_{\tau_0^-},\ \overline{V}^\delta_{\tau_0^-})\,\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right] \\
&= \tilde{f}(x,\bar{x}) + \int_0^{\bar{x}} (\mathcal{A}-q)\tilde{f}(y,\bar{x})\Big[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-y) - W^{(q)}(x-y)\Big]\mathrm{d}y \\
&\quad + \frac{\sigma^2}{2}\left[f(0,\bar{x}) - \tilde{f}(0+,\bar{x})\right]\left[W^{(q)\prime}(x) - \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)\prime}(\bar{x})\right] \\
&\quad + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{\int_{\bar{x}}^a\int_0^a (\mathcal{A}-q)\tilde{f}(y,z)\left\{\Big[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\Big]\mathrm{d}y\right. \\
&\quad \left. + W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\} l_{\bar{x}}(z)\,\mathrm{d}z \\
&\quad - \int_{\bar{x}}^a l_{\bar{x}}(z)\,\Gamma^\delta\tilde{f}(z,z)\,\mathrm{d}z - \tilde{f}(a,a)\,\exp\left\{-\int_{\bar{x}}^a \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\} \\
&\quad + \mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\tau_0^-}\left[f(0,\overline{V}^\delta_{\tau_0^-}) - \tilde{f}(0+,\overline{V}^\delta_{\tau_0^-})\right]\mathbf{1}_{\left\{V^\delta_{\tau_0^-}=0,\ \tau_0^-<\tau_a^+\right\}}\right]\Bigg\},
\end{aligned} \tag{4.24}$$

where $l_{\bar{x}}(z)$ is given by (4.19).

Proof The proof follows by Theorem 4.1.4 and Lemma 4.1.5.□

4.2 Applications

In this section, we give some applications of (4.24). The first example gives a new identity in the literature for the process V^δ, the two-sided exit problem. The second and last examples give analytic expressions of the expected discounted function of the maximum surplus prior to ruin and the expected discounted amount of ruin, respectively, for the natural tax process V^δ, when ruin occurs before reaching some level $a > 0$. By looking at a recent article, [64], we realized that expressions (4.48) and

(4.31) below, have been derived for a latent tax process U^γ with γ constant. Their work depends on some recent results in the draw-down literature, [60, Equations 7, 9]. We will see that our expressions in both examples are more general compared to the ones given in [64], since we derive them for a general natural tax function δ. These examples are useful in many situations and we present them in the following Lemmas.

Recall that

$$\overline{W}^{(q)}(y) = \int_0^y W^{(q)}(r)\,\mathrm{d}r, \tag{4.25}$$

and

$$\overline{Z}^{(q)}(y) = \int_0^y Z^{(q)}(r)\,\mathrm{d}r = y + q\int_0^y \int_0^z W^{(q)}(r)\,\mathrm{d}r\,\mathrm{d}z. \tag{4.26}$$

Lemma 4.2.1 *For $x, \bar{x} \in [0,a]$, the solution of the below two-sided exit problem for a natural tax process V^δ has the following expression*

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}\right] \\
&\quad = Z^{(q)}(x) - Z^{(q)}(a)\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\exp\left\{-\int_{\bar{x}}^a \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\,\mathrm{d}r\right\} \\
&\quad\quad - q\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^a \frac{\delta(z)}{(1-\delta(z))}\,W^{(q)}(z)\,\exp\left\{-\int_{\bar{x}}^z \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\,\mathrm{d}z.
\end{aligned} \tag{4.27}$$

Proof The function $\tilde{f}(y,z) = f(y,z) = 1$ is in $\mathcal{S}_{[0,a]\times[0,a]}$. So, we can substitute it in (4.24) and get

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}\right] \\
&\quad = 1 - q\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}} W^{(q)}(\bar{x}-y)\,\mathrm{d}y + q\int_0^x W^{(q)}(x-y)\,\mathrm{d}y \\
&\quad\quad - \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\exp\left\{-\int_{\bar{x}}^a \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\,\mathrm{d}r\right\} \\
&\quad\quad + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^a\int_0^a -q\left\{\left[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right]\mathrm{d}y\right. \\
&\quad\quad\quad \left. + W^{(q)}(0+)\delta_z(\mathrm{d}y)\right\}\,l_{\bar{x}}(z)\,\mathrm{d}z.
\end{aligned} \tag{4.28}$$

By using

$$\int_0^z -W^{(q)\prime}(z-y)\,\mathrm{d}y = W^{(q)}(0+) - W^{(q)}(z),$$

and the change of variable $z-y=s$, we have that

$$\begin{aligned}
&\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{a}\int_{0}^{a} -q\left\{\left[W^{(q)\prime}(z-y)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\,W^{(q)}(z-y)\right]\mathrm{d}y\right.\\
&\quad\left.+\,W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\} l_{\bar{x}}(z)\,\mathrm{d}z\\
&= q\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\left\{\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\Big(\int_0^z W^{(q)}(s)\mathrm{d}s\Big)\,l_{\bar{x}}(z)\,\mathrm{d}z\right.\\
&\quad\left.-\int_{\bar{x}}^{a} l_{\bar{x}}(z)\,W^{(q)}(z)\,\mathrm{d}z\right\}. \qquad (4.29)
\end{aligned}$$

Then, in the first integral of the right-hand side of (4.29), we use integration by parts with $u=\displaystyle\int_0^z W^{(q)}(s)\,\mathrm{d}s$ and $\mathrm{d}v = l_{\bar{x}}(z)\,\dfrac{W^{(q)\prime}(z)}{W^{(q)}(z)}\,\mathrm{d}z$, which implies that

$$\begin{aligned}
&\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{a}\int_{0}^{a} -q\left\{\left[W^{(q)\prime}(z-y)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right]\mathrm{d}y\right.\\
&\quad\left.+\,W^{(q)}(0+)\delta_z(\mathrm{d}y)\right\} l_{\bar{x}}(z)\,\mathrm{d}z\\
&= -\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\,\mathrm{d}r\right\}\int_0^a W^{(q)}(r)\,\mathrm{d}r\\
&\quad+\int_0^{\bar{x}} W^{(q)}(r)\,\mathrm{d}r+\int_{\bar{x}}^{a} W^{(q)}(z)\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}z\\
&\quad-\int_{\bar{x}}^{a} l_{\bar{x}}(z)\,W^{(q)}(z)\,\mathrm{d}z. \qquad (4.30)
\end{aligned}$$

After that, substitute (4.30) in (4.28) and with some calculations, we get (4.27).□

Lemma 4.2.2 *Suppose that h is an absolutely continuous function with a locally bounded density on $[\bar{x},a]$. Then, for $x,\bar{x}\in[0,a]$, the following expression holds for a natural tax process V^δ:*

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\,h(\overline{V}^{\delta}_{\tau_0^-})\,\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\
&= h(\bar{x})\,Z^{(q)}(x)+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\left\{-h(a)\,Z^{(q)}(a)\,\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\right.\\
&\quad+\int_{\bar{x}}^{a} h'(s)\exp\left\{-\int_{\bar{x}}^{s}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}Z^{(q)}(s)\,\mathrm{d}s\\
&\quad\left.-\int_{\bar{x}}^{a}\frac{\delta(s)}{1-\delta(s)}\,h(s)\,q\,W^{(q)}(s)\exp\left\{-\int_{\bar{x}}^{s}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}s\right\}. \qquad (4.31)
\end{aligned}$$

Proof Since $\tilde{f}(y,z)=f(y,z)=h(z)$ is in $\mathcal{S}_{[0,a]\times[0,a]}$, we can substitute it in (4.24). Then, we use (4.8) to get $\mathcal{A}f(y,z)=0$ and (4.9) to get $\Gamma^\delta h(z)=-h'(z)(1-\delta(z))$ and have

$$
\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\,h(\overline{V}^{\delta}_{\tau_0^-})\,\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\
&= h(\bar{x}) - q\,h(\bar{x})\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}} W^{(q)}(r)\,\mathrm{d}r + q\,h(\bar{x})\int_0^{x} W^{(q)}(r)\,\mathrm{d}r\\
&\quad+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{\int_{\bar{x}}^{a}\int_0^{a} -q\,h(z)\left\{\left[W^{(q)\prime}(z-y)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right]\mathrm{d}y\right.\\
&\qquad\left.+\,W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\}l_{\bar{x}}(z)\,\mathrm{d}z\\
&\quad+\int_{\bar{x}}^{a} l_{\bar{x}}(z)\,h'(z)(1-\delta(z))\,\mathrm{d}z - h(a)\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\Bigg\},
\end{aligned}
\tag{4.32}
$$

where $l_{\bar{x}}(z)$ is given by (4.19). We can rewrite (4.32) as

$$
\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\,h(\overline{V}^{\delta}_{\tau_0^-})\,\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\
&= h(\bar{x})\,Z^{(q)}(x) - q\,h(\bar{x})\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}} W^{(q)}(r)\,\mathrm{d}r\\
&\quad+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{a} h'(z)\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}z\\
&\quad-\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,h(a)\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\\
&\quad+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{a}\int_0^{a} -q\,h(z)\left\{\left[W^{(q)\prime}(z-y)-\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\right]\mathrm{d}y\right.\\
&\qquad\left.+\,W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\}l_{\bar{x}}(z)\,\mathrm{d}z.
\end{aligned}
\tag{4.33}
$$

The last integral on the RHS of (4.33) becomes

$$
\begin{aligned}
&\int_{\bar{x}}^{a} -h(z)\,q\,W^{(q)}(z)\,\frac{1}{1-\delta(z)}\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}z\\
&+\int_{\bar{x}}^{a}\frac{1}{1-\delta(z)}\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\,h(z)\,q\Big(\int_0^{z} W^{(q)}(r)\,\mathrm{d}r\Big)\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}z.
\end{aligned}
\tag{4.34}
$$

Use integration by parts in the second term of (4.34), then we can rewrite (4.34) as

$$
\begin{aligned}
&\int_{\bar{x}}^{a} -h(z)\,q\,W^{(q)}(z)\,\frac{1}{1-\delta(z)}\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\mathrm{d}z\\
&\quad- h(a)\,q\Big(\int_0^{a} W^{(q)}(r)\,\mathrm{d}r\Big)\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\} + h(\bar{x})\,q\int_0^{\bar{x}} W^{(q)}(r)\,\mathrm{d}r\\
&\quad+\int_{\bar{x}}^{a}\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}h'(z)\,q\Big(\int_0^{z} W^{(q)}(r)\,\mathrm{d}r\Big)\mathrm{d}z\\
&\quad+\int_{\bar{x}}^{a}\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}h(z)\,q\,W^{(q)}(z)\,\mathrm{d}z.
\end{aligned}
\tag{4.35}
$$

Then substitute (4.35) in (4.33) and with easy computations, we get the required expression in (4.31).□

Remark 8 Expression (4.31) can be derived in another way using [60, Equations 7]. This has been done in [64] for a constant tax, and here we extend it for a general tax function. To show that, we are going to use the relation between the draw-down time of the risk process X without tax and the latent tax process U^γ, which is explained in [65, Section 4.1]. Then, by Theorem 3.2.3, it is a relation with the natural tax process V^δ. A general draw-down function ζ is a measurable function $\zeta : [0,\infty) \to (-\infty,\infty)$ satisfying $\zeta(z) < z$ for all $z \geq 0$. For $x \in [0,\infty)$ and $z \in [x,\infty)$, let $\zeta(z) := \int_x^z \gamma(r)\,\mathrm{d}r$. By the change of variable $r = \overline{X}_t$, the latent tax process U_t^γ can be written as

$$U_t^\gamma = X_t - \zeta(\overline{X}_t).$$

Let $\overline{\zeta}(z) := z - \zeta(z)$, then by recalling (3.9), we have that

$$\begin{aligned}
\bar{\gamma}_x(s) &= x + \int_x^s (1-\gamma(y))\,\mathrm{d}y \\
&= s - \int_x^s \gamma(y)\,\mathrm{d}y \\
&= \overline{\zeta}(s),
\end{aligned} \tag{4.36}$$

and hence,

$$\overline{U}_t^\gamma = \bar{\gamma}_x(\overline{X}_t) = \overline{\zeta}(\overline{X}_t).$$

For a latent tax process with a latent tax rate γ, recall the first passage times $\sigma_a^+ = \inf\{t > 0 : U_t^\gamma > a\}$ and $\sigma_0^- = \inf\{t > 0 : U_t^\gamma < 0\}$. Also, recall that $\rho_a^+ = \inf\{t \geq 0 : X_t > a\}$ and $\rho_0^- = \inf\{t \geq 0 : X_t < 0\}$. We are going to compute the following expectation

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\, \mathbf{1}_{\{\sigma_0^- < \sigma_a^+\}}\right],$$

where h is an absolutely continuous function with a locally bounded density on $[\bar{x}, a]$. Since before reaching $\bar{x}$, the process U^γ is just the Lévy process X, then by the strong

Markov property of $(X, \overline{X})$, we have that

$$\begin{aligned}\mathbb{E}_{x,\bar{x}}\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] &= \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}\, h(\bar{x})\,\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] \\ &\quad + \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_{\bar{x}}^+}\,\mathbf{1}_{\{\rho_{\bar{x}}^+<\rho_0^-\}}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] \\ &= h(\bar{x})\,\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}\,\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] \\ &\quad + \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_{\bar{x}}^+}\,\mathbf{1}_{\{\rho_{\bar{x}}^+<\rho_0^-\}}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right].\end{aligned} \tag{4.37}$$

In order to find (4.37), we first find the following expression for any $x = \bar{x}$,

$$\mathbb{E}_{x,x}\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right].$$

Since

$$\sigma_0^- = \inf\{t \geq 0 : U_t^\gamma < 0\} = \inf\{t \geq 0 : X_t < \zeta(\overline{X}_t)\} = \rho_\zeta^-,$$

and

$$\begin{aligned}\sigma_a^+ = \inf\{t \geq 0 : U_t^\gamma > a\} &= \inf\{t \geq 0 : \overline{U}_t^\gamma > a\} \\ &= \inf\{t \geq 0 : \overline{\zeta}(\overline{X}_t) > a\} \\ &= \inf\{t \geq 0 : \overline{X}_t > \overline{\zeta}^{-1}(a)\} = \rho^+_{\overline{\zeta}^{-1}(a)},\end{aligned}$$

then

$$\mathbb{E}_x\left[e^{-q\sigma_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] = \mathbb{E}_x\left[e^{-q\rho_\zeta^-}\, h(\overline{\zeta}(\overline{X}_{\rho_\zeta^-}))\,\mathbf{1}_{\left\{\rho_\zeta^-<\rho^+_{\overline{\zeta}^{-1}(a)}\right\}}\right]. \tag{4.38}$$

Let $\phi = h \circ \overline{\zeta}$, then by the formula [60, Equation 7], (4.38) becomes

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\tau_0^-}\, h(\overline{U}^\gamma_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] &= \mathbb{E}_x\left[e^{-q\rho_\zeta^-}\, \phi(\overline{X}_{\rho_\zeta^-})\,\mathbf{1}_{\left\{\rho_\zeta^-<\rho^+_{\overline{\zeta}^{-1}(a)}\right\}}\right] \\ &= \int_x^{\overline{\zeta}^{-1}(a)} \phi(s)\exp\left\{-\int_x^s \frac{W^{(q)\prime}(\overline{\zeta}(z))}{W^{(q)}(\overline{\zeta}(z))}\,\mathrm{d}z\right\} \\ &\qquad \times \left[\frac{W^{(q)\prime}(\overline{\zeta}(s))}{W^{(q)}(\overline{\zeta}(s))}\, Z^{(q)}(\overline{\zeta}(s)) - qW^{(q)}(\overline{\zeta}(s))\right]\mathrm{d}s.\end{aligned} \tag{4.39}$$

Now, use the change of variable $r = \overline{\zeta}(z)$, the right hand side of (4.39) becomes

$$\begin{aligned}&\int_x^{\overline{\zeta}^{-1}(a)} \phi(s)\exp\left\{-\int_x^{\overline{\zeta}(s)} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\overline{\zeta}^{-1}(r)))}\,\mathrm{d}r\right\} \\ &\qquad \times \left[\frac{W^{(q)\prime}(\overline{\zeta}(s))}{W^{(q)}(\overline{\zeta}(s))}\, Z^{(q)}(\overline{\zeta}(s)) - qW^{(q)}(\overline{\zeta}(s))\right]\mathrm{d}s.\end{aligned}$$

Then again, use the change of variables $t = \overline{\zeta}(s)$ and get

$$\int_x^a \phi(\overline{\zeta}^{-1}(t)) \exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\overline{\zeta}^{-1}(r)))}\,\mathrm{d}r\right\} \\ \times \frac{1}{1-\gamma(\overline{\zeta}^{-1}(r))}\left[\frac{W^{(q)\prime}(t)}{W^{(q)}(t)}\,Z^{(q)}(t) - qW^{(q)}(t)\right]\mathrm{d}t,$$

then by using that $\phi = h \circ \overline{\zeta}$ we get the final expression

$$\mathbb{E}_{x,x}\left[e^{-q\sigma_0^-}\,h(\overline{U}^{\gamma}_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] = \int_x^a h(t)\exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\overline{\zeta}^{-1}(r)))}\,\mathrm{d}r\right\} \\ \times \frac{1}{1-\gamma(\overline{\zeta}^{-1}(t))}\left[\frac{W^{(q)\prime}(t)}{W^{(q)}(t)}\,Z^{(q)}(t) - qW^{(q)}(t)\right]\mathrm{d}t. \tag{4.40}$$

By (4.36), we can rewrite (4.40) as

$$\mathbb{E}_{x,x}\left[e^{-q\sigma_0^-}\,h(\overline{U}^{\gamma}_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] = \int_x^a h(t)\exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_x^{-1}(r)))}\,\mathrm{d}r\right\} \\ \times \frac{1}{1-\gamma(\bar{\gamma}_x^{-1}(t))}\left[\frac{W^{(q)\prime}(t)}{W^{(q)}(t)}\,Z^{(q)}(t) - qW^{(q)}(t)\right]\mathrm{d}t. \tag{4.41}$$

Then, we can use integration by parts on (4.41) and with some easy computations, we get

$$\begin{aligned} &\mathbb{E}_{x,x}\left[e^{-q\sigma_0^-}\,h(\overline{U}^{\gamma}_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right] \\ &= -h(a)\,Z^{(q)}(a)\,\exp\left\{-\int_x^a \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_x^{-1}(r)))}\,\mathrm{d}r\right\} \\ &\quad + h(x)\,Z^{(q)}(x) + \int_x^a h'(t)\,Z^{(q)}(t)\,\exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_x^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t \\ &\quad - \int_x^a \frac{\gamma(\bar{\gamma}_x^{-1}(t))}{1-\gamma(\bar{\gamma}_x^{-1}(t))} h(t)\,qW^{(q)}(t)\,\exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_x^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t. \end{aligned} \tag{4.42}$$

Next, we use (2.7), (2.8) and (4.42) in (4.37) to find that

$$
\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\sigma_0^-}\, h(\overline{U}^{\gamma}_{\sigma_0^-})\,\mathbf{1}_{\{\sigma_0^-<\sigma_a^+\}}\right]\\
&\quad= h(\bar{x})\,\left[Z^{(q)}(x)-Z^{(q)}(\bar{x})\,\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\right]\\
&\qquad+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{-h(a)\,Z^{(q)}(a)\,\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\\
&\qquad\quad+h(\bar{x})\,Z^{(q)}(\bar{x})+\int_{\bar{x}}^{a}h'(t)\,Z^{(q)}(t)\,\exp\left\{-\int_{\bar{x}}^{t}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t\\
&\qquad\quad-\int_{\bar{x}}^{a}\frac{\gamma(\bar{\gamma}_{\bar{x}}^{-1}(t))}{1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(t))}h(t)\,qW^{(q)}(t)\,\exp\left\{-\int_{\bar{x}}^{t}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t\Bigg\}\\
&\quad= h(\bar{x})\,Z^{(q)}(x)+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{-h(a)\,Z^{(q)}(a)\,\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\\
&\qquad+\int_{\bar{x}}^{a}h'(t)\,Z^{(q)}(t)\,\exp\left\{-\int_{\bar{x}}^{t}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t\\
&\qquad-\int_{\bar{x}}^{a}\frac{\gamma(\bar{\gamma}_{\bar{x}}^{-1}(t))}{1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(t))}h(t)\,qW^{(q)}(t)\,\exp\left\{-\int_{\bar{x}}^{t}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_{\bar{x}}^{-1}(r)))}\,\mathrm{d}r\right\}\mathrm{d}t\Bigg\}.
\end{aligned}
\tag{4.43}
$$

The final step is to use Theorem 3.2.3 part (ii) and (4.43) in order to recover expression (4.31).

Before we move to the final example in this section, we recall that,

$$
\begin{aligned}
\mathcal{A}f(y) &= \mu f'(y)+\frac{\sigma^2}{2}f''(y)\\
&\quad+\int_{0+}^{\infty}\left[f(y-\theta)-f(y)+\theta f'(y)\mathbf{1}_{\{0<\theta\le 1\}}\right]\nu(\mathrm{d}\theta).
\end{aligned}
$$

We introduce the function $l:(0,\infty)\to\mathbb{R}$ by

$$
l(x)=(\mathcal{A}-q)f(x),\ \text{where } f(x)=x\mathbf{1}_{\{x\ge 0\}},
$$

then we can prove that,

$$
l'(x)=(\mathcal{A}-q)g(x),\ \text{where } g(x)=\mathbf{1}_{\{x\ge 0\}}. \tag{4.44}
$$

Indeed,

$$\begin{aligned}
l(x) &= (\mathcal{A} - q)f(x) \\
&= \mu \mathbf{1}_{\{x>0\}} + \int_{0^+}^{\infty} \left[(x-\theta)\,\mathbf{1}_{\{x-\theta\geq 0\}} - x\,\mathbf{1}_{\{x\geq 0\}} + \theta\,\mathbf{1}_{\{x>0\}}\mathbf{1}_{\{0<\theta\leq 1\}}\right] \nu(\mathrm{d}\theta) \\
&\quad - q\,x\mathbf{1}_{\{x\geq 0\}} \\
&= \mu + \int_{0^+}^{\infty} \left[(x-\theta)\,\mathbf{1}_{\{x-\theta\geq 0\}} - x + \theta\,\mathbf{1}_{\{0<\theta\leq 1\}}\right] \nu(\mathrm{d}\theta) - q\,x, \qquad (4.45)
\end{aligned}$$

where the last equality is true since $x > 0$.

Then by (4.45),

$$\begin{aligned}
l'(x) &= \int_{0^+}^{\infty} \left[\mathbf{1}_{\{x>\theta\}} - \mathbf{1}_{\{x>0\}}\right] \nu(\mathrm{d}\theta) - q\,\mathbf{1}_{\{x>0\}} \\
&= -\int_{x}^{\infty} \nu(\mathrm{d}\theta) - q \\
&= -\nu(x,\infty) - q, \qquad (4.46)
\end{aligned}$$

where we get the last equality because $x > 0$. Now, the right hand side of (4.44), for $x > 0$ is,

$$\begin{aligned}
(\mathcal{A} - q)g(x) &= \int_{0^+}^{\infty} \left[\mathbf{1}_{\{x\geq\theta\}} - \mathbf{1}_{\{x\geq 0\}}\right] \nu(\mathrm{d}\theta) - q\,\mathbf{1}_{\{x\geq 0\}} \\
&= -\int_{x}^{\infty} \nu(\mathrm{d}\theta) - q \\
&= -\nu(x,\infty) - q,
\end{aligned}$$

hence, (4.44) is satisfied. Also, for $c < 0$, let $f_c(z) = z\,\mathbf{1}_{\{z\geq c\}}$, and $g_c(z) = \mathbf{1}_{\{z\geq c\}}$, then for $z > c$,

$$\begin{aligned}
(\mathcal{A} - q)f_c(z) &= (\mathcal{A} - q)f_c(z) - c\,(\mathcal{A} - q)\,g_c(z) + c\,(\mathcal{A} - q)\,g_c(z) \\
&= (\mathcal{A} - q)\,[f_c - c\,g_c]\,(z) + c\,(\mathcal{A} - q)\,g_c(z) \\
&= l(z-c) + c\,l'(z-c),
\end{aligned}$$

where we use (4.45) and (4.46) such that the last line follows from spatial homogeneity of $\mathcal{A}$. That is, for $x \in \mathbb{R}$, where $f_x(z) = f(x+z)$, we have that $\mathcal{A}f_x(z) = \mathcal{A}f(x+z)$.

Remark 9 For $z \in \mathbb{R}$ such that $z > c$,

$$\begin{aligned}\lim_{-c\uparrow\infty}\left[l(z-c)+c\,l'(z-c)\right] &= \lim_{-c\uparrow\infty}\Big\{\mu-(z-c)\int_{z-c}^{\infty}\nu(\mathrm{d}\theta)-\int_{1}^{z-c}\theta\nu(\mathrm{d}\theta)-q(z-c)\\ &\qquad -c\int_{z-c}^{\infty}\theta\nu(\mathrm{d}\theta)-q\,c\Big\}\\ &= \lim_{-c\uparrow\infty}\Big\{\mu-z\int_{z-c}^{\infty}\nu(\mathrm{d}\theta)-\int_{1}^{z-c}\theta\nu(\mathrm{d}\theta)-q\,z\Big\}\\ &= \mu-\int_{1}^{\infty}\theta\nu(\mathrm{d}\theta)-q\,z\\ &= \psi'(0^{+})-q\,z. \end{aligned} \tag{4.47}$$

Lemma 4.2.3 *Suppose that $\psi'(0+) > -\infty$, then for $x, \bar{x} \in [0, a]$, the following expression holds for a natural tax process V^{δ}*

$$\begin{aligned}&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\;(-V^{\delta}_{\tau_0^-})\;\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\ &\quad= -\overline{Z}^{(q)}(x)+\psi'(0+)\,\overline{W}^{(q)}(x)\\ &\qquad+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{\left[\overline{Z}^{(q)}(a)-\psi'(0+)\,\overline{W}^{(q)}(a)\right]\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\\ &\qquad+\int_{\bar{x}}^{a}Z^{(q)}(s)\,\delta(s)\,l_{\bar{x}}(s)\,\mathrm{d}s\\ &\qquad-\psi'(0+)\int_{\bar{x}}^{a}W^{(q)}(s)\,\delta(s)\,l_{\bar{x}}(s)\,\mathrm{d}s\Bigg\},\end{aligned} \tag{4.48}$$

where $l_{\bar{x}}(s)$ is given by (4.19).

Proof Let $\tilde{f}(y,z) = f(y,z) = -y\,\mathbf{1}_{\{y\geq -n\}}$, for $n \geq 1$. Let $g(y) = y\,\mathbf{1}_{\{y\geq -n\}}$ and $h(z) = -1$. Then, clearly g and h satisfy the first and third conditions of Definition 4.1.1 on $[0, a]$. For the second condition, since g is a bounded function, then for any $\lambda > 0$ and all $s \in (0, a)$

$$\Big|\int_{\lambda}^{\infty} g(s-\theta)\;\nu(\mathrm{d}\theta)\Big| \leq a\int_{\lambda}^{\infty}\nu(\mathrm{d}\theta),$$

where the last integral is bounded (see [9, p.29]). So, $\tilde{f} \in \mathcal{S}_{[0,a]\times[0,a]}$, and therefore, we can substitute $\tilde{f}$ in (4.24) and get that

$$
\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}\left(-V^\delta_{\tau_0^-}\mathbf{1}_{\left\{V^\delta_{\tau_0^-}\geq -n\right\}}\right)\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\
&= -x\,\mathbf{1}_{\{x\geq -n\}} - \int_0^{\bar{x}}(\mathcal{A}-q)f_{-n}(y)\Big[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-y) - W^{(q)}(x-y)\Big]\mathrm{d}y\\
&\quad + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{-\int_{\bar{x}}^a\int_0^a(\mathcal{A}-q)f_{-n}(y)\left\{\Big[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\Big]\mathrm{d}y\right.\\
&\quad \left. + W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\} l_{\bar{x}}(z)\,\mathrm{d}z\\
&\quad + \int_{\bar{x}}^a l_{\bar{x}}(z)\,\delta(z)\,\mathrm{d}z + a\,\exp\left\{-\int_{\bar{x}}^a\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\Bigg\},
\end{aligned}
\tag{4.49}
$$

where

$$f_{-n}(x) := x\,\mathbf{1}_{\{x\geq -n\}},$$

$$g_{-n}(x) := \mathbf{1}_{\{x\geq -n\}},$$

and so for $x > -n$,

$$(\mathcal{A}-q)f_{-n}(x) = l(x+n) - n\,l'(x+n).$$

By Remark 9, with $c = -n$ and for $x \in [0,a]$, $n \geq 1$,

$$l(x+n) - n\,l'(x+n) = \mu - x\int_{x+n}^\infty \nu(\mathrm{d}\theta) - \int_1^{x+n}\theta\nu(\mathrm{d}\theta) - q\,x,$$

and hence

$$
\begin{aligned}
&\left|\mu - x\int_{x+n}^\infty \nu(\mathrm{d}\theta) - \int_1^{x+n}\theta\nu(\mathrm{d}\theta) - q\,x\right|\\
&\leq \mu + q\,x + x\int_{x+n}^\infty \nu(\mathrm{d}\theta) + \int_1^{x+n}\theta\nu(\mathrm{d}\theta) + x\int_1^{x+n}\nu(\mathrm{d}\theta) - x\int_1^{x+n}\nu(\mathrm{d}\theta)\\
&= \mu + q\,x + x\int_1^\infty \nu(\mathrm{d}\theta) - \int_1^{x+n}(x-\theta)\nu(\mathrm{d}\theta)\\
&\leq \mu + q\,x + x\int_1^\infty \nu(\mathrm{d}\theta) - \int_1^{x+1}(x-\theta)\nu(\mathrm{d}\theta),
\end{aligned}
\tag{4.50}
$$

where $g(x) := \mu + q\,x + x\int_1^\infty \nu(\mathrm{d}\theta) - \int_1^{x+1}(x-\theta)\nu(\mathrm{d}\theta)$ is bounded on $[0,a]$. Take limit as $n \uparrow \infty$, and use (4.47) with $c = -n$, and (4.50), then by monotone convergence theorem on the LHS of (4.49), and the dominated convergence theorem on RHS of

(4.49), we get

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_0^-}(-V^{\delta}_{\tau_0^-})\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right] \\
&= -x - \int_0^{\bar{x}} (\psi'(0+) - qy)\Big[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-y) - W^{(q)}(x-y)\Big]\mathrm{d}y \\
&\quad + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Bigg\{-\int_{\bar{x}}^a\int_0^a(\psi'(0+) - qy)\left\{\Big[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\Big]\mathrm{d}y\right. \\
&\quad \left. + W^{(q)}(0+)\,\delta_z(\mathrm{d}y)\right\} l_{\bar{x}}(z)\,\mathrm{d}z \\
&\quad + \int_{\bar{x}}^a l_{\bar{x}}(z)\,\delta(z)\,\mathrm{d}z + a\,\exp\left\{-\int_{\bar{x}}^a\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\Bigg\}. \qquad (4.51)
\end{aligned}$$

Use that

$$\int_0^{\bar{x}} y\,W^{(q)}(\bar{x}-y)\,\mathrm{d}y = \int_0^{\bar{x}}\int_0^Y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}Y. \qquad (4.52)$$

where this is true by making first the change of variable $Y = \bar{x} - y$ and then use integral by parts. So by (4.52), the first integral term in (4.51) becomes

$$\begin{aligned}
&-\int_0^{\bar{x}}(\psi'(0+) - qy)\Big[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-y) - W^{(q)}(x-y)\Big]\mathrm{d}y \\
&= -\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\psi'(0+)\int_0^{\bar{x}}W^{(q)}(y)\,\mathrm{d}y + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}q\int_0^{\bar{x}}\int_0^Y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}Y \\
&\quad + \psi'(0+)\int_0^x W^{(q)}(s)\,\mathrm{d}s - q\int_0^x\int_0^Y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}Y, \qquad (4.53)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^a(\psi'(0+) - qy)\Big[W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}W^{(q)}(z-y)\Big]\mathrm{d}y \\
&\quad + \int_0^a W^{(q)}(0+)(\psi'(0+) - qy)\,\delta_z(\mathrm{d}y) \\
&= \psi'(0+)\,W^{(q)}(z) - \psi'(0+)\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\int_0^z W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_0^z W^{(q)}(y)\,\mathrm{d}y + \frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\,q\int_0^z\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y. \qquad (4.54)
\end{aligned}$$

Then, we use (4.54) to find the second integral term in (4.51)

$$
\begin{aligned}
&-\int_{\bar{x}}^{a} l_{\bar{x}}(z)\Big\{\psi'(0+)\,W^{(q)}(z)-\psi'(0+)\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\int_0^z W^{(q)}(y)\,\mathrm{d}y\\
&-q\int_0^z W^{(q)}(y)\,\mathrm{d}y+\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}q\int_0^z\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y\Big\}\mathrm{d}z\\
&=-\int_{\bar{x}}^{a}\frac{1}{(1-\delta(z))}\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\,\Big\{\psi'(0+)\,W^{(q)}(z)\\
&-\psi'(0+)\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\int_0^z W^{(q)}(y)\,\mathrm{d}y\\
&-q\int_0^z W^{(q)}(y)\,\mathrm{d}y+\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}q\int_0^z\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y\Big\}\mathrm{d}z. \qquad (4.55)
\end{aligned}
$$

Use integration by parts twice in (4.55), the first time with $u=q\int_0^z\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y$ and $\mathrm{d}v=-\frac{1}{(1-\delta(z))}\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}$ in the integral

$$
-\int_{\bar{x}}^{a}\frac{1}{(1-\delta(z))}\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}q\Big(\int_0^z\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y\Big)\mathrm{d}z,
$$

and the second time with $u=\int_0^z W^{(q)}(s)\,\mathrm{d}s$ and

$$
\mathrm{d}v=\frac{1}{(1-\delta(z))}\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}
$$

in the integral

$$
\int_{\bar{x}}^{a}\frac{1}{(1-\delta(z))}\,\exp\left\{-\int_{\bar{x}}^{z}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\,\frac{W^{(q)\prime}(z)}{W^{(q)}(z)}\Big(\int_0^z W^{(q)}(s)\,\mathrm{d}s\Big)\,\mathrm{d}z.
$$

With some calculations, (4.55) equals

$$
\begin{aligned}
&\int_{\bar{x}}^{a}\delta(z)\,l_{\bar{x}}(z)\,\Big(q\int_0^z W^{(q)}(y)\,\mathrm{d}y\Big)\,\mathrm{d}z\\
&\quad+q\Big(\int_0^a\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y\Big)\,\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\\
&\quad-q\int_0^{\bar{x}}\int_0^y W^{(q)}(s)\,\mathrm{d}s\,\mathrm{d}y\\
&\quad-\psi'(0+)\int_{\bar{x}}^{a}W^{(q)}(z)\,\delta(z)\,l_{\bar{x}}(z)\,\mathrm{d}z\\
&\quad-\psi'(0+)\,\Big(\int_0^a W^{(q)}(y)\,\mathrm{d}y\Big)\,\exp\left\{-\int_{\bar{x}}^{a}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta(r))}\mathrm{d}r\right\}\\
&\quad+\psi'(0+)\int_0^{\bar{x}}W^{(q)}(y)\,\mathrm{d}y. \qquad (4.56)
\end{aligned}
$$

Now, substitute (4.53) and (4.56) in (4.51), do some easy computations, then by (4.25) and (4.26) we find the required representation (4.48).$\square$

Remark 10 Note that, similar to Example 4.2.2, we can derive (4.48) in a different way, by using [60, Equation 9] and find that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\tau_0^-}\,(-U^\gamma_{\tau_0^-})\,\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right]\\
&=\int_x^a \exp\left\{-\int_x^t \frac{W^{(q)\prime}(r)}{W^{(q)}(r)\,(1-\gamma(\bar{\gamma}_x^{-1}(r)))}\,\mathrm{d}r\right\}\\
&\qquad\times\frac{1}{1-\gamma(\bar{\gamma}_x(t))}\left[Z^{(q)}(t)-\psi'(0+)\,W^{(q)}(t)-\frac{\overline{Z}^{(q)}(t)-\psi'(0+)\,\overline{W}^{(q)}(t)}{W^{(q)}(t)}\,W^{(q)\prime}(t)\right]\mathrm{d}t.
\end{aligned}\tag{4.57}$$

Again, if we use integration by parts in (4.57) and Theorem (3.2.3) part (ii), then we can recover expression (4.48).

4.3 The approach

In the tax processes literature, different arguments have been used to obtain the fluctuation identities of interest. Mainly, there were two approaches. The first one is deriving an ODE in x for the required quantity. This has been studied in some articles using different methods. For example, in [6] and [53], authors looked at the effect of a small increment above the initial point and derived the desired ODE through a discrete approximation. In [2], an integro-differential equation was derived, while [3] used a probability argument. The other way is using excursion theory, as in deriving the solution of the two sided exit problem in [6] and as in proving all of the main results in [33] and [32]. In this section, we explain our approach to obtain the expression of some required identity in the model, by deducing an ODE from some available PDE, which will be explained below through an example. This approach was motivated by our work in Chapter 6, while we were trying to find the net present value of taxation when there are capital injections in the model. At the beginning, we used identity (4.24). Afterwards, we realised that we can not verify the required conditions in Theorem 4.1.4 to use that identity before applying it. Therefore, we had to find another approach. However, in a very recent article, [64], authors used the draw-down literature for providing similar examples for applying (4.24) for constant tax rate, and we extended their method for general tax function (see Remark 8). According to authors work in [64], we recognised that it is possible to check explicitly the necessary conditions in order to apply their identities and provide an alternative way to derive similar examples to ours.

We give now an example to illustrate the steps in our approach, which will be used in Chapters 6 and 7. Suppose we need to obtain an expression for

$$f(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right].$$

First, by the strong Markov property of $(V^\delta, \overline{V}^\delta)$ we have that

$$\left(e^{-q(t\wedge\tau_0^-\wedge\tau_a^+)}\, f(V^\delta_{t\wedge\tau_0^-\wedge\tau_a^+}, \overline{V}^\delta_{t\wedge\tau_0^-\wedge\tau_a^+})\right)_{t\geq 0}$$

is a martingale and

$$f(x,\bar{x}) = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\, f(\bar{x},\bar{x}). \tag{4.58}$$

Our aim now is to derive a first order linear ODE for $f(\bar{x},\bar{x})$. The first step is by assuming that f is smooth enough so that we can apply Itô's formula or a generalisation of it and have that:

$$\begin{aligned}
&e^{-q(t\wedge\tau_0^-\wedge\tau_a^+)} f(V^\delta_{t\wedge\tau_0^-\wedge\tau_a^+}, \overline{V}^\delta_{t\wedge\tau_0^-\wedge\tau_a^+}) - f(x,\bar{x}) \\
&\qquad = M_t + \int_{0+}^{t\wedge\tau_0^-\wedge\tau_a^+} e^{-qs}(\mathcal{A}-q) f(V^\delta_{s-}, \overline{V}^\delta_s)\mathrm{d}s \\
&\qquad\quad + \int_{0+}^{t\wedge\tau_0^-\wedge\tau_a^+} e^{-qs}\left[\frac{\partial f}{\partial \bar{x}}(V^\delta_{s-}, \overline{V}^\delta_s)(1-\delta(\overline{V}^\delta_s)) - \frac{\partial f}{\partial x}(V^\delta_{s-}, \overline{V}^\delta_s)\delta(\overline{V}^\delta_s)\right]\mathrm{d}\overline{X}_s,
\end{aligned}$$

where M_t represents some martingale. This suggests that

$$(\mathcal{A}-q)f(x,\bar{x}) = 0 \quad , 0 \leq x \leq \bar{x},$$

and

$$\frac{\partial f}{\partial \bar{x}}(x,\bar{x})(1-\delta(\bar{x})) - \frac{\partial f}{\partial x}(x,\bar{x})\delta(\bar{x}) = 0 \text{ when } x = \bar{x}. \tag{4.59}$$

By (4.58), we get to the second step in the approach, where the PDE (4.59) leads to a first order linear ODE in $f(\bar{x},\bar{x})$

$$\frac{\partial f}{\partial \bar{x}}(\bar{x},\bar{x}) - \frac{1}{1-\delta(\bar{x})}\frac{W^{(q)\prime}(\bar{x})}{W^{(q)}(\bar{x})}\, f(\bar{x},\bar{x}) = 0, \tag{4.60}$$

with the boundary condition

$$f(a,a) = 1.$$

Then by solving this ODE, we get the solution

$$f(\bar{x},\bar{x}) = \exp\left\{-\int_{\bar{x}}^{a} \frac{W^{(q)\prime}(y)}{W^{(q)}(y)(1-\delta(y))}\,\mathrm{d}y\right\}. \tag{4.61}$$

The third step is to prove that (4.61) is the correct solution of the ODE (4.60). In order to do so, let $\hat{f}(\bar{x},\bar{x})$ be the solution of the ODE (4.60), and for $0 \le x \le \bar{x}$ form

$$\hat{f}(x,\bar{x}) = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\hat{f}(\bar{x},\bar{x}) = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\exp\left\{-\int_{\bar{x}}^{a} \frac{W^{(q)\prime}(y)}{W^{(q)}(y)(1-\delta(y))}\,\mathrm{d}y\right\}.$$

Then one can verify that $\hat{f}(x,\bar{x})$ is smooth, $(\mathcal{A}-q)\hat{f}(x,\bar{x}) = 0$ for $0 \le x \le \bar{x}$, and $\hat{f}(x,\bar{x})$ satisfies the PDE (4.59). Then by an application of Itô's formula, we can prove that:

$$\hat{f}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right] = f(x,\bar{x}).$$

We can notice from the previous example that our approach gives a direct way of deriving the relevant ODE of the required identity. Moreover, as you will see in the next chapters, the proof involves similar steps to solving optimal taxation problems.

Chapter 5

Optimal taxation for natural tax processes

5.1 Introduction

From a tax authority point of view, it is natural to ask what is the maximum expected tax revenue they can have, and what tax policy that achieves this. Authors in [2] addressed this problem in the context of loss-carry-forward taxation in the setting of a classical Cramér-Lundberg process with a constant tax rate. In their work, optimal implementation delay of taxation was studied, which is the problem of finding the optimal threshold surplus level for starting taxation to maximise the expected accumulated (discounted) tax payments. After that, in [6], the same problem is generalised into the setting of a spectrally negative Lévy process. In [57], authors studied an optimal control problem for the latent tax process U^γ, given by

$$\sup_{\gamma\in\Pi}\mathbb{E}_x\left[\int_0^{\sigma_0^-} e^{-qt}\gamma(\overline{X}_t)\,\mathrm{d}\overline{X}_t\right], \tag{5.1}$$

where $\sigma_0^- = \inf\{t \geq 0 : U_t^\gamma < 0\}$, $q > 0$ is a discount factor and Π is the set of measurable functions $\gamma\colon [0,\infty) \to [\alpha,\beta]$, where $0 \leq \alpha \leq \beta < 1$ are fixed. It is noticeable that the optimal control problem addressed in both of [2] and [6] are contained in the one considered in [57]. The motivation for this chapter was to give another proof for the solution of the optimal control problem considered in [57], using the natural tax process definition and results that we introduced in Chapter 3. In fact, in this chapter, we generalise the work in [57] by defining an optimal control problem with a larger class of admissible strategies than the one considered in (5.1). We solve a general optimal tax control problem such that it contains (5.1). To find the solution of our optimal control problem, we adapted the method used for solving optimal dividends problems such as [44], [35] and [45]. This method is described simply as, calculating the tax value function of a barrier tax strategy for an arbitrary barrier, then trying to maximise this value and choosing the optimal (maximiser) barrier, and at the end putting this specific choice through a verification lemma in order to prove its optimality in the general class of predictable tax rates.

We define our optimal control problem in this chapter as the following. For a spectrally negative Lévy process X, consider the aggregate surplus process of an insurance company to be given by

$$V_t^H = X_t - \int_{0+}^{t} H_s \,\mathrm{d}\overline{X}_s, \tag{5.2}$$

where H is a left continuous tax rate process, which is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and with values in $[0,1)$. Let $X_0 = x$ and $\overline{X}_0 = \bar{x}$, then we define the value function starting at any $x, \bar{x}$ such that $0 \leq x \leq \bar{x}$ as

$$v^H(x,\bar{x}) := \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qs} H_s \,\mathrm{d}\overline{X}_s\right], \tag{5.3}$$

where $\tau_0^- = \inf\{t \geq 0 : V_t^H < 0\}$. Let Π be the set of all admissible policies, that is, the set of all left continuous and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ processes, $(H_t)_{t\geq 0}$, such that $0 \leq \alpha \leq H_t \leq \beta < 1$ for any $t \geq 0$, then we introduce the optimal control problem

$$v^*(x,\bar{x}) = \sup_{H\in\Pi} v^H(x,\bar{x}), \tag{5.4}$$

where an optimal tax rate policy $H^* \in \Pi$ is such that $v^*(x,\bar{x}) = v^{H^*}(x,\bar{x})$, for all $0 \leq x \leq \bar{x}$.

It is clear that, the admissible set of strategies Π in (5.4) is much larger than the one considered in (5.1). Furthermore, we point out here to the relation between the work in [57], and our work, which is taken from our published paper [1], as follows. Denote by γ^* the function $\gamma \in \Pi$ which maximises (5.1), if it exists. In [57, Theorem 3.1], Wang and Hu state that γ^* should satisfy the equation

$$\gamma^*(\overline{X}_t) = \eta\left(x + \int_x^{\overline{X}_t} (1-\gamma^*(y))\,\mathrm{d}y\right) = \eta(\overline{U}_t^{\gamma^*}),$$

for some function η which they call the *optimal decision rule*. On the other hand, let δ be a function satisfying the assumptions of Theorem 3.2.3ii with $x = \bar{x}$, and define γ_x^δ as in that result. If we write $\xi = \gamma_x^\delta$, then, by the definition of γ_x^δ together with (3.10) and (3.11), we have the relation

$$\xi(\overline{X}_t) = \delta\left(x + \int_x^{\overline{X}_t} (1 - \xi(y))\,\mathrm{d}y\right) = \delta(\overline{U}_t^\xi).$$

It follows from Theorem 3.2.3, that the relationship between Wang and Hu's optimal decision rule η and optimal tax rate γ^*, is nothing other than the relationship between a particular natural tax rate δ and the equivalent latent tax rate γ_x^δ. Our results clarify that this connection is a sensible one even outside of the optimal control context, and make clear under exactly which conditions this connection is valid. Wang and Hu go on to show that η must be piecewise constant, and in particular $\eta = f^b$, as defined in (3.13), where b is specified in terms of scale functions of the Lévy process but is independent of x; see section 4 and equation (5.15) in their work (in which b is denoted u_0). Combining this with our result, we see that Wang and Hu's solution of the optimal control problem (5.1) is actually a tax process with the piecewise constant natural tax rate f^b, or equivalently the piecewise constant latent tax rate $f^{\tilde{b}(x)}$, where $\tilde{b}(x)$ depends on x as in (3.2.4). Moreover, our work shows this optimality directly without going through the latent tax process.

Before we give an overview for the current chapter, we recall some definitions as given in [45, p. 4]. The tail of the Lévy measure is the function $x \mapsto \nu(x, \infty)$, where $x \in (0, \infty)$, and we say that a function $f\colon (0, \infty) \to (0, \infty)$ is log-convex if $\log \circ f$ is convex on $(0, \infty)$.

Our main result is Theorem 5.2.7, which states that, under the assumption that the tail of the Lévy measure is log-convex, the solution of the optimal control problem (5.4) is piecewise constant, and characterises the switching point.

The structure of this chapter is presented as follows. In Section 5.2, we find the tax value function for a piecewise constant natural tax rate. We state and prove the verification lemma that we need to verify, at the end of the section, optimality for the solution of (5.4). Comparing our method in the proof with the one given in [57, Proposition 2.1], the latter proof depends on deriving a Hamilton-Jacobi-Bellman equation, which is a first order ODE, that is satisfied by the optimal tax value function, while our proof starts with an Itô expansion for a tax value function that satisfies some conditions, and continue by using these assumptions to prove that this value function agrees with (5.4). Moreover, to reach the final stage of the proof of optimality, we need to prove some lemmas under the condition that the tail of the Lévy measure is log-convex. Note that, the assumption in [57] was that each scale function is three times differentiable and its first derivative is a strictly convex function, which is fulfilled if the tail of the Lévy measure is log-convex. In Section 5.3, we explain the relation between our solution for (5.4) in the case $x = \bar{x}$ and the one for (5.1) in [57].

5.2 Optimal control problem

We start this section by defining a piecewise constant natural tax rate function δ^b, which we can call in other words an $(\alpha \longmapsto \beta)$-tax strategy at level b:

$$\delta^b(z) = \begin{cases} \alpha, & z \leq b, \\ \beta, & z > b, \end{cases} \tag{5.5}$$

where $b \geq x = X_0$ and $0 \leq \alpha \leq \beta < 1$. In the next result, we find the corresponding tax value function v^{δ^b}.

Proposition 5.2.1 *Suppose we have the natural tax process*

$$V_t^{\delta^b} = X_t - \int_{0+}^{t} \delta^b(\overline{V}_r^{\delta^b}) \mathrm{d}\bar{X}_r, \quad V_0^{\delta^b} = x, \ \overline{V}_0^{\delta^b} = \bar{x}.$$

Then, for any $0 \leq x \leq \bar{x}$ and $q \geq 0$, we have

$$\begin{aligned} & v^{\delta^b}(x, \bar{x}) \\ & \quad = \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr} \delta^b(\overline{V}_r^{\delta^b}) \mathrm{d}\bar{X}_r\right] \\ & \quad = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})} \begin{cases} v_{\alpha,q}(\bar{x}) - \left[\dfrac{W^{(q)}(\bar{x})}{W^{(q)}(b)}\right]^{1/(1-\alpha)} [v_{\alpha,q}(b) - v_{\beta,q}(b)] & \text{if } 0 \leq \bar{x} \leq b \\ v_{\beta,q}(\bar{x}) & \text{if } \bar{x} > b, \end{cases} \end{aligned} \tag{5.6}$$

where for any $0 \leq \gamma < 1$ and $x \geq 0$,

$$v_{\gamma,q}(x) := \frac{\gamma}{1-\gamma} \int_x^{\infty} \left(\frac{W^{(q)}(x)}{W^{(q)}(s)}\right)^{1/(1-\gamma)} \mathrm{d}s. \tag{5.7}$$

PROOF Since δ^b is an increasing measurable function, then by Example 3.3.1 we know that ODE (3.4) with respect to this given function δ^b has a unique solution. Moreover, it is clear from (3.4) and as $\delta^b \in [\alpha, \beta]$ that $y_{\bar{x}}^{\delta^b}(\infty) = \infty$. Thus, we can use now Corollary 3.2.7 to get

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qr}\delta(\overline{V}_r^{\delta^b})\mathrm{d}\overline{X}_r\right] \\
&\quad = \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_{\bar{x}}^{\infty}\exp\left\{-\int_{\bar{x}}^{y}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta^b(r))}\mathrm{d}r\right\}\frac{\delta^b(y)}{1-\delta^b(y)}\mathrm{d}y.
\end{aligned}$$

For the case, $0 \leq \bar{x} \leq b$, we have

$$\begin{aligned}
&\int_{\bar{x}}^{\infty}\exp\left\{-\int_{\bar{x}}^{y}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta^b(r))}\mathrm{d}r\right\}\frac{\delta^b(y)}{1-\delta^b(y)}\mathrm{d}y \\
&= \frac{\alpha}{1-\alpha}\int_{\bar{x}}^{b}\exp\left\{-\frac{1}{1-\alpha}\int_{\bar{x}}^{y}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)}\mathrm{d}r\right\}\mathrm{d}r \\
&+ \frac{\beta}{1-\beta}\int_{b}^{\infty}\exp\left\{-\left[\frac{1}{1-\alpha}\int_{\bar{x}}^{b}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)}\mathrm{d}r + \frac{1}{1-\beta}\int_{b}^{y}\frac{W^{(q)\prime}(r)}{W^{(q)}(r)}\mathrm{d}r\right]\right\}\mathrm{d}y \\
&= \frac{\alpha}{1-\alpha}\int_{\bar{x}}^{b}\left[\frac{W^{(q)}(\bar{x})}{W^{(q)}(y)}\right]^{1/(1-\alpha)}\mathrm{d}y + \frac{\beta}{1-\beta}\left[\frac{W^{(q)}(\bar{x})}{W^{(q)}(b)}\right]^{1/(1-\alpha)}\int_{b}^{\infty}\left[\frac{W^{(q)}(b)}{W^{(q)}(y)}\right]^{1/(1-\beta)}\mathrm{d}y \\
&= v_{\alpha,q}(\bar{x}) - \left[\frac{W^{(q)}(\bar{x})}{W^{(q)}(b)}\right]^{1/(1-\alpha)}\left[v_{\alpha,q}(b) - v_{\beta,q}(b)\right],
\end{aligned}$$

where to get the last equality, we add and subtract

$$\frac{\alpha}{1-\alpha}\int_{b}^{\infty}\left[\frac{W^{(q)}(\bar{x})}{W^{(q)}(y)}\right]^{1/(1-\alpha)}\mathrm{d}y,$$

and use (5.7).

For the case, $\bar{x} > b$,

$$\int_{\bar{x}}^{\infty} \exp\left\{-\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)(1-\delta^b(r))}\mathrm{d}r\right\} \frac{\delta^b(y)}{1-\delta^b(y)}\mathrm{d}y$$

$$= \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty} \exp\left\{-\frac{1}{1-\beta}\int_{\bar{x}}^{y} \frac{W^{(q)\prime}(r)}{W^{(q)}(r)}\mathrm{d}r\right\}\mathrm{d}y$$

$$= \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty} \exp\left\{\log\left[\frac{W^{(q)}(y)}{W^{(q)}(\bar{x})}\right]^{-1/(1-\beta)}\right\}\mathrm{d}y$$

$$= \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty} \left[\frac{W^{(q)}(\bar{x})}{W^{(q)}(y)}\right]^{1/(1-\beta)}\mathrm{d}y$$

$$= v_{\beta,q}(\bar{x}).$$

Hence, the statement is proved.□

We can rewrite (5.6) in the following way

$$v^{\delta^b}(x,\bar{x}) = W^{(q)}(x)[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left[\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s + C(b\vee\bar{x})\right], \tag{5.8}$$

where

$$\begin{aligned} C(b) = {} & \frac{\beta}{1-\beta}[W^{(q)}(b)]^{1/(1-\beta)-1/(1-\alpha)}\int_{b}^{\infty}[W^{(q)}(s)]^{-1/(1-\beta)}\mathrm{d}s \\ & -\frac{\alpha}{1-\alpha}\int_{b}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s. \end{aligned} \tag{5.9}$$

The next lemma gives the conditions we need for having the solution of the optimal control problem (5.4).

Lemma 5.2.2 (The verification lemma) *Let V^H be the tax process given in* (5.2) *for any H. Let $w(x,\bar{x}) := v^{\hat{H}}(x,\bar{x})$ be the value function defined in* (5.3)*, where $\hat{H}$ is an admissible policy. Suppose that $w(x,\bar{x}) \in \mathcal{S}_{[1/n,n]\times[1/n,n]}$ for each $n \in \mathbb{N}$, $x \mapsto w(x,\bar{x})$ is right-continuous at $x = 0$ for every $\bar{x} > 0$, $x \mapsto w(x,x)$ is right-continuous at $x = 0$, and w satisfies the following conditions for all $0 < x \le \bar{x}$:*

(I) $(\mathcal{A}-q)w(x,\bar{x}) = 0$,

(II) There exist Radon-Nikodým densities $\dfrac{\partial w}{\partial x}$ *and* $\dfrac{\partial w}{\partial \bar{x}}$ *such that*

$$H\frac{\partial w}{\partial x}(\bar{x},\bar{x}) + (H-1)\frac{\partial w}{\partial \bar{x}}(\bar{x},\bar{x}) - H \geq 0, \quad \textit{for all} \quad H \in [\alpha,\beta],$$

then $w(x,\bar{x}) = v^*(x,\bar{x})$ *for all* $x \leq \bar{x}$.

Proof We start the proof with the case, $0 < x \leq \bar{x}$. By Lemma 3.2.1, we have

$$\overline{V}^H_t = \overline{X}_t - \int_{0^+}^t H_s \,\mathrm{d}\overline{X}_s.$$

We recall the notations: $\Delta X_s = X_s - X_{s-}$ and $\Delta w(V^H_s, \overline{V}^H_s) = w(V^H_s, \overline{V}^H_s) - w(V^H_{s-}, \overline{V}^H_s)$. Thus, clearly we find that

$$\begin{aligned}
&\sum_{0<s\leq t} e^{-qs}\left[\Delta w(V^H_s,\overline{V}^H_s) - \frac{\partial w}{\partial x}(V^H_{s-},\overline{V}^H_s)\Delta X_s\right] \\
&\quad = \sum_{0<s\leq t} e^{-qs}\left[\Delta w(V^H_{s-}+\Delta X_s,\overline{V}^H_s) - \frac{\partial w}{\partial x}(V^H_{s-},\overline{V}^H_s)\Delta X_s\right].
\end{aligned} \tag{5.10}$$

Let $\tau^-_{1/n} = \inf\{t > 0 : V^H_t < 1/n\}$, $\tau^+_n = \inf\{t > 0 : V^H_t > n\}$, $\kappa^-_{1/n} = \inf\{t > 0 : \overline{V}^H_t < 1/n\}$, and $\kappa^+_n = \inf\{t > 0 : \overline{V}^H_t > n\}$. This implies that, $\kappa^-_{1/n} = \infty$ as $\overline{V}^H$ is increasing and we assumed starting with case where $0 < x \leq \bar{x}$. Also, clearly $\kappa^+_n = \tau^+_n$. Let $T_n = \tau^-_{1/n} \wedge \tau^+_n$. Since V^H and $\overline{V}^H$ are semi-martingales, $\overline{V}^H$ is a continuous process of bounded variation and $w \in \mathcal{S}_{[1/n,n]\times[1/n,n]}$, then we can use Corollary 4.1.3 to get the expansion

$$\begin{aligned}
&e^{-q(t\wedge T_n)} w(V^H_{t\wedge T_n}, \overline{V}^H_{t\wedge T}) - w(x,\bar{x}) \\
&= \int_{0^+}^{t\wedge T_n} -qe^{-qs}\, w(V^H_{s-},\overline{V}^H_s)\,\mathrm{d}s + \int_{0^+}^{t\wedge T_n} e^{-qs}\,\frac{\partial w}{\partial x}(V^H_{s-},\overline{V}^H_s)\,\mathrm{d}V^H_s \\
&\quad + \int_{0^+}^{t\wedge T_n} e^{-qs}\,\frac{\partial w}{\partial \bar{x}}(V^H_{s-},\overline{V}^H_s)\,\mathrm{d}\overline{V}^H_s + \frac{1}{2}\int_{0^+}^{t\wedge T_n} e^{-qs}\,\frac{\partial^2 w}{\partial x^2}(V^H_{s-},\overline{V}^H_s)\,\mathrm{d}\left[V^H,V^H\right]^c_s \\
&\quad + \sum_{0<s\leq t\wedge T_n} e^{-qs}\left[\Delta w(V^H_s,\overline{V}^H_s) - \frac{\partial w}{\partial x}(V^H_{s-},\overline{V}^H_s)\,\Delta V^H_s\right].
\end{aligned} \tag{5.11}$$

Use that $\left[V^H, V^H\right]_s^c = \sigma^2 s$, $\Delta V_s^H = \Delta X_s$, and by (5.10), with some calculations, we can rewrite (5.11) as,

$$
\begin{aligned}
& w(x,\bar{x}) \\
&= e^{-q(t\wedge T_n)} w(V^H_{t\wedge T}, \overline{V}^H_{t\wedge T_n}) - \int_{0+}^{t\wedge T_n} e^{-qs}\,(\mathcal{A}-q) w(V^H_{s-}, \overline{V}^H_s)\,\mathrm{d}s \\
&\quad - \int_{0+}^{t\wedge T_n} e^{-qs}\,\frac{\partial w}{\partial x}(V^H_{s-}, \overline{V}^H_s)\,\mathrm{d}\left[X_s - \mu s - \sum_{0<u\le s} \Delta X_u \mathbf{1}_{\{|\Delta X_u|>1\}}\right] \\
&\quad - \Bigg\{ \sum_{0<s\le t\wedge T_n} e^{-qs}\left[w(V^H_{s^-} + \Delta X_s, \overline{V}^H_s) - w(V^H_{s^-}, \overline{V}^H_s) - \frac{\partial w}{\partial x}(V^H_{s-}, \overline{V}^H_s)\,\Delta X_s\,\mathbf{1}_{\{|\Delta X_s|\le 1\}}\right] \\
&\quad - \int_{0+}^{t\wedge T_n}\int_{0+}^{\infty} e^{-qs}\left[w(V^H_{s-} - \theta, \overline{V}^H_s) - w(V^H_{s-}, \overline{V}^H_s) + \theta\,\frac{\partial w}{\partial x}(V^H_{s-}, \overline{V}^H_s)\,\mathbf{1}_{\{0<\theta\le 1\}}\right]\nu(\mathrm{d}\theta)\,\mathrm{d}s \Bigg\} \\
&\quad + \int_{0+}^{t\wedge T_n} e^{-qs}\left[\frac{\partial w}{\partial x}(V^H_{s-}, \overline{V}^H_s)\,H - \frac{\partial w}{\partial \bar{x}}(V^H_{s-}, \overline{V}^H_s)\,(1-H)\right]\mathrm{d}\overline{X}_s. \qquad (5.12)
\end{aligned}
$$

On the right hand side of (5.12), by the Lévy-Itô decomposition (2.3), the second integral is a zero-mean martingale, and by Theorem 2.1.5, the expression between the curly-brackets is also a zero-mean martingale. Let $M_{t\wedge T_n}$ represents the sum of the two martingales, which is also a zero-mean martingale. In the last integral, since the integrand is only counted when $X = \overline{X}$, and by Lemma 3.2.1, this happens at the same times as V^H and $\overline{V}^H$ are equal, then we can use condition (II). Since $w(x,\bar{x}) \ge 0$ for any $(x,\bar{x})$ and by conditions (I) and (II), we get

$$
w(x,\bar{x}) \ge \int_{0+}^{t\wedge T_n} e^{-qs}\,H_s\,\mathrm{d}\overline{X}_s + M_{t\wedge T_n}.
$$

Take expectation, let t and n go to infinity, and since the tax revenue is monotone in t, then we can use the monotone convergence theorem to find that

$$
w(x,\bar{x}) \ge E_{x,\bar{x}}\left[\int_{0+}^{\tau_0^-} e^{-qs}\,H_s\,\mathrm{d}\overline{X}_s\right],
$$

for any admissible strategy H. Hence $w(x,\bar{x}) \ge v^*(x,\bar{x})$, for all $0 < x \le \bar{x}$. Since from the definition of v^*, $w(x,\bar{x}) \le v^*(x,\bar{x})$, therefore, we get that $w(x,\bar{x}) = v^*(x,\bar{x})$ for all $0 < x \le \bar{x}$.

For the case $x = 0$ and $\bar{x} > 0$, since v^* is an increasing function in the initial capital, we have the following:

$$
v^*(0,\bar{x}) \le v^*(x,\bar{x}) \le w(x,\bar{x}),
$$

then by taking $\lim_{x\downarrow 0}$ and as w is right-continuous by the assumptions we have that

$$v^*(0,\bar{x}) \leq w(0,\bar{x}).$$

For the case $(0,0)$, in the unbounded variation case the process get ruined immediately and thus $v^*(0,0)=0\leq w(0,0)$ as $w\geq 0$. In the bounded variation case, for some $\varepsilon>0$ and any admissible strategy H at $(0,0)$, that is under $\mathbb{P}_{0,0}$, there is an admissible strategy $\tilde{H}$ at $(\varepsilon,\varepsilon)$, that is under $\mathbb{P}_{\varepsilon,\varepsilon}$, such that V^H under $\mathbb{P}_{0,0}$ has the same law as $V^{\tilde{H}}-\varepsilon$ under $\mathbb{P}_{\varepsilon,\varepsilon}$. In other words, with H at $(0,0)$ and $\tilde{H}$ at $(\varepsilon,\varepsilon)$, taxes are collected at the same rate and at the same times. Since, $\varepsilon>0$, when starting at $(\varepsilon,\varepsilon)$, ruin happens later and therefore we get the inequality

$$\sup_{H\in\Pi} v^H(0,0) \leq \sup_{\tilde{H}\in\Pi} v^{\tilde{H}}(\varepsilon,\varepsilon),$$

that is,

$$v^*(0,0)\leq v^*(\varepsilon,\varepsilon),$$

which justifies the first inequality in the following

$$v^*(0,0)\leq \lim_{\varepsilon\downarrow 0} v^*(\varepsilon,\varepsilon) \leq \lim_{\varepsilon\downarrow 0} w(\varepsilon,\varepsilon) = w(0,0),$$

where the second inequality follows from the first part of the proof and the last equality from the right-continuity of w. Hence, the proof is complete.□

We recall Theorem 1.2 in [45], which we will use to find the solution of the optimal control problem (5.4).

Theorem 5.2.3 *Suppose the tail of the Lévy measure is log-convex. Then, for all* $q\geq 0$, $W^{(q)}$ *has a log-convex first derivative.*

In order to prove the optimality for our solution of (5.4), we need the following results.

Lemma 5.2.4 *Let*

$$a^* = \sup\left\{a\geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x\geq 0\right\},$$

and for any $x>0$

$$Q(x) = [W^{(q)}(x)]^{-1/(1-\alpha)}\frac{W^{(q)}(x)}{W^{(q)\prime}(x)} - \frac{\alpha}{1-\alpha}\int_x^\infty [W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s. \tag{5.13}$$

Suppose the tail of the Lévy measure is log-convex. Then Q *is strictly increasing on* $(0,a^*)$ *and strictly decreasing on* (a^*,∞).

PROOF By (2.9), $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$, which implies that $a^* < \infty$ and it is the unique point that $W^{(q)\prime}$ attains its minimum. Also, by the assumption and Theorem 5.2.3, for all $q \geq 0$, $W^{(q)\prime}$ is log-convex. That is, the second derivative exists and $W^{(q)\prime}$ is convex. Therefore, $W^{(q)\prime}$ is strictly decreasing on $(0, a^*)$ and strictly increasing on (a^*, ∞). Consequently, $W^{(q)\prime\prime}$ is strictly negative on $(0, a^*)$ and strictly positive on (a^*, ∞). Since for any $x > 0$,

$$Q'(x) = -\frac{[W^{(q)}(x)]^{-\alpha/(1-\alpha)}}{[W^{(q)\prime}(x)]^2} W^{(q)\prime\prime}(x),$$

then Q is strictly increasing on $(0, a^*)$ and strictly decreasing on (a^*, ∞).□

Lemma 5.2.5 *Let C be defined as in (5.9) and $b^* = \sup\{b \geq 0 : C(b) \geq C(x)$ for all $x \geq 0\}$. Suppose the tail of the Lévy measure is log-convex. Then, $b^* < \infty$, C is strictly increasing on $(0, b^*)$ and strictly decreasing on (b^*, ∞), and therefore, it follows that b^* is the only point where C has a local/global maximum.*

PROOF To show that $b^* < \infty$, it is enough to prove that $\lim_{b\to\infty} C(b) = 0$. Given that

$$\lim_{b\to\infty} [W^{(q)}(b)]^{-1/(1-\alpha)} = 0, \tag{5.14}$$

and

$$\lim_{b\to\infty} \int_b^\infty [W^{(q)}(s)]^{-1/(1-\alpha)} \mathrm{d}s = 0, \tag{5.15}$$

we can find the next limit by using L'Hôpital's rule

$$\begin{aligned} \lim_{b\to\infty} \int_b^\infty \left[\frac{W^{(q)}(b)}{W^{(q)}(s)}\right]^{1/(1-\beta)} \mathrm{d}s &= \lim_{b\to\infty} \frac{\int_b^\infty [W^{(q)}(s)]^{-1/(1-\beta)}\mathrm{d}s}{[W^{(q)}(b)]^{-1/(1-\beta)}} \\ &= \lim_{b\to\infty} (1-\beta)\frac{W^{(q)}(b)}{W^{(q)\prime}(b)} \\ &= \frac{1-\beta}{\Phi(q)}, \end{aligned} \tag{5.16}$$

where $\Phi(q)$ is finite as defined in (2.4). Since

$$\begin{aligned} C(b) = & \frac{\beta}{1-\beta}[W^{(q)}(b)]^{1/(1-\beta)-1/(1-\alpha)} \int_b^\infty [W^{(q)}(s)]^{-1/(1-\beta)}\mathrm{d}s \\ & - \frac{\alpha}{1-\alpha}\int_b^\infty [W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s, \end{aligned}$$

then by (5.16), (5.14) and (5.15), it is clearly that $\lim_{b\to\infty} C(b) = 0$. The derivative of C can be calculated easily and get that

$$C'(b) = f(b)[C(b) - Q(b)],$$

where Q is the function given by (5.13) and f is given by

$$f(b) = \left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right) \frac{W^{(q)\prime}(b)}{W^{(q)}(b)}.$$

This implies that

$$C'(b) > 0(< 0, = 0) \text{ iff } C(b) > Q(b) \left(< Q(b), = Q(b)\right), \tag{5.17}$$

and we have the following cases:

Case I Suppose that $C(0) < Q(0)$, then by (5.17), $C'(0) < 0$, that is, C is strictly decreasing after the starting point until it crosses Q. On the other hand, by Lemma 5.2.4, Q is strictly decreasing on (a^*, ∞). Suppose on the contrary that C and Q intersect, say at $\hat{b}$, then, $C(\hat{b}) = Q(\hat{b})$, but $C'(\hat{b}) = 0$ while $Q'(\hat{b}) < 0$. Thus, there exists sufficiently small $\epsilon > 0$ such that $C(b) > Q(b)$ for $b \in (\hat{b}, \hat{b} + \epsilon)$. By (5.17), this means that C strictly increases until it crosses Q again. Since Q is strictly decreasing on (a^*, ∞), C and Q will not intersect again. This implies that C increases ultimately, which is a contradiction to that $\lim_{b\to\infty} C(b) = 0$. Therefore, C and Q can not intersect in this case. Hence, C is strictly decreasing on $(0, \infty)$, and in this case $b^* = 0$.

Case II Suppose that $C(0) > Q(0)$, then by (5.17), $C'(0) > 0$, that is, C strictly increases after the starting point until it crosses Q and so $b^* > 0$. Note that, in this case, $a^* > 0$, as if $a^* = 0$, C will increase ultimately, which contradicts that $\lim_{b\to\infty} C(b) = 0$. Thus, C can not increase ultimately and should intersects Q at some point. At the point of intersection, say $\hat{b}$, $C(\hat{b}) = Q(\hat{b})$, we will have two cases:

(i)If $\hat{b} < a^*$, then $C'(\hat{b}) = 0$ while $Q'(\hat{b}) > 0$, which implies that there exists sufficiently small $\epsilon > 0$ such that $C(b) < Q(b)$ for $b \in (\hat{b}, \hat{b} + \epsilon)$. By (5.17), this means that C is strictly decreasing in this interval until it crosses Q again. In a similar argument to case I, we can see that $\hat{b}$ is the only intersection point. That is, $\hat{b} = b^*$, where $C(b) > Q(b)$ for $b < b^*$, and $C(b) < Q(b)$ for $b > b^*$.

(ii) If $\hat{b} = a^*$, then $C'(\hat{b}) = 0$ and $Q'(\hat{b}) = 0$, and also in a similar argument to case I, C can not increase after the intersection point $\hat{b}$. So, C strictly decreases after $\hat{b}$, and hence, there is only one intersection point $\hat{b} = b^* = a^*$, where $C(b) > Q(b)$ for $b < b^*$ and $C(b) < Q(b)$ for $b > b^*$.

Case III Suppose that $C(0) = Q(0)$. If $a^* > 0$, then either C decreases after the starting point and hence we have a similar argument to case I, or C increases after the starting point and hence we have a similar argument to case II. If $a^* = 0$, then C decreases after the starting point and the case is similar to case I, hence, $b^* = 0$.

So, we conclude that either C intersects Q at exactly one point $b^* \leq a^*$, or C decreases strictly on $(0, \infty)$, and hence, the statement is proved.$\square$

Now, we are ready to give the solution of (5.4) and prove its optimality. Before we do that, we recall some results in order to use them in the proof. The next result combines Theorems A and B in [55, pp.9-11].

Theorem 5.2.6 *A function $f\colon (a,b) \mapsto \mathbb{R}$ is convex if and only if it is absolutely continuous with an increasing density.*

Recall the operator $\mathcal{A}$ defined in (4.8),

$$\begin{aligned}\mathcal{A}f(y,z) &= \mu\frac{\partial f}{\partial y}(y,z) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2}(y,z) \\ &\quad + \int_{0+}^{\infty}\left[f(y-\theta,z) - f(y,z) + \theta\frac{\partial f}{\partial y}(y,z)\mathbf{1}_{\{0<\theta\leq 1\}}\right]\nu(\mathrm{d}\theta).\end{aligned}$$

Also, we will need computations in the following remark to complete the proof of optimality.

Remark 11 We compute the partial derivatives of the value function given by (5.8) for any $b \geq 0$.

$$\frac{\partial v^{\delta^b}(x,\bar{x})}{\partial x} = W^{(q)\prime}(x)[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left[\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-(1/(1-\alpha))}\mathrm{d}s + C(b\vee\bar{x})\right] \tag{5.18}$$

and

$$\frac{\partial v^{\delta^b}(x,\bar{x})}{\partial \bar{x}} = \begin{cases} K_1(x,\bar{x}) & \text{if } 0 \leq \bar{x} \leq b \\ \\ K_2(x,\bar{x}) & \text{if } \bar{x} > b, \end{cases} \tag{5.19}$$

where

$$\begin{aligned}K_1(x,\bar{x}) &= \left(\frac{\alpha}{1-\alpha}\right)^2\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)\prime}(\bar{x})[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s \\ &\quad + \frac{\alpha}{1-\alpha}\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\left[W^{(q)\prime}(\bar{x})(W^{(q)}(\bar{x}))^{\alpha/(1-\alpha)}C(b) - 1\right],\end{aligned}$$

and

$$K_2(x,\bar{x}) = \frac{\beta}{1-\beta}\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\Big\{\frac{\beta}{1-\beta}W^{(q)\prime}(\bar{x})[W^{(q)}(\bar{x})]^{\beta/(1-\beta)}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\beta)}\mathrm{d}s - 1\Big\}.$$

Theorem 5.2.7 *Suppose the tail of the Lévy measure is log-convex, then an $(\alpha \longmapsto \beta)$-tax strategy at level b^* is an optimal tax rate policy.*

Proof We need to prove that $v^{\delta^{b^*}}$ satisfies the conditions of the verification lemma (5.2.2) in order for δ^{b^*} to be an optimal strategy. First, we have to prove that the value function $v^{\delta^{b^*}}(x,\bar{x}) \in \mathcal{S}_{[1/n,n]\times[1/n,n]}$ for each $n \in \mathbb{N}$. Recall from (5.8) that,

$$v^{\delta^{b^*}}(x,\bar{x}) = W^{(q)}(x)[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left[\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s + C(b^*\vee\bar{x})\right], \tag{5.20}$$

where

$$C(b) = \frac{\beta}{1-\beta}[W^{(q)}(b)]^{1/(1-\beta)-1/(1-\alpha)}\int_{b}^{\infty}[W^{(q)}(s)]^{-1/(1-\beta)}\mathrm{d}s - \frac{\alpha}{1-\alpha}\int_{b}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s.$$

Let $g(x) = W^{(q)}(x)$ and

$$h(\bar{x}) = [W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left[\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s + C(b^*\vee\bar{x})\right],$$

then by (5.20),

$$v^{\delta^{b^*}}(x,\bar{x}) = g(x)\,h(\bar{x}). \tag{5.21}$$

By the assumption that the tail of the Lévy measure is log-convex and Theorem 5.2.3, $W^{(q)\prime}$ is convex. Thus, by Theorem 5.2.6, we see clearly that g satisfies condition I of Definition 4.1.1 of $\mathcal{S}_{[1/n,n]\times[1/n,n]}$ for each $n \in \mathbb{N}$. For condition II of Definition 4.1.1, choose $\lambda = n$ so that $\theta \geq \lambda = n \geq s$ and thus $s - \theta \leq 0$. This implies that $W^{(q)}(s-\theta) = 0$, and hence the condition is satisfied for all $s \in (1/n, n)$. For the function h, it has a density and we denote it by h', which is bounded and integrable on $[1/n, n]$ for each $n \in \mathbb{N}$, and such that for any $\bar{x} \in [1/n, n]$ and for each $n \in \mathbb{N}$,

$$h(\bar{x}) - h(1/n) = \int_{1/n}^{\bar{x}} h'(t)\mathrm{d}t,$$

where h', when $\bar{x} \leq b^*$, is given by

$$h'(\bar{x}) = \frac{\alpha}{1-\alpha} \frac{W^{(q)\prime}(\bar{x})}{W^{(q)}(\bar{x})} [W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)} \left[\frac{\alpha}{1-\alpha} \int_{\bar{x}}^{\infty} [W^{(q)}(s)]^{-1/(1-\alpha)} \mathrm{d}s + C(b^*) \right] - \frac{\alpha}{1-\alpha} \frac{1}{W^{(q)}(\bar{x})},$$

and similarly for the case $\bar{x} > b^*$. This implies that h satisfies condition III of Definition 4.1.1 of $\mathcal{S}_{[1/n,n]\times[1/n,n]}$ for each $n \in \mathbb{N}$. Therefore, $v^{\delta^{b^*}}(x,\bar{x})$ belongs to $\mathcal{S}_{[1/n,n]\times[1/n,n]}$ for each $n \in \mathbb{N}$.

For any $0 < x \leq \bar{x}$, we use (5.21), then by the martingale property (2.13) and from [19, p. 136], we get that

$$(\mathcal{A} - q)v^{\delta^{b^*}}(x,\bar{x}) = h(\bar{x})(\mathcal{A} - q)W^{(q)}(x) = 0.$$

Hence, condition I is verified.

For condition II, we have two cases:

Case I For $0 < \bar{x} \leq b^*$, the condition is satisfied for $v^{\delta^{b^*}}(x,\bar{x})$ if for any $H \in [\alpha, \beta]$,

$$H \frac{\partial v^{\delta^{b^*}}}{\partial x}(\bar{x},\bar{x}) + (H-1)\frac{\partial v^{\delta^{b^*}}}{\partial \bar{x}}(\bar{x},\bar{x}) - H \geq 0,$$

but this happens if and only if

$$\left[\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}\Big|_{x=\bar{x}} - 1 \right] \left[\frac{H-\alpha}{1-\alpha} \right] \geq 0.$$

Since $H \in [\alpha, \beta]$, it is always true that

$$\left[\frac{H-\alpha}{1-\alpha} \right] \geq 0,$$

therefore, in order to satisfy the condition, we should have that

$$\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}\Big|_{x=\bar{x}} \geq 1.$$

Since $0 < \bar{x} \leq b^*$, and by Lemma 5.2.5, $C' \geq 0$ on $(0, b^*]$. By (5.17), $C'(\bar{x}) \geq 0$ is satisfied if and only if $C(\bar{x}) \geq Q(\bar{x})$, but as $C(b^*) \geq C(\bar{x})$, then we get that $C(b^*) \geq Q(\bar{x})$. By (5.13), we have

$$\begin{aligned} & C(b^*) \geq Q(\bar{x}) \\ \Longleftrightarrow\ & C(b^*) \geq [W^{(q)}(\bar{x})]^{-1/(1-\alpha)} \frac{W^{(q)}(\bar{x})}{W^{(q)\prime}(\bar{x})} - \frac{\alpha}{1-\alpha} \int_{\bar{x}}^{\infty} [W^{(q)}(s)]^{-1/(1-\alpha)} \mathrm{d}s, \end{aligned}$$

but this happens if and only if

$$\frac{\alpha}{1-\alpha}W^{(q)\prime}(\bar{x})[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\int_{\bar{x}}^{\infty}[W^{(q)}(s)]^{-1/(1-\alpha)}\mathrm{d}s$$
$$+W^{(q)\prime}(\bar{x})[W^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}C(b^*)\geq 1.$$

By (5.18), we can write the last inequality as

$$\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}}\geq 1.$$

That is, $v^{\delta^{b^*}}(x,\bar{x})$ satisfies II in the first case.

Case II For $\bar{x}>b^*$, condition II in the verification lemma is satisfied for $v^{\delta^{b^*}}(x,\bar{x})$ if for any $H\in[\alpha,\beta]$,

$$H\frac{\partial v^{\delta^{b^*}}}{\partial x}(\bar{x},\bar{x})+(H-1)\frac{\partial v^{\delta^{b^*}}}{\partial \bar{x}}(\bar{x},\bar{x})-H\geq 0,$$

but in this case, it happens if and only if

$$\left[\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}}-1\right]\left[\frac{H-\beta}{1-\beta}\right]\geq 0.$$

Since $H\leq\beta$, the last inequality is satisfied if and only if

$$\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}}<1.$$

By Lemma 5.2.5 and as $\bar{x}>b^*$, we have that $C'(\bar{x})<0$. By (5.17), this is equivalent to $C(\bar{x})<Q(\bar{x})$. By a similar argument to case I, we can have that

$$C(\bar{x})<Q(\bar{x})\quad\Longleftrightarrow\quad\frac{\partial v^{\delta^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}}<1,$$

that is, $v^{\delta^{b^*}}(x,\bar{x})$ satisfies II in the second case. Hence, δ^{b^*} is an optimal strategy.□

5.3 Relation with Wang and Hu's work

As the title of this section indicates, we explain the relation between our results and the one given by authors in [57].

Proposition 5.3.1 *The two cases considered in Sections 4 and 5 in [57] for the optimal strategy, and the corresponding optimal value function, agrees with our solution of* (5.4) *in the case* $x=\bar{x}$.

Proof We recall first [57, Equation (4.5)], which is

$$\frac{\beta}{1-\beta}[W^{(q)}(y)]^{-1/(1-\beta)} = -\left(\frac{[W^{(q)}(y)]^{1-(1/(1-\beta))}}{W^{(q)\prime}(y)}\right)' - \frac{[W^{(q)}(y)]^{1-(1/(1-\beta))}W^{(q)\prime\prime}(y)}{[W^{(q)\prime}(y)]^2}. \tag{5.22}$$

To find b^*, by Lemma 5.2.5, we know that either $b^* = 0$ or $b^* > 0$ is such that $C(b^*) = Q(b^*)$, which implies by (5.9) and (5.13) that

$$\frac{\beta}{1-\beta}\int_{b^*}^{\infty}\left[W^{(q)}(s)\right]^{-1/(1-\beta)}\mathrm{d}s = \frac{[W^{(q)}(b^*)]^{1-(1/(1-\beta))}}{W^{(q)\prime}(b^*)}. \tag{5.23}$$

By (5.22), we have that

$$\begin{aligned}\frac{\beta}{1-\beta}\int_{b^*}^{\infty}\left[W^{(q)}(s)\right]^{-1/(1-\beta)}\mathrm{d}s = &\frac{[W^{(q)}(b^*)]^{1-(1/(1-\beta))}}{W^{(q)\prime}(b^*)}\\ &-\int_{b^*}^{\infty}\frac{[W^{(q)}(s)]^{1-(1/(1-\beta))}W^{(q)\prime\prime}(s)}{[W^{(q)\prime}(s)]^2}\mathrm{d}s.\end{aligned}$$

Therefore, by (5.23), we find that b^* is the solution to

$$\int_{b^*}^{\infty}\frac{[W^{(q)}(s)]^{1-(1/(1-\beta))}W^{(q)\prime\prime}(s)}{[W^{(q)\prime}(s)]^2}\mathrm{d}s = 0, \tag{5.24}$$

which agrees with [57, (5.15)]. Note that, from the proof of Lemma 5.2.4, we know that $W^{(q)\prime\prime}$ changes its sign from negative to positive only once. Therefore, the existence of a unique solution $b^* > 0$ of (5.24) is guaranteed by the condition

$$\int_{0}^{\infty}\frac{[W^{(q)}(s)]^{1-(1/(1-\beta))}W^{(q)\prime\prime}(s)}{[W^{(q)\prime}(s)]^2}\mathrm{d}s < 0.$$

This implies that in the case

$$\int_{0}^{\infty}\frac{[W^{(q)}(s)]^{1-(1/(1-\beta))}W^{(q)\prime\prime}(s)}{[W^{(q)\prime}(s)]^2}\mathrm{d}s \geq 0,$$

then $b^* = 0$ and the optimal strategy is to pay tax at the maximum rate β. These conditions agrees with [57].

In order to show that our optimal value function $v^{\delta^{b^*}}(x,x)$ for any $x \geq 0$ equals the corresponding one in [57, (5.7)], we use the tax value function given by (5.8), (5.9) and (5.13). Also, recall that when $b^* > 0$, as $C'(b^*) = 0$, and by (5.17), $C(b^*) = Q(b^*)$, then we can use Q instead of C at b^*. That is, for $x \leq b^*$, we have that

$$\begin{aligned}v^{\delta^{b^*}}(x,x) &= \frac{\alpha}{1-\alpha}\int_x^{\infty}\left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\alpha)}\mathrm{d}s + [W^{(q)}(x)]^{1/(1-\alpha)}C(b^*)\\ &= \frac{\alpha}{1-\alpha}\int_x^{b^*}\left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\alpha)}\mathrm{d}s + \frac{[W^{(q)}(b^*)]^{1-(1/(1-\alpha))}}{W^{(q)\prime}(b^*)}[W^{(q)}(x)]^{1/(1-\alpha)},\end{aligned}$$

which agrees with the optimal value function for the first case, as given in [57, Equation 5.7].

For $x > b^*$, we have that

$$\begin{aligned} v^{\delta^{b^*}}(x,x) &= \frac{\alpha}{1-\alpha}\int_x^\infty \left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\alpha)} \mathrm{d}s + [W^{(q)}(x)]^{1/(1-\alpha)} C(x) \\ &= \frac{\beta}{1-\beta}\int_x^\infty \left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\beta)} \mathrm{d}s \\ &= \frac{\beta}{1-\beta}\int_{b^*}^\infty \left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\beta)} \mathrm{d}s - \frac{\beta}{1-\beta}\int_{b^*}^x \left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\beta)} \mathrm{d}s. \quad (5.25) \end{aligned}$$

Then, we use (5.23) in (5.25), and see that the optimal tax value function in the case $x > b^*$, is given by

$$v^{\delta^{b^*}}(x,x) = \frac{[W^{(q)}(b^*)]^{1-(1/(1-\beta))}}{W^{(q)\prime}(b^*)}[W^{(q)}(x)]^{1/(1-\beta)} - \frac{\beta}{1-\beta}\int_{b^*}^x \left[\frac{W^{(q)}(x)}{W^{(q)}(s)}\right]^{1/(1-\beta)} \mathrm{d}s,$$

which agrees with the corresponding optimal tax value for the second case in [57, (5.7)].□

Chapter 6

Natural Taxation with forced bail-out

6.1 Introduction

It is natural to consider the case when a tax authority supports an insurance company by bail-out loans to continue its businesses. These are the capital injections made by the government to keep the insurance company solvent. Then it would be important for the government to know what is the net tax profit (tax value) and to look at the maximum value and find the optimal strategy that can achieves this. We are thinking about the case where bail-out is unlimited, so no ruin occurs. In the context of dividends, many authors studied this problem. For example, [21] and [48] looked at the problem in the setting of a Cramér-Lundberg process. Similarly, [40] studied the problem but with a Brownian motion risk process model. In [10], the problem is generalised into a spectrally negative Lévy risk process. In this chapter, we study tax processes with the addition of a bail-out process K. This type of taxation has been investigated recently in many articles, such as [3], [4] and [64]. We recall first the tax process defined in Chapter 5. Given a spectrally negative Lévy process X, we define a tax process by

$$V_t^H = X_t - \int_{0+}^t H_s \,\mathrm{d}\overline{X}_s, \quad t \geq 0,$$

where $(H_t)_{t\geq 0}$ is a left-continuous adapted process such that $0 \leq \alpha \leq H_t \leq \beta < 1$. Since tax contributions are made whenever V^H reaches a new maximum, this taxation structure is of loss-carry-forward type. In a similar way, with the addition of a bail-out process K, we define the tax process as,

$$V_t^{\bar{\pi}} = (X+K)_t - \int_{0^+}^t H_s \, \mathrm{d}\overline{(X+K)}_s, \quad t \geq 0. \tag{6.1}$$

We say that $\bar{\pi} := (H, K)$ is an admissible policy if both $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ are left-continuous adapted processes such that for each $t \geq 0$, $0 \leq \alpha \leq H_t \leq \beta < 1$, and the bail-out process K is an increasing process that represents the injection to the capital such that $\overline{(X+K)}$ is continuous and $V_{t^+}^{\bar{\pi}} \geq 0$. Similarly, the tax contributions in (6.1) are made whenever the process $(X+K)$ reaches a new maximum, which is whenever $V^{\bar{\pi}}$ reaches a new maximum; see Lemma 6.2.2 below. As a result, the taxation in (6.1) can be seen as of loss-carry-forward type. In this model, ruin is not allowed. Therefore, we call K in this case, *forced bail-out* and the process $V^{\bar{\pi}}$ is the *tax process with forced bail-out*.

For $x \leq \bar{x}$ such that $\bar{x} > 0$, the net profit (or the value function) of taxation in this model is given by

$$v^{\bar{\pi}}(x, \bar{x}) = \mathbb{E}_{x,\bar{x}} \left[\int_0^\infty e^{-qs} H_s \, \mathrm{d}\overline{(X+K)}_s - \eta \int_0^\infty e^{-qs} \, \mathrm{d}K_s \right],$$

where $\eta \geq 1$ is a bail-out cost factor.

Our aim in this chapter, is to solve the optimal control problem

$$v^*(x, \bar{x}) = \sup_{\bar{\pi} \in \overline{\Pi}} v^{\bar{\pi}}(x, \bar{x}), \tag{6.2}$$

where $\overline{\Pi}$ is the set of all admissible policies. An optimal tax policy $\bar{\pi}^* = (H^*, K^*) \in \overline{\Pi}$ is such that $v^*(x, \bar{x}) = v^{\bar{\pi}^*}(x, \bar{x})$ for all $x \leq \bar{x}$ such that $\bar{x} > 0$. Note that, recently, [64] studied an optimal control problem for a tax process with capital injections, but in the case where there is a delay in tax. That is, they optimise over the class of $(\alpha \longmapsto \beta)$-tax strategies, see (5.5), when $\alpha = 0$ only. Also, they optimise over the bail-out process K where the injection to the capital is only made whenever the surplus is below zero. In contrast, here we solve (6.2) and prove optimality among a larger class of admissible strategies. Note that, from an intuitive point of view, the optimal bail-out strategy would be when capital injections are made only if necessary. This is when the surplus of the insurance company becomes negative, and the government injects the capital back to the level zero in order for the company to operate again. This bail-out strategy is intuitively optimal. This is because when the bail-out happens earlier than needed, that is to say to a strictly positive level, the government can get more taxes. However, the taxes will be paid-out later, so this tax income will be more highly discounted than the capital injections that has been made earlier. Clearly, this would make a loss for the government and hence is not an optimal strategy.

Remark 12 We exclude the case $\bar{x} = 0$ and leave it for further study, as it is not clear how to construct the tax process with bail-out in the case $\bar{x} = 0$, when taxation and injections happen simultaneously.

Our main results in this chapter are, Theorem 6.2.4, which gives an explicit expression for the net present value of taxation of the natural tax process with forced bail-out, $V^{\delta,\infty}$, which will be defined rigorously below. Also, Theorem 6.2.6 gives an explicit expression for the net present value of capital injections of $V^{\delta,\infty}$. Our results agree with some results in literature for special cases, when δ is constant or zero. Then, Theorem 6.3.4, states that the solution of the optimal control problem (6.2) is piecewise constant tax together with the minimal capital injections, which is the injections back to zero whenever the surplus becomes strictly negative. Note that, in Theorem 6.3.4, a condition for the Lévy measure is not needed, compared to the optimal control tax problem considered in Chapter 5.

This chapter is organised as follows. In Section 6.2, we define the process $V^{\delta,\infty}$, through an algorithm and show it is well-defined. Then, we derive the net profit of taxation for $V^{\delta,\infty}$. In Section 6.3, we solve our optimal control problem (6.2).

6.2 Value function

Let $X_0 = x, \overline{X}_0 = \bar{x} > 0$, where $x \leq \bar{x}$ and $\delta\colon [\bar{x}, \infty) \to [0, 1)$ be a measurable function such that there exists a unique solution $y^{\delta}_{\bar{x}}$ to (3.4). In this section, we study a natural tax process with a forced bail-out and we call it $V^{\delta,\infty}$. The process $V^{\delta,\infty}$ is a process refracted from above with rate δ and reflected from below at zero. Such a process can be defined by using a one-sided refraction from above and a one-sided reflection from below locally, and then gluing segments of paths together. The corresponding bail-out process, K^{δ}, is the injection to the capital whenever the process becomes negative.

We define the process $V^{\delta,\infty}$ rigorously through the following algorithm and Proposition 6.2.1.

Algorithm

Initialization, $(n = 0)$:

- For $t \geq 0$, let $V_t^{(0)} = X_t - \inf_{0 \leq s \leq t}(X_s \wedge 0)$, $(K^\delta)_t^{(0)} = 0$, $z^{(1)} = \bar{x}$ and $T_0 = 0$.
- Let $\Delta_0 = T_1 = \inf\left\{t \geq 0 : V_t^{(0)} > \bar{x}\right\}$.

Step $(n\,\mathbf{to}\,n+1)$:

- Let $X_t^{(n)} = V_{\Delta_{n-1}}^{(n-1)} + X_{T_n+t} - X_{T_n}$ for $t \geq 0$ and $n \geq 1$.
- **If** $X_0^{(n)} = z^{(n)}$, **then** for $t \geq 0$, $(K^\delta)_t^{(n)} = 0$, and
$$V_t^{(n)} = X_t^{(n)} - \int_{0_+}^t \delta(\overline{V}_s^{(n)})\, \mathrm{d}\overline{X_s^{(n)}},$$
where $\overline{V}_t^{(n)} = \sup_{0 \leq s \leq t} V_s^{(n)}$. Put
$$\Delta_n = \inf\left\{t \geq 0 : V_t^{(n)} < 0\right\}.$$
Note that, $\overline{X^{(n)}}$ is continuous since $X^{(n)}$ does not have upward jumps. Therefore, by Theorem 3.2.3, $V^{(n)}$ is well defined.
  - **If** $\Delta_n < \infty$, **then** let $T_{n+1} = T_n + \Delta_n$, $z^{(n+1)} = \overline{V_{\Delta_n}^{(n)}}$ and go to step $(n\,\text{to}\,n+1)$, **otherwise** stop.
- **If** $X_0^{(n)} < 0$, **then** for $t \geq 0$
$$(K^\delta)_t^{(n)} = -X_0^{(n)} - \inf_{0 \leq s < t}\left\{(X_s^{(n)} - X_0^{(n)}) \wedge 0\right\}$$
and
$$V_t^{(n)} = X_t^{(n)} + (K^\delta)_t^{(n)}.$$
Put
$$\Delta_n = \inf\left\{t \geq 0 : V_t^{(n)} > z^{(n)}\right\}.$$
  - **If** $\Delta_n < \infty$, **then** let $T_{n+1} = T_n + \Delta_n$, $z^{(n+1)} = z^{(n)}$ and go to step (n to n+1), **otherwise** stop.

Finally, for $t \in (T_n, T_{n+1}]$, we set:

$$V_t^{\delta,\infty} = V_{t-T_n}^{(n)}, \text{ and } K_t^\delta = \sum_{i=0}^{n-1} (K^\delta)_{\Delta_i}^{(i)} + (K^\delta)_{t-T_n}^{(n)}. \tag{6.3}$$

Proposition 6.2.1 *For almost every sample path of X, the pair of processes $(V_t^{\delta,\infty}, K_t^\delta)$ defined in (6.3) is the unique solution of the following integral equation:*

$$V_t^{\delta,\infty} = (X + K^\delta)_t - \int_{0^+}^t \delta(\overline{V}_s^{\delta,\infty}) \, \mathrm{d}\overline{(X + K^\delta)}_s, \quad t \geq 0, \tag{6.4}$$

such that $V_{t^+}^{\delta,\infty} \geq 0$.

Proof We show that $(V_t^{\delta,\infty}, K_t^\delta)$ given by (6.3) is well defined for all $t \geq 0$. Since $T_0 \leq T_1 \leq T_2 \leq ...$, then it is enough to show that $\lim_{n\to\infty} T_n = \infty$ almost surely. Note that,

$$T_n = \sum_{i=0}^{n-1} \Delta_i.$$

From the algorithm, for $n \geq 1$, the up-crossings T_{2n+1} times are when

$$\Delta_{2n} = \inf\left\{t \geq 0 : V_t^{(2n)} > z^{(2n)}\right\}.$$

Since, for any $n \geq 1$, $z^{(n)}$ is increasing, then

$$\Delta_2 = \inf\left\{t \geq 0 : V_t^{(2)} > z^{(2)}\right\} > 0.$$

From the strong Markov property of $(V^{\delta,\infty}, \overline{V}^{\delta,\infty})$, $(\Delta_{2n})_{n\geq 1}$ is a sequence of independent random variables. Note that, for $n \geq 1$, Δ_{2n} is bigger in stochastic order than Δ_2. That is,

$$\mathbb{P}_{x,\bar{x}}\left(\Delta_{2n} \geq t\right) \geq \mathbb{P}_{x,\bar{x}}\left(\Delta_2 \geq t\right).$$

Therefore, by taking independent identically distributed copies of Δ_2, we get that

$$\mathbb{P}_{x,\bar{x}}\left(\lim_{n\to\infty} T_n = \infty\right) \geq \mathbb{P}_{x,\bar{x}}\left(\sum_{n\geq 1} \Delta_{2n} = \infty\right) \geq \mathbb{P}_{x,\bar{x}}\left(\sum_{n\geq 1} \Delta_2^{(n)} = \infty\right) = 1,$$

where the last equality follows from the fact that the sum of independent identically distributed positive random variables converges to infinity. This implies that $\lim_{n\to\infty} T_n = \infty$ almost surely.

Note that, from the algorithm construction, $\overline{(X + K^\delta)}$ is continuous. This is because X is a spectrally negative Lévy process, thus the upward jumps in $(X + K^\delta)$

come from K^δ. So, when there is a bail-out, the surplus process $V^{\delta,\infty}$ follows the maximum that it has previously. Moreover, since the tax is paid whenever the process $(X+K^\delta)$ reaches its maximum, then K^δ does not increase in that case, which implies that $\overline{(X+K^\delta)}$ does not increase. Therefore, $\overline{(X+K^\delta)}$ satisfies the assumptions of Theorem 3.2.3, which proves uniqueness of the solution.

Now we show by induction, that $(V_t^{\delta,\infty}, K_t^\delta)$ defined in (6.3) satisfies (6.4). First step is to prove that the statement is true for $n = 0, 1$. Since $V_{\Delta_0}^{(0)} = V_{T_1}^{(0)} = z^{(1)}$, i.e tax starts in intervals with odd index, so on $(0, T_1]$ there is no tax, and for $t \in (T_1, T_2]$, there is no injection, so

$$K_t^\delta = (K^\delta)_{t-T_1}^{(1)} = 0. \tag{6.5}$$

Also,

$$X_{t-T_1}^{(1)} = V_{\Delta_0}^{(0)} + X_{T_1+t-T_1} - X_{T_1} = X_t, \tag{6.6}$$

because $\Delta_0 = T_1$ and $V_{T_1}^{(0)} = X_{T_1}$. Therefore,

$$\begin{aligned} V_t^{\delta,\infty} = V_{t-T_1}^{(1)} &= X_{t-T_1}^{(1)} - \int_{0+}^{t-T_1} \delta(\overline{V}_s^{(1)})\, \mathrm{d}\overline{X_s^{(1)}} \\ &= X_t - \int_{T_1}^{t} \delta(\overline{V}_{s-T_1}^{(1)})\, \mathrm{d}\overline{X_{s-T_1}^{(1)}} \\ &= X_t - \int_{0+}^{t} \delta(\overline{V}_{s-T_1}^{(1)})\, \mathrm{d}\overline{X_s} \\ &= X_t + K_t^\delta - \int_{0+}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)}_s, \end{aligned}$$

where we use (6.6) in the second and third equality, and also in the third equality we use that there is no tax in $(0, T_1]$. The fourth equality comes from (6.5).

For the second step in the proof, we suppose that for all $m \le n$ and $t \in (T_m, T_{m+1}]$, the pair of processes $(V_t^{\delta,\infty}, K_t^\delta)$ defined by

$$V_t^{\delta,\infty} = V_{t-T_m}^{(m)} \text{ and } K_t^\delta = \sum_{i=0}^{m-1} (K^\delta)_{\Delta_i}^{(i)} + (K^\delta)_{t-T_m}^{(m)}, \tag{6.7}$$

satisfies (6.4).

The third step is to prove that (6.3) satisfies (6.4) for $n+1$, that is, for $t \in (T_{n+1}, T_{n+2}]$ the pair of processes $(V_t^{\delta,\infty}, K_t^\delta)$ defined by

$$V_t^{\delta,\infty} = V_{t-T_{n+1}}^{(n+1)}, \text{ and } K_t^\delta = \sum_{i=0}^{n} (K^\delta)_{\Delta_i}^{(i)} + (K^\delta)_{t-T_{n+1}}^{(n+1)},$$

satisfies (6.4). We first recall that, for $t \in (T_{n+1}, T_{n+2}]$,

$$\begin{aligned} X^{(n+1)}_{t-T_{n+1}} &= V^{(n)}_{\Delta_n} + X_{T_{n+1}+t-T_{n+1}} - X_{T_{n+1}} \\ &= V^{(n)}_{T_{n+1}-T_n} + X_t - X_{T_{n+1}}. \end{aligned} \tag{6.8}$$

Let us suppose that we are in the case $X^{(n+1)}_0 = z^{(n+1)}$, then for $t \geq 0$, $(K^\delta)^{(n+1)}_t = 0$ and hence for $t \in (T_{n+1}, T_{n+2}]$

$$(K^\delta)^{(n+1)}_{t-T_{n+1}} = 0. \tag{6.9}$$

Then, use (6.7) with $m = n$ and $t = T_{n+1}$ to get that

$$V^{\delta,\infty}_{T_{n+1}} = V^{(n)}_{T_{n+1}-T_n} \quad \text{and} \quad K^\delta_{T_{n+1}} = \sum_{i=0}^{n-1} (K^\delta)^{(i)}_{\Delta_i} + (K^\delta)^{(n)}_{T_{n+1}-T_n} = \sum_{i=0}^{n} (K^\delta)^{(i)}_{\Delta_i} \tag{6.10}$$

satisfies (6.4). Thus, (6.8) becomes

$$\begin{aligned} X^{(n+1)}_{t-T_{n+1}} &= V^{\delta,\infty}_{T_{n+1}} + X_t - X_{T_{n+1}} \\ &= X_{T_{n+1}} + K^\delta_{T_{n+1}} - \int_{0^+}^{T_{n+1}} \delta(\overline{V}^{\delta,\infty}_s) \,\mathrm{d}\overline{(X+K^\delta)}_s + X_t - X_{T_{n+1}} \\ &= X_t + \sum_{i=0}^{n} (K^\delta)^{(i)}_{\Delta_i} - \int_{0^+}^{T_{n+1}} \delta(\overline{V}^{\delta,\infty}_s) \,\mathrm{d}\overline{(X+K^\delta)}_s \\ &= X_t + \sum_{i=0}^{n} (K^\delta)^{(i)}_{\Delta_i} + (K^\delta)^{(n+1)}_{t-T_{n+1}} - \int_{0^+}^{T_{n+1}} \delta(\overline{V}^{\delta,\infty}_s) \,\mathrm{d}\overline{(X+K^\delta)}_s \\ &= X_t + K^\delta_t - \int_{0^+}^{T_{n+1}} \delta(\overline{V}^{\delta,\infty}_s) \,\mathrm{d}\overline{(X+K^\delta)}_s, \end{aligned} \tag{6.11}$$

where in the second equality we use that $(V^{\delta,\infty}_{T_{n+1}}, K^\delta_{T_{n+1}})$ given by (6.10) satisfies (6.4) and in the fourth equality, we use (6.9). For $s \in (T_{n+1}, t]$, we have that

$$\overline{X^{(n+1)}_{s-T_{n+1}}} = \sup_{T_{n+1} \leq s' \leq s} X^{(n+1)}_{s'-T_{n+1}} = \sup_{T_{n+1} \leq s' \leq s} (X_{s'} + K^\delta_{s'}) = \overline{(X+K^\delta)}_s. \tag{6.12}$$

Therefore,

$$
\begin{aligned}
V_t^{\delta,\infty} = V_{t-T_{n+1}}^{(n+1)} &= X_{t-T_{n+1}}^{(n+1)} - \int_{0^+}^{t-T_{n+1}} \delta(\overline{V}_s^{(n+1)})\, \mathrm{d}\overline{X_s^{(n+1)}} \\
&= X_t + K_t^\delta - \int_{0^+}^{T_{n+1}} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)}_s \\
&\quad - \int_{T_{n+1}}^{t} \delta(\overline{V}_{s-T_{n+1}}^{(n+1)})\, \mathrm{d}\overline{X_{s-T_{n+1}}^{(n+1)}} \\
&= X_t + K_t^\delta - \int_{0^+}^{T_{n+1}} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)}_s \\
&\quad - \int_{T_{n+1}}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)_s} \\
&= X_t + K_t^\delta - \int_{0^+}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)}_s,
\end{aligned}
$$

where we use (6.11) and (6.12).

Now suppose that we are in the case $X_0^{(n+1)} < 0$, so

$$
\begin{aligned}
V_t^{\delta,\infty} = V_{t-T_{n+1}}^{(n+1)} &= X_{t-T_{n+1}}^{(n+1)} + (K^\delta)_{t-T_{n+1}}^{(n+1)} \\
&= X_t + V_{T_{n+1}}^{\delta,\infty} - X_{T_{n+1}} + (K^\delta)_{t-T_{n+1}}^{(n+1)} \\
&= X_t + X_{T_{n+1}} + K_{T_{n+1}}^\delta - \int_{0^+}^{T_{n+1}} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)_s} \\
&\quad - X_{T_{n+1}} + (K^\delta)_{t-T_{n+1}}^{(n+1)} \\
&= X_t + \sum_{i=0}^{n} (K^\delta)_{\Delta_i}^{(i)} + (K^\delta)_{t-T_{n+1}}^{(n+1)} - \int_{0^+}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)_s} \\
&= X_t + K_t^\delta - \int_{0^+}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)_s},
\end{aligned}
\tag{6.13}
$$

where we use (6.8) and (6.10) in the second equality, and in the third equality that $(V_{T_{n+1}}^{\delta,\infty}, K_{T_{n+1}}^\delta)$ given by (6.10) satisfies (6.4). In the fourth equality of (6.13), we use again (6.10) and that $\int_{T_{n+1}}^{t} \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X+K^\delta)_s} = 0$ as in the interval $(T_{n+1}, T_{n+2}]$, injections only that happens.

The last part of the proof is to show that for $n \geq 1$ and $t \in (T_n, T_{n+1}]$, $V_{t^+}^{\delta,\infty} \geq 0$, which is clear from the construction of the algorithm. Hence, the statement is proved.□

In the next two subsections, we find the analytic expression of the net present tax value function for the process (6.4), which is defined for any $x \leq \bar{x}$ and $\bar{x} > 0$ by

$$
v^{\delta,\infty}(x,\bar{x}) = v_{\mathrm{tax}}^{\delta,\infty}(x,\bar{x}) - \eta\, v_{\mathrm{inj}}^{\delta,\infty}(x,\bar{x}), \tag{6.14}
$$

where $\eta \geq 1$ a bail-out cost factor,

$$v_{\text{tax}}^{\delta,\infty}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\infty} e^{-qs}\,\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^{\delta})}_s\right],$$

and

$$v_{\text{inj}}^{\delta,\infty}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0}^{\infty} e^{-qs}\ \mathrm{d}K_s^{\delta}\right].$$

Recall that for $f \in \mathcal{S}_{[b,a]\times[c,d]}$, $y \in [b,a]$ and $z \in [c,d]$,

$$\Gamma^{\delta} f(y,z) = \frac{\partial}{\partial y} f(y,z)\ \delta(z) - \frac{\partial}{\partial z} f(y,z)(1-\delta(z)). \tag{6.15}$$

6.2.1 The tax value function

As the title explain, we find separately in this subsection the expected accumulated discounted tax payments for the natural tax process with forced bail-out, $V^{\delta,\infty}$. First, we note that, since the tax starts when $x = \bar{x}$, then by using the strong Markov property of $(V^{\delta,\infty}, \overline{V}^{\delta,\infty})$,

$$\begin{aligned}
v_{\text{tax}}^{\delta,\infty}(x,\bar{x}) &= \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\infty} e^{-qs}\,\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^{\delta})}_s\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_{\bar{x}}^+}\,\mathbb{E}_{x,\bar{x}}\left[e^{q\tau_{\bar{x}}^+}\int_{\tau_{\bar{x}}^+}^{\infty} e^{-qs}\,\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^{\delta})}_s|\mathcal{F}_{\tau_{\bar{x}}^+}\right]\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_{\bar{x}}^+}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\infty} e^{-qs}\,\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^{\delta})}_s\right] \\
&= \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\,v_{\text{tax}}^{\delta,\infty}(\bar{x},\bar{x}),
\end{aligned} \tag{6.16}$$

where in the last equality we use Lemma 2.2.7 since the process before reaching level $\bar{x}$ is just the reflected process at zero. For convenience of calculations, we consider first that $x \leq \bar{x} \leq a$ and find the expression of

$$v_{\text{tax},a}^{\delta,\infty}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_a^+} e^{-qs}\,\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}(\overline{X+K^{\delta}})_s\right], \tag{6.17}$$

then we can take the limit as a goes to ∞ in order to get the required expression in (6.14). To find (6.17), we will use our approach explained in Section 4.3. Before we do that, we state and prove an important result which will be used in the proofs of some of the next results.

Lemma 6.2.2 *For the process*

$$V_t^{\bar{\pi}} = (X+K)_t - \int_{0+}^{t} H_s\ \mathrm{d}\overline{(X+K)}_s, \quad t \geq 0,$$

we have that

$$\overline{V}^{\bar{\pi}}_t = \overline{(X+K)}_t - \int_{0^+}^t H_s \, \mathrm{d}\overline{(X+K)}_s.$$

Moreover, $\{t \geq 0 : V^{\bar{\pi}}_t = \overline{V}^{\bar{\pi}}_t\}$ *agree precisely with* $\{t \geq 0 : (X+K)_t = \overline{(X+K)}_t\}$.

Proof The proof is similar to the one for Lemma 3.2.1.□

The next lemma gives the conditions under which a function is the tax value function given by (6.17).

Lemma 6.2.3 *Let* $V^{\delta,\infty}$ *be the natural tax process with forced bail-out given in* (6.4). *For fixed* $a > 0$, *suppose* f *is a function with domain* $D_f = (-\infty, a] \times (0, a]$ *and satisfying the following conditions:*

(I) For each $n \geq 1$, $f \in \mathcal{S}_{[-n,a]\times[\frac{1}{n},a]}$ *such that* f *is of the form,* $f(x,\bar{x}) = g(x)\,h(\bar{x})$, *where* g *and* h *satisfy the conditions of Definition 4.1.1.*

(II) For $x < 0$, $f(x,\bar{x}) = f(0,\bar{x})$.

(III) $f(a,a) = 0$.

(IV) $(\mathcal{A} - q)f(x,\bar{x}) = 0$ *for* $0 < x \leq \bar{x} \leq a$.

(V) $(\mathcal{A} - q)f(0,\bar{x}) = 0$ *for* $0 < \bar{x} \leq a$.

(VI) There exists a locally bounded density for h *such that*

$$\Gamma^\delta f(\bar{x},\bar{x}) = \delta(\bar{x}) \quad \textit{for all} \;\; 0 < \bar{x} \leq a,$$

where Γ^δ *is given by* (6.15). *Then,*

$$f(x,\bar{x}) = v^{\delta,\infty}_{tax,a}(x,\bar{x}), \quad \textit{for} \;\; x \leq \bar{x} \leq a \;\; \textit{and} \;\; \bar{x} > 0.$$

Proof Let $\widetilde{V}, \widetilde{K}$ be the right-continuous modifications of $V^{\delta,\infty}$ and K^δ. Define

$$\tau^-_{-n} := \inf\left\{t \geq 0 : \widetilde{V}_t < -n\right\}, \quad \tau^+_a := \inf\left\{t \geq 0 : \widetilde{V}_t > a\right\}$$

and

$$\kappa^-_{\frac{1}{n}} := \inf\left\{t \geq 0 : \overline{\widetilde{V}}_t < \frac{1}{n}\right\}.$$

Since $\widetilde{V}$ is greater or equal to zero, then $T = \tau_{-n}^{-} \wedge \tau_a^{+} \wedge \kappa_{\frac{1}{n}}^{-} = \tau_a^{+} \wedge \kappa_{\frac{1}{n}}^{-}$. By condition (I), $f \in \mathcal{S}_{[-n,a]\times[\frac{1}{n},a]}$, and since $\widetilde{V}$ and $\overline{\widetilde{V}}$ are semi-martingales and as $\overline{\widetilde{V}}$ is a continuous process of bounded variation, then we can use Corollary 4.1.3 and get that

$$
\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&= \int_{0+}^{t\wedge T} -q e^{-qs} f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\widetilde{V}_s \\
&\quad + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial \bar{x}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\overline{\widetilde{V}}_s + \frac{1}{2}\int_{0+}^{t\wedge T} e^{-qs} \frac{\partial^2 f}{\partial x^2}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\left[\widetilde{V}, \widetilde{V}\right]_s^c \\
&\quad + \sum_{0<s\le t} e^{-qs}\left[\Delta f(\widetilde{V}_s, \overline{\widetilde{V}}_s) - \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \Delta \widetilde{V}_s\right],
\end{aligned}
$$

where we use the notations: $\Delta\widetilde{V}_s = \widetilde{V}_s - \widetilde{V}_{s^-}$ and for a stochastic process Z, $(\widetilde{Z}_s)^c = \widetilde{Z}_s - \sum_{0<u\le s} \Delta\widetilde{Z}_u$.

Note that,

$$
\begin{aligned}
\Delta f(\widetilde{V}_s, \overline{\widetilde{V}}_s) &= f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) \\
&= f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) \\
&\quad + f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s).
\end{aligned}
$$

Also, note that

$$
\left[\widetilde{V}, \widetilde{V}\right]_s^c = \sigma^2 s. \tag{6.18}
$$

Now using definition of the operator $\mathcal{A}$ in (4.8), (6.4) and (6.18), we get

$$
\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&= M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs}\,(\mathcal{A} - q)\, f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \mathrm{d}\widetilde{K}_s \\
&\quad + \sum_{0<s\le t\wedge T} e^{-qs}\left\{f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) - \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\Delta\widetilde{K}_s\right\} \\
&\quad + \int_{0+}^{t\wedge T} e^{-qs}\left[\frac{\partial f}{\partial \bar{x}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)(1 - \delta(\overline{\widetilde{V}}_s)) - \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\delta(\overline{\widetilde{V}}_s)\right] \mathrm{d}\overline{(X + \widetilde{K})}_s,
\end{aligned} \tag{6.19}
$$

where $M_{t\wedge T}$ is the sum of the two zero mean martingales $M_{t\wedge T}^1$ and $M_{t\wedge T}^2$ given respectively by

$$
M_{t\wedge T}^1 = \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\left[X_s - \mu s - \sum_{0<u\le s} \Delta X_u \mathbf{1}_{\{|\Delta X_u|>1\}}\right],
$$

and

$$M^2_{t\wedge T} = \sum_{0<s\leq t\wedge T} e^{-qs}\Big\{f(\widetilde{V}_{s^-}+\Delta X_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) - \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \Delta X_s \,\mathbf{1}_{\{|\Delta X_s|\leq 1\}}\Big\}$$
$$- \int_{0^+}^{t\wedge T}\int_{0^+}^{\infty} e^{-qs}\Big\{f(\widetilde{V}_{s-}-\theta, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) + \theta\,\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\,\mathbf{1}_{\{0<\theta\leq 1\}}\Big\}\nu(\mathrm{d}\theta)\,\mathrm{d}s.$$

Since the last integral in (6.19) is counted only when $X+\widetilde{K} = \overline{(X+\widetilde{K})}$, which is by Lemma 6.2.2, when $\widetilde{V} = \overline{\widetilde{V}}$, and by (6.15), then we can rewrite the expansion in (6.19) as

$$\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0)\\
&\quad = M_{t\wedge T} + \int_{0^+}^{t\wedge T} e^{-qs}\,(\mathcal{A}-q) f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\,\mathrm{d}s + \int_{0^+}^{t\wedge T} e^{-qs}\,\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\,\mathrm{d}\widetilde{K}_s\\
&\qquad + \sum_{0<s\leq t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-}+\Delta X_s+\Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-}+\Delta X_s, \overline{\widetilde{V}}_s) - \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\Delta\widetilde{K}_s\right]\\
&\qquad - \int_{0^+}^{t\wedge T} e^{-qs}\left[\Gamma^\delta f(\overline{\widetilde{V}}_s, \overline{\widetilde{V}}_s)\right]\mathrm{d}\overline{(X+\widetilde{K})}_s.
\end{aligned} \tag{6.20}$$

On the right hand side of (6.20), by conditions (IV) and (V), the first integral term vanishes. Use condition (VI), and combine the second integral term together with the last term of the summation term in (6.20) and use that $(\widetilde{K}_s)^c = \widetilde{K}_s - \sum_{0<u\leq s}\Delta\widetilde{K}_u$, we get

$$\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0)\\
&\quad = M_{t\wedge T} + \int_{0^+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c\\
&\qquad + \sum_{0<s\leq t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-}+\Delta X_s+\Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-}+\Delta X_s, \overline{\widetilde{V}}_s)\right]\\
&\qquad - \int_{0^+}^{t\wedge T} e^{-qs}\,\delta(\overline{\widetilde{V}}_s)\,\mathrm{d}\overline{(X+\widetilde{K})}_s.
\end{aligned} \tag{6.21}$$

Since $\overline{(X+\widetilde{K})} = \overline{(X+K^\delta)}$, then $\overline{\widetilde{V}} = \overline{V}^{\delta,\infty}$. We define $\widetilde{V}_{0^-} := V_0^{\delta,\infty} = x$, therefore, (6.21) can be rewritten as

$$
\begin{aligned}
& e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\le s\le t\wedge T} e^{-qs} \left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\qquad - \left[f(\widetilde{V}_0, \overline{\widetilde{V}}_0) - f(\widetilde{V}_{0-}, \overline{\widetilde{V}}_0) \right] \\
&\qquad - \int_{0+}^{t\wedge T} e^{-qs}\, \delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^\delta)}_s.
\end{aligned} \tag{6.22}
$$

Since $f(\widetilde{V}_{0-}, \overline{\widetilde{V}}_0) = f(x, \bar{x})$, then (6.22) becomes

$$
\begin{aligned}
& e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x, \bar{x}) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\le s\le t\wedge T} e^{-qs} \left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\qquad - \int_{0+}^{t\wedge T} e^{-qs}\, \delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^\delta)}_s.
\end{aligned} \tag{6.23}
$$

Note first that, if $\Delta\widetilde{K}_s > 0$, by the mean value theorem, $\exists\zeta\colon \Omega \to \mathbb{R}$ such that $\zeta \in (\widetilde{V}_{s^-} + \Delta X_s, \widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s) \subseteq (-\infty, 0)$ and

$$
f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) = \frac{\partial f}{\partial x}(\zeta, \overline{\widetilde{V}}_s)\ \Delta\widetilde{K}_s. \tag{6.24}
$$

By condition (II), $\dfrac{\partial f}{\partial x}(x,y) = 0$, for $x < 0$, this implies that (6.24) becomes zero. If we are in the bounded variation case, then $(\widetilde{K}_s)^c = 0$. In the unbounded variation case, $(\widetilde{K}_s)^c$ changes when $\widetilde{V}_{s^-} = 0$, but since for each $n \geq 1$, $g \in \mathcal{C}^1[-n, a]$, then $\dfrac{\partial f}{\partial x}(x,y)|_{x=0} = \lim_{x\uparrow 0} \dfrac{\partial f}{\partial x}(x,y) = 0$, and hence this term equals zero.

Take expectation in (6.23). Note that, since $\widetilde{V}$ is greater or equal zero and by conditions on f, g is bounded on $[0,a]$ and $\overline{\widetilde{V}}$ can not go below zero, then we can use bounded convergence theorem on the left hand side of (6.23). So, by letting t and n go to infinity and using the monotone convergence theorem on the right hand side of (6.23), after taking expectation, and by condition (III) that $f(a,a) = 0$, we find that

$$
f(x, \bar{x}) = v^{\delta,\infty}_{\mathrm{tax},a}(x, \bar{x}). \tag{6.25}
$$

□

The next point is to guess the candidate expression of $v^{\delta,\infty}_{\mathrm{tax},a}(x,\bar{x})$. We use Lemma 6.2.3. First, by (6.16),

$$f(x,\bar{x}) = v^{\delta,\infty}_{\mathrm{tax},a}(x,\bar{x}) = \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\mathrm{tax},a}(\bar{x},\bar{x}). \tag{6.26}$$

Then, we use (6.26) and that f satisfies conditions (VI) and (III), so we get the following ODE

$$\frac{\partial}{\partial \bar{x}} v^{\delta,\infty}_{\mathrm{tax},a}(\bar{x},\bar{x}) - \frac{1}{1-\delta(\bar{x})}\frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\mathrm{tax},a}(\bar{x},\bar{x}) = -\frac{\delta(\bar{x})}{1-\delta(\bar{x})},$$

with the condition

$$v^{\delta,\infty}_{\mathrm{tax},a}(a,a) = 0.$$

Thus, solving this ODE by integrating factor method, we can find that

$$\begin{aligned} v^{\delta,\infty}_{\mathrm{tax},a}(\bar{x},\bar{x}) &= \exp\left\{\int_a^{\bar{x}} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \\ &\quad \times \left\{-\int_a^{\bar{x}} \exp\left\{-\int_a^{y} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y + C\right\}, \end{aligned}$$

where C is a constant. To find it, we use the initial condition and find that $C = 0$. Therefore, we get that the candidate expression should be

$$v^{\delta,\infty}_{\mathrm{tax},a}(x,\bar{x}) = \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y. \tag{6.27}$$

In the next theorem, we prove that (6.27) is the correct expression.

Theorem 6.2.4 *For $\bar{x} > 0$, let $\delta : [\bar{x},\infty) \to [0,1)$ be a natural tax rate function such that $1/(1-\delta(s))$, for all $s \in [\bar{x},\infty)$, is locally bounded. The net present value of taxation for the process $V^{\delta,\infty}$, for all $x \leq \bar{x}$, is given by*

$$v^{\delta,\infty}_{tax}(x,\bar{x}) = \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} \int_{\bar{x}}^{\infty} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y. \tag{6.28}$$

Proof Let

$$f(x,\bar{x}) = g(x)\ h(\bar{x}),$$

where $g(x) = Z^{(q)}(x)$, and

$$h(\bar{x}) = \frac{1}{Z^{(q)}(\bar{x})} \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y,$$

which can be written as

$$h(\bar{x}) = \frac{1}{Z^{(q)}(\bar{x})} \exp\left\{-\int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}$$
$$\times \int_{\bar{x}}^{a} \exp\left\{-\int_{a}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y.$$

We only need to show that f satisfies the conditions in Lemma 6.2.3, then by (6.25) and taking the limit as a goes to infinity, the statement is proved. By assumption on $1/(1-\delta(s))$, and the properties of the scale functions mentioned below Theorem 2.2.1, it is clear that for each $n \geq 1$, g and h satisfy the first and third conditions in Definition 4.1.1 on $[-n, a]$ and $[\frac{1}{n}, a]$ respectively. For the second condition of Definition 4.1.1, it is true that there exists $\lambda > 0$ such that $s \mapsto \int_{\lambda}^{\infty} Z^{(q)}(s-\theta)\ \nu(\mathrm{d}\theta)$ is bounded on $(-n, a)$ for each $n \geq 1$. This is because, by choosing $\lambda = a$, then $\theta \geq \lambda = a \geq s$. Thus, $s - \theta \leq 0$ which implies that $Z^{(q)}(s-\theta) = 1$, and from the definition of Lévy measure (see [9, p.29]), $\nu(\epsilon, \infty) < \infty$ for all $\epsilon > 0$, the condition is satisfied. So, condition (I) is verified. Since $g(x) = 1$ for $x < 0$, then (II) is satisfied. Also, as $h(a) = 0$, then $f(a, a) = 0$, this satisfies condition (III). By the martingale property (2.14) and [50, p.193], we have that for $x > 0$

$$(\mathcal{A} - q)f(x, \bar{x}) = h(\bar{x})\,(\mathcal{A} - q)Z^{(q)}(x) = 0, \tag{6.29}$$

which verifies condition (IV). By continuity of $Z^{(q)}$, right-continuity of $Z^{(q)\prime}$ and $Z^{(q)\prime\prime}$ at 0,

$$(\mathcal{A} - q)f(0, \bar{x}) = h(\bar{x})\,(\mathcal{A} - q)Z^{(q)}(0) = 0,$$

which verifies condition (V). For the last condition in Lemma 6.2.3, since

$$\frac{\partial f(x, \bar{x})}{\partial x}\Big|_{x=\bar{x}} = Z^{(q)\prime}(\bar{x})\ h(\bar{x}), \tag{6.30}$$

and

$$\frac{\partial f(x, \bar{x})}{\partial \bar{x}}\Big|_{x=\bar{x}} = Z^{(q)}(\bar{x})\ h'(\bar{x}), \tag{6.31}$$

where

$$h'(\bar{x}) = -\frac{1}{Z^{(q)}(\bar{x})} \frac{\delta(\bar{x})}{1-\delta(\bar{x})}$$
$$+ \frac{\delta(\bar{x})}{1-\delta(\bar{x})} \exp\left\{-\int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{Z^{(q)\prime}(\bar{x})}{[Z^{(q)}(\bar{x})]^2} A(\bar{x}), \tag{6.32}$$

and

$$A(\bar{x}) = \int_{\bar{x}}^{a} \exp\left\{-\int_{a}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y.$$

By (6.30), (6.31) and (6.32) we find that $\Gamma^{\delta} f(\bar{x}, \bar{x}) = \delta(\bar{x})$ for all $0 < \bar{x} \leq a$.□

6.2.2 The injection value function

As the title explained, we find in this subsection the expected accumulated discounted capital injections for the natural tax process with forced bail-out, $V^{\delta,\infty}$. It is given by

$$v_{\text{inj}}^{\delta,\infty}(x, \bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_0^{\tau_{\bar{x}}^+} e^{-qs}\, \mathrm{d}K_s^{\delta}\right] + \mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+}^{\infty} e^{-qs}\, \mathrm{d}K_s^{\delta}\right].$$

From [10, p.12], we find that

$$\mathbb{E}_{x,\bar{x}}\left[\int_0^{\tau_{\bar{x}}^+} e^{-qs}\, \mathrm{d}K_s^{\delta}\right] = -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0^+)}{q}\right], \tag{6.33}$$

where $\overline{Z}^{(q)}(x) = \int_0^x Z^{(q)}(r)\mathrm{d}r$. Then, similar to (6.16),

$$\mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+}^{\infty} e^{-qs}\, \mathrm{d}K_s^{\delta}\right] = \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\ v_{\text{inj}}^{\delta,\infty}(\bar{x}, \bar{x}),$$

therefore,

$$\begin{aligned} v_{\text{inj}}^{\delta,\infty}(x, \bar{x}) &= -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0^+)}{q}\right] \\ &\quad + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\ v_{\text{inj}}^{\delta,\infty}(\bar{x}, \bar{x}). \end{aligned} \tag{6.34}$$

For convenience of calculations, we consider first that $x \leq \bar{x} \leq a$ and find the expression of

$$v_{\text{inj},a}^{\delta,\infty}(x, \bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_0^{\tau_a^+} e^{-qs}\, \mathrm{d}K_s^{\delta}\right],$$

then we can take the limit as a goes to ∞ in order to get the required expression in (6.14). With similar steps to the previous section, we can find the net present value of injections, $v_{\text{inj}}^{\delta,\infty}$, as follows.

Lemma 6.2.5 *Let $V^{\delta,\infty}$ be the natural tax process with forced bail-out given in (6.4). For fixed $a > 0$, suppose f is a function with domain $D_f = (-\infty, a] \times (0, a]$ and satisfying the following conditions:*

(I) For each $n \geq 1$, $f \in \mathcal{S}_{[-n,a]\times[\frac{1}{n},a]}$ *such that* f *is of the form,* $f(x,\bar{x}) = f_1(x,\bar{x}) + f_2(x,\bar{x})$, *where for each* $i = 1, 2$, $f_i(x,\bar{x}) = g_i(x)\,h_i(\bar{x})$, *and each* g_i *and* h_i *satisfy the conditions of Definition 4.1.1.*

(II) For $x < 0$, *let* $f(x,\bar{x}) = -x + f(0,\bar{x})$.

(III) $f(a,a) = 0$.

(IV) $(\mathcal{A} - q)f(x,\bar{x}) = 0$ *for* $0 < x \leq \bar{x} \leq a$.

(V) $(\mathcal{A} - q)f(0,\bar{x}) = 0$ *for* $0 < \bar{x} \leq a$.

(VI) There exists a locally bounded density for each h_i *such that*

$$\Gamma^\delta f(\bar{x},\bar{x}) = 0 \quad \textit{for all} \quad 0 < \bar{x} \leq a,$$

where Γ^δ *is given by* (6.15). *Then,*

$$f(x,\bar{x}) = v^{\delta,\infty}_{\mathit{inj},a}(x,\bar{x}), \ \textit{for } x \leq \bar{x} \leq a \textit{ and } \bar{x} > 0. \tag{6.35}$$

Proof We follow the same steps as in the proof of Lemma 6.2.3 up to (6.20). Use that $(\widetilde{K}_s)^c = \widetilde{K}_s - \sum_{0<u\leq s} \Delta\widetilde{K}_u$, we get

$$\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs}(\mathcal{A}-q)f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0<s\leq t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s)\right] \\
&\qquad - \int_{0+}^{t\wedge T} e^{-qs}\left[\Gamma^\delta f(\overline{\widetilde{V}}_s, \overline{\widetilde{V}}_s)\right]\mathrm{d}\overline{(X+\widetilde{K})}_s.
\end{aligned}$$

Use conditions (IV), (V) and (VI), and that $f(\widetilde{V}_{0^-}, \overline{\widetilde{V}}_0) = f(x,\bar{x})$ to get

$$\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x,\bar{x}) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\leq s\leq t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s)\right],
\end{aligned} \tag{6.36}$$

where $\Delta\widetilde{K}_0 = \widetilde{K}_0 - \widetilde{K}_{0^-}$ such that $\widetilde{K}_{0^-} := 0$. Note first that, if $\Delta\widetilde{K}_s > 0$, by the mean value theorem, $\exists\zeta\colon \Omega \to \mathbb{R}$ such that $\zeta \in (\widetilde{V}_{s^-} + \Delta X_s, \widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s) \subseteq (-\infty, 0)$

and

$$f(\widetilde{V}_{s-}+\Delta X_s+\Delta\widetilde{K}_s,\overline{\widetilde{V}}_s)-f(\widetilde{V}_{s-}+\Delta X_s,\overline{\widetilde{V}}_s)=\frac{\partial f}{\partial x}(\zeta,\overline{\widetilde{V}}_s)\,\Delta\widetilde{K}_s. \tag{6.37}$$

By condition (II), for any $x<0$, $\frac{\partial f}{\partial x}(x,\bar{x})=-1$, this implies that the summation term in (6.36) becomes $-\sum_{0\le s\le t\wedge T}e^{-qs}\Delta\widetilde{K}_s$. Then the summation term becomes $-\sum_{0<s\le t\wedge T}e^{-qs}\Delta\widetilde{K}_s-\widetilde{K}_0$. If we are in the bounded variation case, then $(\widetilde{K}_s)^c=0$, that is, $\widetilde{K}_s=\sum_{0<u\le s}\Delta\widetilde{K}_u$, and hence, in the right hand side of (6.36), we have the integral term

$$-\int_{0+}^{t\wedge T}e^{-qs}\mathrm{d}\widetilde{K}_s.$$

In the unbounded variation case, $(\widetilde{K}_s)^c$ changes when $\widetilde{V}_{s-}=0$, but since for each $n\ge 1$, $g_1,g_2\in\mathcal{C}^1[-n,a]$, $\frac{\partial f}{\partial x}(x,y)|_{x=0}=\lim_{x\uparrow 0}\frac{\partial f}{\partial x}(x,y)|_{x<0}=-1$, and therefore, we get the term

$$-\int_{0+}^{t\wedge T}e^{-qs}\mathrm{d}(\widetilde{K}_s)^c.$$

Add this term to the one resulted from (6.37), then also in this case, we have the integral term

$$-\int_{0+}^{t\wedge T}e^{-qs}\mathrm{d}\widetilde{K}_s.$$

Since

$$\int_{0+}^{t\wedge T}e^{-qs}\mathrm{d}\widetilde{K}_s=\int_{0}^{t\wedge T}e^{-qs}\mathrm{d}K^\delta_s-K^\delta_{0+},$$

then (6.36) becomes

$$\begin{aligned}&e^{-q(t\wedge T)}f(\widetilde{V}_{t\wedge T},\overline{\widetilde{V}}_{t\wedge T})-f(x,\bar{x})\\&\quad=M_{t\wedge T}-\int_0^{t\wedge T}e^{-qs}\mathrm{d}K^\delta_s+K^\delta_{0+}-\widetilde{K}_0.\end{aligned}$$

Since $K^\delta_{0+}=\widetilde{K}_0$, then by taking expectation, letting t and n go to infinity, using the bounded convergence theorem on the left hand side and the monotone convergence theorem on the right hand side, and by condition (III) that $f(a,a)=0$, we find that

$$f(x,\bar{x})=v^{\delta,\infty}_{\mathrm{inj},a}(x,\bar{x}).$$

□

The next point is to guess the candidate expression of $v^{\delta,\infty}_{\text{inj},a}$. For that, we use (6.35) and (6.34), so that

$$f(x,\bar{x}) = v^{\delta,\infty}_{\text{inj},a}(x,\bar{x}) = -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0^+)}{q}\right] + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}). \tag{6.38}$$

Now we find that

$$\frac{\partial}{\partial x} v^{\delta,\infty}_{\text{inj},a}(x,\bar{x})|_{x=\bar{x}} = -Z^{(q)}(\bar{x}) + \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0+)}{q}\right] + \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}), \tag{6.39}$$

and

$$\frac{\partial}{\partial \bar{x}} v^{\delta,\infty}_{\text{inj},a}(x,\bar{x})|_{x=\bar{x}} = -\frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0+)}{q}\right] + Z^{(q)}(\bar{x}) - \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) + \frac{\partial}{\partial \bar{x}} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}). \tag{6.40}$$

Then, we use (6.38), (6.39), (6.40) and that f satisfies conditions (VI) and (III), so we get the following ODE

$$\frac{\partial}{\partial \bar{x}} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) - \frac{1}{1-\delta(\bar{x})}\frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) = -\frac{1}{1-\delta(\bar{x})} Z^{(q)}(\bar{x}) + \frac{1}{1-\delta(\bar{x})}\frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0+)}{q}\right],$$

with the condition

$$v^{\delta,\infty}_{\text{inj},a}(a,a) = 0.$$

Thus, solving this ODE by integrating method, we can get the solution

$$v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) = \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)}\left[Z^{(q)}(y) - \frac{Z^{(q)\prime}(y)}{Z^{(q)}(y)}\left(\overline{Z}^{(q)}(y) + \frac{\psi'(0+)}{q}\right)\right] \times \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\,\mathrm{d}s\right\}\mathrm{d}y. \tag{6.41}$$

We consider (6.41) as the first expression of $v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x})$. Then, we can write (6.41) as

$$\begin{aligned} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) &= \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)} Z^{(q)}(y) \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \,\mathrm{d}s\right\} \mathrm{d}y \\ &\quad - \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)} \frac{Z^{(q)\prime}(y)}{Z^{(q)}(y)} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \\ &\quad \times \left(\overline{Z}^{(q)}(y) + \frac{\psi'(0+)}{q}\right) \mathrm{d}y \\ &= \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)} Z^{(q)}(y) \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \mathrm{d}y \\ &\quad + \left\{\left(\overline{Z}^{(q)}(y) + \frac{\psi'(0+)}{q}\right) \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}\Big|_{y=\bar{x}}^{y=a} \right. \\ &\quad \left. - \int_{\bar{x}}^{a} Z^{(q)}(y) \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \mathrm{d}y\right\}, \end{aligned}$$

where we use integration by parts in the second integral of the first equality to get a second expression of $v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x})$,

$$\begin{aligned} v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x}) = &\left[\overline{Z}^{(q)}(a) + \frac{\psi'(0+)}{q}\right] \exp\left\{-\int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} - \left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0+)}{q}\right] \\ &+ \int_{\bar{x}}^{a} \frac{\delta(y)}{1-\delta(y)} Z^{(q)}(y) \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \mathrm{d}y. \end{aligned} \tag{6.42}$$

Since

$$\frac{1}{1-\delta(s)} = \frac{\delta(s)}{1-\delta(s)} + 1,$$

then

$$\begin{aligned} &\exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \\ &\quad = \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \exp\left\{-\int_{\bar{x}}^{y} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \\ &\quad = \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)} \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}. \end{aligned} \tag{6.43}$$

So, we can use (6.43) to rewrite (6.42) and get a third expression of $v^{\delta,\infty}_{\text{inj},a}(\bar{x},\bar{x})$, which we consider it as the candidate expression.

That is, the candidate expression for $v^{\delta,\infty}_{\mathrm{inj},a}(x,\bar{x})$ is given by

$$\begin{aligned} v^{\delta,\infty}_{\mathrm{inj},a}(x,\bar{x}) = &- \left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] \\ &+ Z^{(q)}(x) \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y \\ &+ Z^{(q)}(x) \left\{\frac{\psi'(0+)}{q} \frac{1}{Z^{(q)}(a)} + \frac{\overline{Z}^{(q)}(a)}{Z^{(q)}(a)}\right\} \exp\left\{-\int_{\bar{x}}^{a} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\, \mathrm{d}s\right\}. \end{aligned} \tag{6.44}$$

In the next theorem, we prove that (6.44) is the correct expression.

Theorem 6.2.6 *Let $\psi'(0+) > -\infty$ and for $\bar{x} > 0$, let $\delta : [\bar{x}, \infty) \to [0, 1)$ be a natural tax rate function such that $1/(1 - \delta(s))$, for all $s \in [\bar{x}, \infty)$, is locally bounded. The net present value of capital injections for the process $V^{\delta,\infty}$, for all $x \le \bar{x}$, is given by*

$$\begin{aligned} v^{\delta,\infty}_{inj}(x,\bar{x}) = &- \left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] \\ &+ Z^{(q)}(x) \int_{\bar{x}}^{\infty} \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y \\ &+ \frac{Z^{(q)}(x)}{\Phi(q)} \exp\left\{-\int_{\bar{x}}^{\infty} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}. \end{aligned} \tag{6.45}$$

Proof Let

$$\begin{aligned} f(x,\bar{x}) = &- \left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] \\ &+ Z^{(q)}(x) \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y \\ &+ Z^{(q)}(x) \left\{\frac{\psi'(0+)}{q} \frac{1}{Z^{(q)}(a)} + \frac{\overline{Z}^{(q)}(a)}{Z^{(q)}(a)}\right\} \exp\left\{-\int_{\bar{x}}^{a} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}. \end{aligned}$$

We only need to show that f satisfies the conditions in Lemma 6.2.5, which implies that

$$f(x,\bar{x}) = v^{\delta,\infty}_{\mathrm{inj},a}(x,\bar{x}),$$

and hence,

$$v^{\delta,\infty}_{\mathrm{inj}}(x,\bar{x}) = \lim_{a\uparrow\infty} v^{\delta,\infty}_{\mathrm{inj},a}(x,\bar{x}).$$

Note that f can be written as

$$f(x,\bar{x}) = g_1(x)\ h_1(\bar{x}) + g_2(x)\ h_2(\bar{x}),$$

where

$$g_1(x) = \left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right],\ h_1(\bar{x}) = -1,\ g_2(x) = Z^{(q)}(x),$$

and

$$\begin{aligned} h_2(\bar{x}) = &\int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y \\ &+ \left\{\frac{\psi'(0+)}{q} \frac{1}{Z^{(q)}(a)} + \frac{\overline{Z}^{(q)}(a)}{Z^{(q)}(a)}\right\} \exp\left\{-\int_{\bar{x}}^{a} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\}, \end{aligned}$$

which can be written as

$$h_2(\bar{x}) = \exp\left\{-\int_{\bar{x}}^{a} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} A(\bar{x}),$$

where

$$\begin{aligned} A(\bar{x}) = &\int_{\bar{x}}^{a} \exp\left\{-\int_{a}^{y} \frac{\delta(s)}{1-\delta(s)} \frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)} \mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y \\ &+ \left\{\frac{\psi'(0+)}{q} \frac{1}{Z^{(q)}(a)} + \frac{\overline{Z}^{(q)}(a)}{Z^{(q)}(a)}\right\}. \end{aligned}$$

Now, following the same argument as in the proof of Lemma (6.2.4), we find that for each $n \geq 1$, each g_i and h_i for $i = 1, 2$ satisfies the first and third conditions of Definition 4.1.1 on $[-n, a]$ and $[\frac{1}{n}, a]$, respectively. For the second condition of Definition 4.1.1, we should verify that there exists $\lambda > 0$ such that $s \mapsto \int_{\lambda}^{\infty} \overline{Z}^{(q)}(s - \theta)\ \nu(\mathrm{d}\theta)$ is bounded on $(-n, a)$ for each $n \geq 1$. We choose $\lambda = \max\{a, 1\}$, so $\theta \geq \lambda \geq s$ for all $s \in (-n, a)$ and hence $s - \theta \leq 0$. Since

$$\overline{Z}^{(q)}(x) = \int_0^x Z^{(q)}(z)\mathrm{d}z = x + q\int_0^x \int_0^z W^{(q)}(w)\mathrm{d}w\, \mathrm{d}z,$$

so $\overline{Z}^{(q)}(x) = x$ for $x < 0$. Therefore, we find that

$$\begin{aligned} \int_{\lambda}^{\infty} \overline{Z}^{(q)}(s-\theta)\ \nu(\mathrm{d}\theta) = \int_{\lambda}^{\infty} (s-\theta)\ \nu(\mathrm{d}\theta) &= -\int_{\lambda}^{\infty} (\theta - s)\ \nu(\mathrm{d}\theta) \\ &< \int_{\lambda}^{\infty} \theta\ \nu(\mathrm{d}\theta) \\ &\leq \int_{1}^{\infty} \theta\ \nu(\mathrm{d}\theta), \end{aligned}$$

where the last integral is finite because the assumption $\psi'(0+) > -\infty$ is equivalent to $\int_1^{\infty} \theta\ \nu(\mathrm{d}\theta) < \infty$. Since $\int_{\lambda}^{\infty} \overline{Z}^{(q)}(s-\theta)\ \nu(\mathrm{d}\theta)$ is bounded below by 0, then we proved that for $s \in (-n, a)$,

$$\left|\int_{\lambda}^{\infty} \overline{Z}^{(q)}(s-\theta)\ \nu(\mathrm{d}\theta)\right| < \infty.$$

For condition (III), it is clear that $f(a,a) = 0$. Since for $x < 0$, $\overline{Z}^{(q)}(x) = x$ and $Z^{(q)}(x) = 1$, then condition (II) is satisfied.

For any $x \in (0,\infty)$, by (6.29), we have that

$$(\mathcal{A} - q)Z^{(q)}(x) = 0,$$

and from Lemma (2.2.8) and [13, p.367 (Step 2)], we also have

$$(\mathcal{A} - q)\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] = 0.$$

Therefore, condition (IV) is satisfied. Also, by right-continuity of $Z^{(q)}$ and $\overline{Z}^{(q)}(x)$, and their first and second derivatives, at 0, we see that condition (V) is satisfied.

For condition (VI), we first find

$$\begin{aligned}\frac{\partial f(x,\bar{x})}{\partial x}|_{x=\bar{x}} &= [g_1'(x)\ h_1(\bar{x}) + g_2'(x)\ h_2(\bar{x})]\,|_{x=\bar{x}}\\ &= -Z^{(q)}(\bar{x}) + Z^{(q)\prime}(\bar{x})\ h_2(\bar{x}),\end{aligned}\tag{6.46}$$

and

$$\begin{aligned}\frac{\partial f(x,\bar{x})}{\partial \bar{x}}|_{x=\bar{x}} &= [g_1(x)\ h_1'(\bar{x}) + g_2(x)\ h_2'(\bar{x})]\,|_{x=\bar{x}}\\ &= Z^{(q)}(\bar{x})\ h_2'(\bar{x}),\end{aligned}\tag{6.47}$$

where

$$h_2'(\bar{x}) = \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\frac{\delta(\bar{x})}{1-\delta(\bar{x})}\exp\left\{-\int_{\bar{x}}^{a}\frac{\delta(s)}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\mathrm{d}s\right\}A(\bar{x}) - \frac{\delta(\bar{x})}{1-\delta(\bar{x})}. \tag{6.48}$$

Then, by (6.46), (6.47) and (6.48) we find that $\Gamma^{\delta} f(\bar{x},\bar{x}) = 0.\square$

We conclude this section with the net present value of profit for the process (6.4), for any $x \le \bar{x}$ and $\bar{x} > 0$,

$$\begin{aligned}v^{\delta,\infty}(x,\bar{x}) &= v^{\delta,\infty}_{\text{tax}}(x,\bar{x}) - \eta\, v^{\delta,\infty}_{\text{inj}}(x,\bar{x})\\ &= \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\int_{\bar{x}}^{\infty}\exp\left\{-\int_{\bar{x}}^{y}\frac{1}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\mathrm{d}s\right\}\frac{\delta(y)}{1-\delta(y)}\mathrm{d}y\\ &\quad -\eta\Bigg\{-\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right]\\ &\qquad + Z^{(q)}(x)\int_{\bar{x}}^{\infty}\exp\left\{-\int_{\bar{x}}^{y}\frac{\delta(s)}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\mathrm{d}s\right\}\frac{\delta(y)}{1-\delta(y)}\mathrm{d}y\\ &\quad + \frac{Z^{(q)}(x)}{\Phi(q)}\exp\left\{-\int_{\bar{x}}^{\infty}\frac{\delta(s)}{1-\delta(s)}\frac{Z^{(q)\prime}(s)}{Z^{(q)}(s)}\mathrm{d}s\right\}\Bigg\}.\end{aligned}\tag{6.49}$$

Remark 13 Note that, for $\delta(s)=\gamma\in(0,1)$, and with $x=\bar{x}>0$, (6.49) equals

$$
\begin{aligned}
v^{\gamma,\infty}(x,x) &= \frac{\gamma}{1-\gamma}\int_x^\infty \left[\frac{Z^{(q)}(x)}{Z^{(q)}(y)}\right]^{1/(1-\gamma)} \mathrm{d}y \\
&\quad - \eta\left\{-\left[\overline{Z}^{(q)}(x)+\frac{\psi'(0+)}{q}\right]+Z^{(q)}(x)\frac{\gamma}{1-\gamma}\int_x^\infty \left[\frac{Z^{(q)}(x)}{Z^{(q)}(y)}\right]^{\gamma/(1-\gamma)} \mathrm{d}y\right\}.
\end{aligned}
\tag{6.50}
$$

We found that (6.50) agrees with the result of [4, Theorem 2], which can be shown by a long tedious computations when taking the limit as $\theta\to 0$ in [4, Theorem 2]. Moreover, (6.50) also agrees with a recent study in [64, p.14], when we use the first candidate expression (6.41) with a constant tax rate γ.

Lemma 6.2.7 *For $\gamma\in(0,1)$ and $x>0$,*

$$
\lim_{\gamma\downarrow 0}\frac{\gamma}{1-\gamma}\int_x^\infty \left[\frac{Z^{(q)}(x)}{Z^{(q)}(y)}\right]^{\gamma/(1-\gamma)} \mathrm{d}y = \frac{1}{\Phi(q)}.
$$

Proof Let $\dfrac{\gamma}{1-\gamma}=\alpha$ and $f(y)=\dfrac{Z^{(q)}(x)}{Z^{(q)}(y)}$, then f is decreasing and $f(x)=1$. Then $[f(y)]^\alpha = e^{-\alpha(-\log f(y))}$. Let $U=-\log f(y)$, then as

$$
\frac{\mathrm{d}U}{\mathrm{d}y} = -\frac{f'(y)}{f(y)} > 0,
$$

U has an inverse say $y=g(U)$ so that $g(0)=x$ and $g(\infty)=\infty$. So, substituting this and using integral by parts we get that

$$
\begin{aligned}
\lim_{\gamma\downarrow 0}\frac{\gamma}{1-\gamma}\int_x^\infty \left[\frac{Z^{(q)}(x)}{Z^{(q)}(y)}\right]^{\gamma/(1-\gamma)} \mathrm{d}y &= \lim_{\alpha\downarrow 0}\alpha\int_x^\infty [f(y)]^\alpha\,\mathrm{d}y \\
&= \lim_{\alpha\downarrow 0}\int_0^\infty -\alpha e^{-\alpha U}\frac{f(g(U))}{f'(g(U))}\mathrm{d}U \\
&= \lim_{\alpha\downarrow 0}\left\{\frac{f(g(U))}{f'(g(U))}e^{-\alpha U}\Big|_{U=0}^{U=\infty}\right. \\
&\quad \left. - \int_0^\infty e^{-\alpha U}\left[\frac{f(g(U))}{f'(g(U))}\right]'\mathrm{d}U\right\} \\
&= -\lim_{y\downarrow 0}\frac{f(y)}{f'(y)} \\
&= \frac{1}{\Phi(q)}.
\end{aligned}
$$

□

Remark 14 From Lemma 6.2.6, Remark 13 and Lemma 6.2.7, the net present value of capital injections without taxation when $x = \bar{x} > 0$ is given by

$$v_{\text{inj}}^{0,\infty}(x,x) = -\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] + \frac{Z^{(q)}(x)}{\Phi(q)}. \tag{6.51}$$

We note that (6.51) agrees with the result in [4, Remark in p.4]. Also, by using (2.10), it agrees with [10, (4.4)].

6.3 Optimal control problem

In this section, we solve the optimal control problem that we introduced in (6.2).

Lemma 6.3.1 ***(The verification lemma)***

Let $V^{\bar{\pi}}$ be a tax process with forced bail-out as given by (6.1). Let $\hat{\pi} := (\hat{H}, \hat{K})$ be an admissible policy such that, for each $n \geq 1$, $v^{\hat{\pi}} \in \mathcal{S}_{[-n,n]\times[\frac{1}{n},n]}$. Suppose that $v^{\hat{\pi}}$ satisfies the following conditions:

(I) $v^{\hat{\pi}}(x,\bar{x}) = \eta x + v^{\hat{\pi}}(0,\bar{x})$, *for* $x < 0$,

(II) $(\mathcal{A} - q)v^{\hat{\pi}}(x,\bar{x}) = 0$, *for all* $0 < x \leq \bar{x}$,

(III) $(\mathcal{A} - q)v^{\hat{\pi}}(0,\bar{x}) = 0$, *for all* $0 < \bar{x}$,

(IV) There exist Radon-Nikodým derivatives $\dfrac{\partial v^{\hat{\pi}}}{\partial x}$ *and* $\dfrac{\partial v^{\hat{\pi}}}{\partial \bar{x}}$ *such that*

$$H\frac{\partial v^{\hat{\pi}}}{\partial x}(\bar{x},\bar{x}) - (1-H)\frac{\partial v^{\hat{\pi}}}{\partial \bar{x}}(\bar{x},\bar{x}) \geq H, \quad \text{for all } H \in [\alpha,\beta] \text{ and all } \bar{x} > 0.$$

(V) $\dfrac{\partial v^{\hat{\pi}}}{\partial x}(x,\bar{x}) \leq \eta$, *for all* $0 < x \leq \bar{x}$.

Then, $v^{\hat{\pi}}(x,\bar{x}) = v^*(x,\bar{x})$ *for all* $x \leq \bar{x}$ *and* $\bar{x} > 0$.

Proof Fix $\bar{\pi} := (H, K)$ to be an admissible strategy. We let $\widetilde{V}$ and $\widetilde{K}$ to be the right-continuous modifications of $V^{\bar{\pi}}$ and K, respectively. Define

$$\tau_{-n}^{-} := \inf\left\{t \geq 0 : \widetilde{V}_t < -n\right\}, \quad \tau_n^{+} := \inf\left\{t \geq 0 : \widetilde{V}_t > n\right\}$$

and

$$\kappa_{\frac{1}{n}}^{-} := \inf\left\{t \geq 0 : \overline{\widetilde{V}}_t < \frac{1}{n}\right\}.$$

Let $T = \tau_{-n}^{-} \wedge \tau_{n}^{+} \wedge \kappa_{\frac{1}{n}}^{-} = \tau_{n}^{+} \wedge \kappa_{\frac{1}{n}}^{-}$, as $\widetilde{V}$ always greater or equal to zero. Since $v^{\hat{\pi}} \in \mathcal{S}_{[-n,n]\times[\frac{1}{n},n]}$, $\widetilde{V}$ and $\overline{\widetilde{V}}$ are semi-martingales, and as $\overline{\widetilde{V}}$ is a continuous process of bounded variation, then we can use Corollary 4.1.3 to get that,

$$\begin{aligned}
&e^{-q(t\wedge T)}\, v^{\hat{\pi}}(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - v^{\hat{\pi}}(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&= \int_{0+}^{t\wedge T} -qe^{-qs}\, v^{\hat{\pi}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\widetilde{V}_s \\
&\quad + \int_{0+}^{t\wedge T} e^{-qs}\, \frac{\partial v^{\hat{\pi}}}{\partial \bar{x}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\overline{\widetilde{V}}_s + \frac{1}{2}\int_{0+}^{t\wedge T} e^{-qs}\frac{\partial^2 v^{\hat{\pi}}}{\partial x^2}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\left[\widetilde{V}, \widetilde{V}\right]_s^c \\
&\quad + \sum_{0<s\le t} e^{-qs}\left[\Delta v^{\hat{\pi}}(\widetilde{V}_s, \overline{\widetilde{V}}_s) - \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \Delta\widetilde{V}_s\right].
\end{aligned}$$

since

$$\begin{aligned}
\Delta v^{\hat{\pi}}(\widetilde{V}_s, \overline{\widetilde{V}}_s) &= v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) \\
&= v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) \\
&\quad + v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s).
\end{aligned}$$

Also, note that

$$\left[\widetilde{V}, \widetilde{V}\right]_s^c = \sigma^2 s. \tag{6.52}$$

Now using the operator $\mathcal{A}$ definition in (4.8), (6.1) and (6.52), we get

$$\begin{aligned}
&e^{-q(t\wedge T)}v^{\hat{\pi}}(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - v^{\hat{\pi}}(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&= M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs}\,(\mathcal{A} - q)\, v^{\hat{\pi}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}s + \int_{0+}^{t\wedge T} e^{-qs}\, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\widetilde{K}_s \\
&\quad + \sum_{0<s\le t\wedge T} e^{-qs}\Big\{v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) - \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \Delta\widetilde{K}_s\Big\} \\
&\quad - \int_{0+}^{t\wedge T} e^{-qs}\left[\frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)H_s - \frac{\partial v^{\hat{\pi}}}{\partial \bar{x}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)(1 - H_s)\right]\mathrm{d}\overline{(X + \widetilde{K})}_s,
\end{aligned}$$

where $M_{t\wedge T}$ is the sum of the two zero mean martingales $M^1_{t\wedge T}$ and $M^2_{t\wedge T}$ given respectively by

$$M^1_{t\wedge T} = \int_{0+}^{t\wedge T} e^{-qs}\, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathrm{d}\left[X_s - \mu s - \sum_{0<u\le s} \Delta X_u \mathbf{1}_{\{|\Delta X_u|>1\}}\right],$$

and

$$\begin{aligned}
M^2_{t\wedge T} &= \sum_{0<s\le t\wedge T} e^{-qs}\Big\{v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-}, \overline{\widetilde{V}}_s) - \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \Delta X_s\, \mathbf{1}_{\{|\Delta X_s|\le 1\}}\Big\} \\
&\quad - \int_{0+}^{t\wedge T}\int_{0+}^{\infty} e^{-qs}\Big\{v^{\hat{\pi}}(\widetilde{V}_{s-} - \theta, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) + \theta\, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\, \mathbf{1}_{\{0<\theta\le 1\}}\Big\}\nu(\mathrm{d}\theta)\, \mathrm{d}s,
\end{aligned}$$

where we used the Lévy-Itô decomposition (2.3) and the compensation formula in 2.1.5, respectively. Then, use that $(\widetilde{K}_s)^c = \widetilde{K}_s - \sum_{0<u\leq s} \Delta\widetilde{K}_u$, and get that

$$
\begin{aligned}
&e^{-q(t\wedge T)} v^{\hat{\pi}}(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - v^{\hat{\pi}}(\widetilde{V}_0, \overline{\widetilde{V}}_0) \\
&\quad = M_{t\wedge T} + \int_{0^+}^{t\wedge T} e^{-qs} \, (\mathcal{A}-q) \, v^{\hat{\pi}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \, \mathrm{d}s + \int_{0^+}^{t\wedge T} e^{-qs} \, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \, \mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\leq s\leq t\wedge T} e^{-qs} \left[v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\qquad - \left[v^{\hat{\pi}}(\widetilde{V}_0, \overline{\widetilde{V}}_0) - v^{\hat{\pi}}(\widetilde{V}_{0^-}, \overline{\widetilde{V}}_0) \right] \\
&\qquad - \int_{0^+}^{t\wedge T} e^{-qs} \left[\frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) H_s - \frac{\partial v^{\hat{\pi}}}{\partial \bar{x}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)(1-H_s) \right] \mathrm{d}\overline{(X+\widetilde{K})}_s. \qquad (6.53)
\end{aligned}
$$

Since $v^{\hat{\pi}}(\widetilde{V}_{0^-}, \overline{\widetilde{V}}_0) = v^{\hat{\pi}}(x, \bar{x})$ and by Lemma 6.2.2, the last integral in (6.53) is counted only when $\widetilde{V} = \overline{\widetilde{V}}$, then we can rewrite (6.53) as

$$
\begin{aligned}
v^{\hat{\pi}}(x, \bar{x}) &= -M_{t\wedge T} + e^{-q(t\wedge T)} \, v^{\hat{\pi}}(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - \int_{0^+}^{t\wedge T} e^{-qs} \, (\mathcal{A}-q) \, v^{\hat{\pi}}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \, \mathrm{d}s \\
&\quad - \int_{0^+}^{t\wedge T} e^{-qs} \, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \, \mathrm{d}(\widetilde{K}_s)^c \\
&\quad - \sum_{0\leq s\leq t\wedge T} e^{-qs} \left[v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\quad + \int_{0^+}^{t\wedge T} e^{-qs} \left[\frac{\partial v^{\hat{\pi}}}{\partial x}(\overline{\widetilde{V}}_{s-}, \overline{\widetilde{V}}_s) H_s - \frac{\partial v^{\hat{\pi}}}{\partial \bar{x}}(\overline{\widetilde{V}}_{s-}, \overline{\widetilde{V}}_s)(1-H_s) \right] \mathrm{d}\overline{(X+\widetilde{K})}_s.
\end{aligned}
$$

Use that $v^{\hat{\pi}} \geq 0$, conditions (II), (III) and (IV), then we get

$$
\begin{aligned}
v^{\hat{\pi}}(x, \bar{x}) &\geq -M_{t\wedge T} - \int_{0^+}^{t\wedge T} e^{-qs} \, \frac{\partial v^{\hat{\pi}}}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \, \mathrm{d}(\widetilde{K}_s)^c \\
&\quad - \sum_{0\leq s\leq t\wedge T} e^{-qs} \left[v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\quad + \int_{0^+}^{t\wedge T} e^{-qs} \, H_s \mathrm{d}\overline{(X+\widetilde{K})}_s.
\end{aligned}
$$

Note first that, if $\Delta\widetilde{K}_s > 0$, by the mean value theorem,

$$
v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - v^{\hat{\pi}}(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) = \frac{\partial v^{\hat{\pi}}}{\partial x}(\zeta, \overline{\widetilde{V}}_s) \, \Delta\widetilde{K}_s, \qquad (6.54)
$$

where $\zeta \colon \Omega \to \mathbb{R}$. If $\zeta < 0$, then by condition (I), we have that $\dfrac{\partial v^{\hat{\pi}}}{\partial x}(\zeta, y) = \eta$, and if $\zeta > 0$, then we can use condition (V). In both cases, this implies that the summation term becomes

$$
-\eta \sum_{0\leq s\leq t\wedge T} e^{-qs} \Delta\widetilde{K}_s = -\eta \sum_{0<s\leq t\wedge T} e^{-qs} \Delta\widetilde{K}_s - \eta\widetilde{K}_0.
$$

If we are in the bounded variation case, then $(\widetilde{K}_s)^c = 0$, that is, $\widetilde{K}_s = \sum_{0<u\leq s} \Delta\widetilde{K}_u$, and hence, in the right hand side we get the term

$$-\eta\left[\int_{0+}^{t\wedge T} e^{-qs}\mathrm{d}\widetilde{K}_s + \widetilde{K}_0\right] = -\eta\int_0^{t\wedge T} e^{-qs}\mathrm{d}K_s,$$

and this is because $K_{0+} = \widetilde{K}_0$. In the unbounded variation case, $(\widetilde{K}_s)^c$ changes when $\widetilde{V}_{s^-} = 0$, but since for each $n \geq 1$, from the conditions on $v^{\hat{\pi}}$ as it belongs to the space $\mathcal{S}_{[-n,n]\times[\frac{1}{n},n]}$, then it is a finite sum of multiple functions $g(x)\,h(\bar{x})$ such that $g \in \mathcal{C}^1[-n,n]$, then $\dfrac{\partial v^{\hat{\pi}}}{\partial x}(x,y)|_{x=0} = \lim_{x\uparrow 0}\dfrac{\partial v^{\hat{\pi}}}{\partial x}(x,y) = \eta$, and therefore, we get the term

$$-\eta\int_{0+}^{t\wedge T} e^{-qs}\mathrm{d}(\widetilde{K}_s)^c.$$

Add this term to the one resulted from (6.54), similarly we get the term

$$-\eta\int_0^{t\wedge T} e^{-qs}\mathrm{d}K_s.$$

Then, take the expectation, let t and n go to infinity, and by monotone convergence theorem we get that

$$v^{\hat{\pi}}(x,\bar{x}) \geq E_{x,\bar{x}}\left[\int_{0+}^{\infty} e^{-qs}H_s\mathrm{d}(\overline{X+K})_s - \eta\int_0^{\infty} e^{-qs}\mathrm{d}K_s\right].$$

That is, we proved that for any $\bar{\pi} \in \overline{\Pi}$, and for all $0 < x \leq \bar{x}$, $v^{\hat{\pi}}(x,\bar{x}) \geq v^{\bar{\pi}}(x,\bar{x})$. Hence,

$$v^{\hat{\pi}}(x,\bar{x}) \geq \sup_{\bar{\pi}\in\overline{\Pi}} v^{\bar{\pi}}(x,\bar{x}) = v^*(x,\bar{x}).$$

Since from the definition of v^*, we have that $v^{\hat{\pi}}(x,\bar{x}) \leq v^*(x,\bar{x})$, therefore, $v^{\hat{\pi}}(x,\bar{x}) = v^*(x,\bar{x})$ for all $0 < x \leq \bar{x}$.

For the case $(0,\bar{x})$, where $\bar{x} > 0$, since v^* is an increasing function in the initial capital and as $v^{\hat{\pi}}$ is right-continuous at $(0,\bar{x})$ by the assumptions, we have the following:

$$v^*(0,\bar{x}) \leq \lim_{x\downarrow 0} v^*(x,\bar{x}) \leq \lim_{x\downarrow 0} v^{\hat{\pi}}(x,\bar{x}) = v^{\hat{\pi}}(0,\bar{x}).$$

Hence, the proof is complete.□

Next, we find the expression for the net present value of profit for the process $V^{\delta,\infty}$, when the tax is given by the piecewise constant function (the $(\alpha \longmapsto \beta)$-tax strategy at level b) defined in (5.5).

Case 1: $\alpha = 0$ and $\beta > 0$

By (6.49), we find that for the case $0 < \bar{x} \leq b$, the value function $v^{\bar{\pi}^b}(x, \bar{x})$, where $\bar{\pi}^b := (\delta^b, K^{\delta^b})$, is given by

$$\begin{aligned} v^{\bar{\pi}^b}(x, \bar{x}) = & \frac{Z^{(q)}(x)}{Z^{(q)}(b)} \frac{\beta}{1-\beta} \int_b^\infty \left[\frac{Z^{(q)}(b)}{Z^{(q)}(y)}\right]^{1/(1-\beta)} \mathrm{d}y \\ & - \eta \left\{ -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + Z^{(q)}(x) \frac{\beta}{1-\beta} \int_b^\infty \left[\frac{Z^{(q)}(b)}{Z^{(q)}(y)}\right]^{\beta/(1-\beta)} \mathrm{d}y \right\}, \end{aligned}$$

and for the case $\bar{x} > b$

$$\begin{aligned} v^{\bar{\pi}^b}(x, \bar{x}) = & \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} \frac{\beta}{1-\beta} \int_{\bar{x}}^\infty \left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{1/(1-\beta)} \mathrm{d}y \\ & - \eta \left\{ -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + Z^{(q)}(x) \frac{\beta}{1-\beta} \int_{\bar{x}}^\infty \left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{\beta/(1-\beta)} \mathrm{d}y \right\}. \end{aligned}$$

That is,

$$v^{\bar{\pi}^b}(x, \bar{x}) = Z^{(q)}(x)\, C(b \vee \bar{x}) + \eta \left[\overline{Z}^{(q)}(x) + \frac{\psi'(0^+)}{q}\right], \tag{6.55}$$

where

$$C(b) = \frac{\beta}{1-\beta} \left[Z^{(q)}(b)\right]^{\beta/(1-\beta)} \int_b^\infty \left[Z^{(q)}(y)\right]^{-1/(1-\beta)} \left[1 - \eta Z^{(q)}(y)\right] \mathrm{d}y.$$

Note that, since $\eta \geq 1$ and $Z^{(q)}(y) \geq 1$, $C(b) \leq 0$, moreover, we have that

$$\lim_{b \uparrow \infty} C(b) = -\frac{\eta}{\Phi(q)}.$$

Indeed,

$$\begin{aligned} \lim_{b \uparrow \infty} C(b) &= \frac{\beta}{1-\beta} \lim_{b \uparrow \infty} \frac{-[Z^{(q)}(b)]^{-1/(1-\beta)} + \eta [Z^{(q)}(b)]^{-\beta/(1-\beta)}}{(-\beta/(1-\beta))[Z^{(q)}(b)]^{-1/(1-\beta)} Z^{(q)\prime}(b)} \\ &= \lim_{b \uparrow \infty} \left[\frac{1}{Z^{(q)\prime}(b)} - \eta \frac{Z^{(q)}(b)}{Z^{(q)\prime}(b)}\right] \\ &= -\frac{\eta}{\Phi(q)}. \end{aligned}$$

This means that if we never reach the level b, then we only have bail-out and hence the value function (6.55) agrees with the net present value of injections in a previous literature as we mentioned in Remark 14. Also, we can find that

$$C'(b) = \frac{\beta}{1-\beta} \frac{Z^{(q)\prime}(b)}{Z^{(q)}(b)} \left[C(b) - Q(b)\right],$$

where $Q(b) = \dfrac{1-\eta Z^{(q)}(b)}{Z^{(q)\prime}(b)}$.

Case 2: $\alpha > 0$.

By (6.49), we find that for the case $0 < \bar{x} \le b$, the value function is given by

$$\begin{aligned}
v^{\bar{\pi}^b}(x,\bar{x}) = {} & \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\Bigg\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{b}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{1/(1-\alpha)}\mathrm{d}y \\
& + \frac{\beta}{1-\beta}\int_{b}^{\infty}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(b)}\right]^{1/(1-\alpha)}\left[\frac{Z^{(q)}(b)}{Z^{(q)}(y)}\right]^{1/(1-\beta)}\mathrm{d}y\Bigg\} \\
& - \eta\Bigg\{-\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + Z^{(q)}(x)\Big\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{b}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{\alpha/(1-\alpha)}\mathrm{d}y \\
& + \frac{\beta}{1-\beta}\int_{b}^{\infty}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(b)}\right]^{\alpha/(1-\alpha)}\left[\frac{Z^{(q)}(b)}{Z^{(q)}(y)}\right]^{\beta/(1-\beta)}\mathrm{d}y\Big\}\Bigg\}.
\end{aligned}$$

For the case $\bar{x} > b$,

$$\begin{aligned}
v^{\bar{\pi}^b}(x,\bar{x}) = {} & \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{1/(1-\beta)}\mathrm{d}y \\
& - \eta\left\{-\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + Z^{(q)}(x)\frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}\left[\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(y)}\right]^{\beta/(1-\beta)}\mathrm{d}y\right\}.
\end{aligned}$$

By some easy calculations, we can rewrite the value function as

$$\begin{aligned}
v^{\bar{\pi}^b}(x,\bar{x}) = {} & \left[Z^{(q)}(\bar{x})\right]^{\alpha/(1-\alpha)} Z^{(q)}(x)\Bigg\{\frac{\alpha}{(1-\alpha)}\int_{\bar{x}}^{\infty}\left[Z^{(q)}(y)\right]^{-1/(1-\alpha)}\left[1-\eta Z^{(q)}(y)\right]\mathrm{d}y \\
& + C(b\vee\bar{x})\Bigg\} + \eta\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0^+)}{q}\right], \qquad (6.56)
\end{aligned}$$

where

$$\begin{aligned}
C(b) = {} & \frac{\beta}{1-\beta}\left[Z^{(q)}(b)\right]^{1/(1-\beta)-1/(1-\alpha)}\int_{b}^{\infty}\left[Z^{(q)}(y)\right]^{-1/(1-\beta)}\left[1-\eta Z^{(q)}(y)\right]\mathrm{d}y \\
& - \frac{\alpha}{(1-\alpha)}\int_{b}^{\infty}\left[Z^{(q)}(y)\right]^{-1/(1-\alpha)}\left[1-\eta Z^{(q)}(y)\right]\mathrm{d}y. \qquad (6.57)
\end{aligned}$$

Remark 15 If we let $\alpha \downarrow 0$ in (6.56), then we can see that the value functions in (6.56) and (6.55) are equal.

Also, we can find that

$$C'(b) = \left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\frac{Z^{(q)\prime}(b)}{Z^{(q)}(b)}\left[C(b) - Q(b)\right], \qquad (6.58)$$

where

$$Q(b) = \frac{Z^{(q)}(b)}{Z^{(q)\prime}(b)} \left[Z^{(q)}(b)\right]^{-1/(1-\alpha)} \left[1 - \eta Z^{(q)}(b)\right] \\ - \frac{\alpha}{(1-\alpha)} \int_b^\infty \left[Z^{(q)}(y)\right]^{-1/1-\alpha} \left[1 - \eta Z^{(q)}(y)\right] \mathrm{d}y. \tag{6.59}$$

Lemma 6.3.2 *Let $q > 0$,*

$$a^* = \inf\left\{a \geq 0 : G(a) := [\eta Z^{(q)}(a) - 1] W^{(q)\prime}(a) - \eta q [W^{(q)}(a)]^2 \leq 0\right\}, \tag{6.60}$$

and let Q be given as in (6.59). Then, $a^* < \infty$ and Q is strictly increasing on $(0, a^*)$ and strictly decreasing on (a^*, ∞). Furthermore, for $\eta > 1$, if $\nu(0,\infty) \leq q/(\eta - 1)$ and $\sigma = 0$, then $a^* = 0$; otherwise $a^* > 0$.

Proof First, note that

$$Q'(u) = -\eta \left[Z^{(q)}(u)\right]^{-\alpha/(1-\alpha)} - \frac{Z^{(q)\prime\prime}(u)}{[Z^{(q)\prime}(u)]^2} \left[1 - \eta Z^{(q)}(u)\right] \left[Z^{(q)}(u)\right]^{-\alpha/(1-\alpha)}, \tag{6.61}$$

which can be simplified as

$$Q'(u) = \frac{\left[Z^{(q)}(u)\right]^{-\alpha/(1-\alpha)}}{q W^{(q)}(u)^2} G(u), \tag{6.62}$$

where G is defined as in (6.60). We are going to use the same argument as in [10, pp. 15 and 16]. Let $\hat{\tau}_a := \inf\left\{t \geq 0 : \overline{X}_t - X_t > a\right\}$ and

$$A(a) = \mathbb{E}_0[e^{-q\hat{\tau}_a}] = Z^{(q)}(a) - \frac{q[W^{(q)}(a)]^2}{W^{(q)\prime}(a)},$$

where the last equality is given by (3.10) in [10] with $y = 0$. Then, as $a \mapsto \hat{\tau}_a$ is increasing with $\lim_{a\to\infty} \hat{\tau}_a = \infty$ almost surely, then $A(a)$ decreases to zero almost surely as $a \to \infty$, and hence $[\eta A(a) - 1]$ decreases to -1 almost surely as $a \to \infty$. We can rewrite $G(a)$ as

$$G(a) = [\eta A(a) - 1] W^{(q)\prime}(a).$$

Note that, $W^{(q)\prime}(a) > 0$ for all $a \geq 0$, therefore, if $G(0^+) \leq 0$, then $[\eta A(0^+) - 1] \leq 0$, which implies that $[\eta A(a) - 1] \leq 0$ for all $a > 0$, and hence $G(a) \leq 0$ for all $a > 0$. If $G(0^+) > 0$, as G is continuous and $\lim_{a\to\infty} G(a) = -\infty$, then by the intermediate value theorem, $G(a) = 0$ has a root in $(0, \infty)$, which proves the existence of a^*. Since A is decreasing, this implies that $G(a) \leq 0$ for $a > a^*$. If $a^* > 0$, then $G(a^*) = 0$, and by the definition of a^*, $G(a) > 0$ for $0 < a < a^*$. Consequently, from (6.62), the statement of the lemma is proved. Furthermore, since both $W^{(q)}(0^+) > 0$ and $W^{(q)\prime}(0^+) < \infty$ hold true if $\nu(0,\infty) < \infty$ and $\sigma = 0$, then by easy calculations we find that $G(0^+) \leq 0$ if and only if both $\sigma = 0$ and $\nu(0,\infty) \leq q/(\eta - 1)$.□

Lemma 6.3.3 *Let $b^* = \sup\{b \geq 0 : C(b) \geq C(s) \ \text{ for all } s \geq 0\}$. Then $b^* < \infty$ and C is strictly increasing on $(0, b^*)$ and strictly decreasing on (b^*, ∞).*

Proof First, $b^* < \infty$ since we have that

$$\lim_{b\uparrow\infty} C(b) = 0.$$

Indeed,

$$\begin{aligned}\lim_{b\uparrow\infty} C(b) &= \lim_{b\uparrow\infty} \frac{\beta}{1-\beta} \frac{\displaystyle\int_b^\infty \left[Z^{(q)}(y)\right]^{-1/(1-\beta)} \left[1 - \eta Z^{(q)}(y)\right] \mathrm{d}y}{\left[Z^{(q)}(b)\right]^{1/(1-\alpha)-1/(1-\beta)}} \\ &= C\eta \lim_{b\uparrow\infty} \left[\frac{1}{[Z^{(q)}(b)]^{\alpha/(1-\alpha)}}\right] \left[\frac{W^{(q)}(b)}{W^{(q)\prime}(b)}\right] \\ &= 0,\end{aligned}$$

where C is some constant. From (6.58), we have that

$$C'(b) > 0 (< 0, = 0) \ \text{ if and only if } \ C(b) > Q(b) \left(< Q(b), = Q(b)\right), \tag{6.63}$$

and hence, the rest of the proof is like the proof of Lemma 5.2.5.□

Remark 16 We can prove by simple calculations that

$$\begin{aligned}&\frac{\beta}{1-\beta}[Z^{(q)}(y)]^{-1/(1-\beta)} \left[1 - \eta Z^{(q)}(y)\right] + \eta [Z^{(q)}(y)]^{-\beta/(1-\beta)} \\ &\quad = -\left(\frac{[Z^{(q)}(y)]^{-\beta/(1-\beta)}}{Z^{(q)\prime}(y)} \left[1 - \eta Z^{(q)}(y)\right]\right)' - \frac{[Z^{(q)}(y)]^{-\beta/(1-\beta)} \left[1 - \eta Z^{(q)}(y)\right] Z^{(q)\prime\prime}(y)}{[Z^{(q)\prime}(y)]^2}.\end{aligned} \tag{6.64}$$

For $b^* > 0$, we have that $C'(b^*) = 0$, then by (6.63), $C(b^*) = Q(b^*)$, which implies that

$$\frac{\beta}{1-\beta} \int_{b^*}^\infty \left[Z^{(q)}(s)\right]^{-(1/(1-\beta))} [1 - \eta Z^{(q)}(s)] \mathrm{d}s = \frac{[Z^{(q)}(b^*)]^{-\beta/(1-\beta)}}{Z^{(q)\prime}(b^*)} [1 - \eta Z^{(q)}(b^*)]. \tag{6.65}$$

By (6.64), we have that

$$\begin{aligned}
&\frac{\beta}{1-\beta}\int_{b^*}^{\infty}\left[Z^{(q)}(s)\right]^{-(1/(1-\beta))}[1-\eta Z^{(q)}(s)]\mathrm{d}s \\
&=\frac{[Z^{(q)}(b^*)]^{-\beta/(1-\beta))}}{Z^{(q)\prime}(b^*)}[1-\eta Z^{(q)}(b^*)]-\int_{b^*}^{\infty}\frac{[Z^{(q)}(s)]^{-\beta/(1-\beta)}[1-\eta Z^{(q)}(s)]Z^{(q)\prime\prime}(s)}{[Z^{(q)\prime}(s)]^2}\mathrm{d}s \\
&\quad-\eta\int_{b^*}^{\infty}[Z^{(q)}(s)]^{-\beta/(1-\beta)}\mathrm{d}s.
\end{aligned}$$

By (6.65), we get that b^* is the unique solution to

$$\int_{b^*}^{\infty}\left\{[Z^{(q)}(s)]^{-\beta/(1-\beta)}[1-\eta Z^{(q)}(s)]\frac{Z^{(q)\prime\prime}(s)}{[Z^{(q)\prime}(s)]^2}+\eta[Z^{(q)}(s)]^{-\beta/(1-\beta)}\right\}\mathrm{d}s=0. \tag{6.66}$$

We can see by Lemma (6.3.2), and by finding the derivative and using (6.61) for the left hand side of (6.66), we can see it is a strictly increasing function in b^*. Then, the existence of a unique solution $b^*>0$ is guaranteed by the condition

$$\int_{0}^{\infty}\left\{[Z^{(q)}(s)]^{-\beta/(1-\beta)}[1-\eta Z^{(q)}(s)]\frac{Z^{(q)\prime\prime}(s)}{[Z^{(q)\prime}(s)]^2}+\eta[Z^{(q)}(s)]^{-\beta/(1-\beta)}\right\}\mathrm{d}s<0.$$

Since $b^*\le a^*$, then $a^*=0$ implies that $b^*=0$. Therefore, we find that $b^*=0$ if either $a^*=0$ or the following condition is satisfied:

$$\int_{0}^{\infty}\left\{[Z^{(q)}(s)]^{-\beta/(1-\beta)}[1-\eta Z^{(q)}(s)]\frac{Z^{(q)\prime\prime}(s)}{[Z^{(q)\prime}(s)]^2}+\eta[Z^{(q)}(s)]^{-\beta/(1-\beta)}\right\}\mathrm{d}s\ge 0.$$

Remark 17 We find the partial derivatives of $v^{\bar{\pi}^b}(x,\bar{x})$, given by (6.56) as we will use it in the proof of the next Theorem.

$$\begin{aligned}
\frac{\partial v^{\bar{\pi}^b}}{\partial x}(x,\bar{x})=Z^{(q)\prime}(x)[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\Big\{&\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \\
&+C(b\vee\bar{x})\Big\}+\eta Z^{(q)}(x).
\end{aligned}$$

$$\frac{\partial v^{\bar{\pi}^b}}{\partial \bar{x}}(x,\bar{x})=\begin{cases}F_1(x,\bar{x}) & \text{if } 0<\bar{x}\le b\\ \\ F_2(x,\bar{x}) & \text{if } \bar{x}>b,\end{cases}$$

where

$$\begin{aligned}
F_1(x,\bar{x})=&\left(\frac{\alpha}{1-\alpha}\right)^2\frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}Z^{(q)\prime}(\bar{x})[Z^{(q)}(\bar{x})]^{(\alpha/(1-\alpha))}\int_{\bar{x}}^{\infty}[Z^{(q)}(s)]^{-(1/(1-\alpha))}[1-\eta Z^{(q)}(s)]\mathrm{d}s \\
&+\frac{\alpha}{1-\alpha}\frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\left[Z^{(q)\prime}(\bar{x})[Z^{(q)}(\bar{x})]^{(\alpha/(1-\alpha))}C(b)+\eta Z^{(q)}(\bar{x})-1\right],
\end{aligned}$$

and

$$F_2(x,\bar{x}) = \frac{\beta}{1-\beta}\frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\Big\{\frac{\beta}{1-\beta}Z^{(q)\prime}(\bar{x})[Z^{(q)}(\bar{x})]^{(\beta/(1-\beta))}\int_{\bar{x}}^{\infty}[Z^{(q)}(s)]^{-(1/(1-\beta))}[1-\eta Z^{(q)}(s)]\mathrm{d}s$$
$$+\eta Z^{(q)}(\bar{x})-1\Big\}.$$

Theorem 6.3.4 *Let $\psi'(0^+) > -\infty$, δ^b as defined in* (5.5) *and b^* as given in Lemma 6.3.3, then $\bar{\pi}^{b^*} := (\delta^{b^*}, K^{\delta^{b^*}})$ is an optimal strategy and $v^{\bar{\pi}^{b^*}}$ is the optimal solution for* (6.2).

Proof We need to prove that $v^{\bar{\pi}^{b^*}}(x,\bar{x})$ satisfies all the conditions of Lemma 6.3.1. First, using (6.56), and by similar arguments to the proofs in Lemma (6.2.4) and Lemma 6.2.6, with using n instead of a, we can show that for each $n \geq 1$, $v^{\bar{\pi}^{b^*}} \in \mathcal{S}_{[-n,n]\times[\frac{1}{n},n]}$.

Condition I. By (6.56), we can verify this condition easily.

Conditions II **and** III. For $0 < x \leq \bar{x}$, we show that $(\mathcal{A}-q)v^{\pi^{b^*}}(x,\bar{x}) = 0$. Recall that, for any $x \in (0,\infty)$, by (6.29), we have that

$$(\mathcal{A}-q)Z^{(q)}(x) = 0, \tag{6.67}$$

and from Lemma (2.2.8) and [13, p.367 (Step 2)], we also have

$$(\mathcal{A}-q)\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right] = 0. \tag{6.68}$$

Also, we use right-continuity of $Z^{(q)}$ and $\overline{Z}^{(q)}(x)$, and their first and second derivatives, at 0, to have (6.67) and (6.68) valid at $x = 0$. Therefore, by the definition of $\mathcal{A}$ and (6.56), for $(x,\bar{x})$ with $x \geq 0$ and such that $\bar{x} > 0$, we have that

$$\begin{aligned}
&(\mathcal{A}-q)v^{\pi^{b^*}}(x,\bar{x}) \\
&= (\mathcal{A}-q)Z^{(q)}(x)\Bigg\{[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\Big\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \\
&\quad + C(b^*\vee\bar{x})\Big\}\Bigg\} + \eta\left\{(\mathcal{A}-q)\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right]\right\} = 0.
\end{aligned}$$

Condition IV. Show that

$$H\,\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(\bar{x},\bar{x}) + (H-1)\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial \bar{x}}(\bar{x},\bar{x}) \geq H, \quad \text{for all } \; H \in [\alpha,\beta] \text{ and all } \bar{x} > 0.$$

For $0 < \bar{x} \le b^*$, this condition is satisfied for $v^{\bar{\pi}^{b^*}}(x, \bar{x})$ and for $H \in [\alpha, \beta]$ when

$$H\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(\bar{x}, \bar{x}) + (H-1)\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial \bar{x}}(\bar{x}, \bar{x}) - H \ge 0,$$

which by some calculations is satisfied if and only if

$$\left[\frac{\partial v^{\bar{\pi}^{b^*}}(x, \bar{x})}{\partial x}|_{x=\bar{x}} - 1\right]\left[\frac{H-\alpha}{1-\alpha}\right] \ge 0.$$

Since $H \in [\alpha, \beta]$, it is always true that

$$\left[\frac{H-\alpha}{1-\alpha}\right] \ge 0,$$

so in order to have the condition satisfied, we should have that

$$\frac{\partial v^{\bar{\pi}^{b^*}}(x, \bar{x})}{\partial x}|_{x=\bar{x}} \ge 1.$$

For $0 < \bar{x} \le b^*$, and by Lemma 6.3.3, $C'(\bar{x}) \ge 0$ on $(0, b^*]$. By (6.63), $C'(\bar{x}) \ge 0$ is satisfied if and only if $C(\bar{x}) \ge Q(\bar{x})$, but as $C(b^*) \ge C(\bar{x})$, then we get that $C(b^*) \ge Q(\bar{x})$. By (6.59), we have that

$$C(b^*) \ge Q(\bar{x})$$

$$\iff$$

$$\left\{ Z^{(q)\prime}(\bar{x})[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(s)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(s)]\mathrm{d}s + C(b^*)\right\} + \eta Z^{(q)}(\bar{x}) \right\} \ge 1,$$

where the last inequality is

$$\frac{\partial v^{\bar{\pi}^{b^*}}(x, \bar{x})}{\partial x}|_{x=\bar{x}} \ge 1.$$

That is, $v^{\bar{\pi}^{b^*}}(x, \bar{x})$ satisfies IV for $0 < \bar{x} \le b^*$.

Similarly, for $\bar{x} > b^*$, condition IV in the verification lemma is satisfied for $v^{\bar{\pi}^{b^*}}(x, \bar{x})$ and for $H \in [\alpha, \beta]$ when

$$H\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(\bar{x}, \bar{x}) + (H-1)\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial \bar{x}}(\bar{x}, \bar{x}) - H \ge 0,$$

which by some calculations is satisfied if and only if

$$\left[\frac{\partial v^{\bar{\pi}^{b^*}}(x, \bar{x})}{\partial x}|_{x=\bar{x}} - 1\right]\left[\frac{H-\beta}{1-\beta}\right] \ge 0.$$

Since $H \leq \beta$, the last inequality is satisfied if and only if

$$\frac{\partial v^{\bar{\pi}^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}} < 1.$$

Now, by Lemma 6.3.3 and as $\bar{x} > b^*$, we have that $C'(\bar{x}) < 0$. By (6.63), this is equivalent to $C(\bar{x}) < Q(\bar{x})$. By a similar argument to the first case above, we can do the calculations and prove that $C(\bar{x}) < Q(\bar{x})$ if and only if $\frac{\partial v^{\bar{\pi}^{b^*}}(x,\bar{x})}{\partial x}|_{x=\bar{x}} < 1$. Hence, $v^{\bar{\pi}^{b^*}}(x,\bar{x})$ satisfies IV for $\bar{x} > b^*$.

Condition V. For $0 < x \leq \bar{x}$, show that

$$\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x}) \leq \eta.$$

Recall that, for $0 < x \leq \bar{x}$,

$$\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x}) = Z^{(q)\prime}(x)[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \right.$$
$$\left. + C(b^* \vee \bar{x})\right\} + \eta Z^{(q)}(x).$$

We fix x and show that $\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})$, in the $\bar{x}$ variable, is strictly increasing on $(0,b^*)$ and strictly decreasing on (b^*,∞), and hence has it's maximum at $\bar{x} = b^*$. For that, we compute the derivative, use (6.57) and (6.59), and by some easy calculations we find that for $\bar{x} \in (0,b^*)$:

$$\frac{1-\alpha}{\alpha}\frac{1}{q\,W^{(q)}(x)}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})\right) = \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\left[Z^{(q)}(\bar{x})\right]^{\alpha/(1-\alpha)}\left[C(b^*) - Q(\bar{x})\right]. \quad (6.69)$$

From Lemma 6.3.3 and (6.63), we have that $C(b^*) > C(\bar{x}) > Q(\bar{x})$. This implies that (6.69) is greater than zero and hence $\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})$, is strictly increasing on $(0,b^*)$. For $\bar{x} \in (b^*,\infty)$,

$$\frac{1-\beta}{\beta}\frac{1}{q\,W^{(q)}(x)}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})\right) = \frac{Z^{(q)\prime}(\bar{x})}{Z^{(q)}(\bar{x})}\left[Z^{(q)}(\bar{x})\right]^{\alpha/(1-\alpha)}\left[C(\bar{x}) - Q(\bar{x})\right], \quad (6.70)$$

so from Lemma 6.3.3, we know that $C'(\bar{x}) < 0$, which is by (6.63) if and only if $C(\bar{x}) < Q(\bar{x})$. This implies that (6.70) is less than zero and hence $\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})$ is strictly decreasing on (b^*,∞).

Now for $b^* > 0$, recall by Remark 16 that

$$\frac{\beta}{1-\beta}\int_{b^*}^{\infty}\left[Z^{(q)}(s)\right]^{-1/(1-\beta)}[1-\eta Z^{(q)}(s)]\mathrm{d}s = \frac{[Z^{(q)}(b^*)]^{-\beta/(1-\beta)}}{Z^{(q)\prime}(b^*)}[1-\eta Z^{(q)}(b^*)]. \tag{6.71}$$

Then,

$$\begin{aligned}\left[\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,\bar{x})-\eta\right]W^{(q)}(b^*) \le \left[\frac{\partial v^{\bar{\pi}^{b^*}}}{\partial x}(x,b^*)-\eta\right]W^{(q)}(b^*) \\ &= q\,W^{(q)}(x)\,W^{(q)}(b^*)[Z^{(q)}(b^*)]^{-1}\frac{Z^{(q)}(b^*)}{Z^{(q)\prime}(b^*)}\left[1-\eta Z^{(q)}(b^*)\right] \\ &\quad + [\eta Z^{(q)}(x)-\eta]W^{(q)}(b^*) \\ &= W^{(q)}(x)(1-\eta)+\eta\,q\left[W^{(q)}(b^*)\,\overline{W}^{(q)}(x)-W^{(q)}(x)\,\overline{W}^{(q)}(b^*)\right] \\ &\le 0, \end{aligned} \tag{6.72}$$

where to get the second equality, we used (6.71), and in last inequality, we used that $\eta \ge 1$ and Lemma 2.2.2. As $W^{(q)}(b^*) > 0$, we conclude that condition (V) is satisfied when $b^* > 0$.

If $b^* = 0$, that is for $\bar{x} \ge x > 0$, recall first that

$$\begin{aligned}\frac{\partial v^{\bar{\pi}^0}}{\partial x}(x,\bar{x}) &= Z^{(q)\prime}(x)[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \right.\\ &\quad \left. + C(0\vee\bar{x})\right\} + \eta Z^{(q)}(x) \\ &= Z^{(q)\prime}(x)[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \right.\\ &\quad \left. + C(\bar{x})\right\} + \eta Z^{(q)}(x) \\ &< Z^{(q)\prime}(x)[Z^{(q)}(\bar{x})]^{\alpha/(1-\alpha)}\left\{\frac{\alpha}{1-\alpha}\int_{\bar{x}}^{\infty}[Z^{(q)}(y)]^{-1/(1-\alpha)}[1-\eta Z^{(q)}(y)]\mathrm{d}y \right.\\ &\quad \left. + Q(\bar{x})\right\} + \eta Z^{(q)}(x) \\ &= Z^{(q)\prime}(x)\frac{[1-\eta Z^{(q)}(\bar{x})]}{Z^{(q)\prime}(\bar{x})} + \eta Z^{(q)}(x),\end{aligned}$$

where in the third inequality, we used (6.63), C is strictly decreasing on $(0,\infty)$ and

(6.59). Therefore, as $W^{(q)}(\bar{x}) > 0$, then similar to (6.72) we have

$$\begin{aligned}\left[\frac{\partial v^{\bar{\pi}^0}}{\partial x}(x,\bar{x}) - \eta\right] W^{(q)}(\bar{x}) &< Z^{(q)\prime}(x)\, W^{(q)}(\bar{x}) \frac{[1-\eta Z^{(q)}(\bar{x})]}{Z^{(q)\prime}(\bar{x})} + \eta Z^{(q)}(x)\, W^{(q)}(\bar{x}) - \eta W^{(q)}(\bar{x}) \\ &= W^{(q)}(x)(1-\eta) + \eta\, q \left[W^{(q)}(\bar{x})\, \overline{W}^{(q)}(x) - W^{(q)}(x)\, \overline{W}^{(q)}(\bar{x})\right] \\ &\leq 0,\end{aligned}$$

which completes the proof.□

Chapter 7

Natural Taxation with a limited bail-out

7.1 Introduction

In some situations, the tax authority may decide to stop providing bail-outs to a financially distressed company. One of the reasons is that, it could be a huge expense for the tax authority to continue bail-outs without a promise of having a solid return in tax revenue. This may bring large amount of losses for that tax authority, and which makes the best solution is to stop bailing out the company and declare its bankruptcy. The tax authority would want to know, in that situation, what is the value of its net profit. In the context of dividends, a recent article studied this problem, which is [22], and found the maximum firm value in that case under the setting of a spectrally negative Lévy process. In this chapter, for the first time in literature, we study the natural tax process $V^{\delta,\infty}$ with a limited bail-out at a parameter $c < 0$.

Recall first the process $V^{\delta,\infty}$ given in Chapter 6 satisfying

$$V_t^{\delta,\infty} = (X + K^\delta)_t - \int_{0^+}^t \delta(\overline{V}_s^{\delta,\infty})\, \mathrm{d}\overline{(X + K^\delta)}_s, \quad t \geq 0.$$

Also, recall that $\tau_a^+ = \inf\left\{t \geq 0 : V_t^{\delta,\infty} > a\right\}$ for $a > 0$ and for $c < 0$, let $\tau_c^- := \inf\left\{t \geq 0 : V_t^{\delta,\infty} < c\right\}$. We study here the process $V^{\delta,-c}$, which is the natural tax process $V^{\delta,\infty}$ with a limited bail-out at a parameter c and denoted by:

$$V_t^{\delta,-c} := V_{t \wedge \tau_c^-}^{\delta,\infty}. \tag{7.1}$$

Note that, even though $V_{t+}^{\delta,\infty}$, the right-continuous version, is always greater or equal zero, the process $V^{\delta,\infty}$ actually attains its overshoots and therefore the time at which it drops below c is actually observable from its path. So, for $V^{\delta,-c}$, bail-outs or capital injections are only allowed if the amount of ruin is above or at level c. Once the ruin level is below c, then the government stops bailing-out, which incurs some ruin-dependent loss in the revenue and which we call *penalty*. Define a *penalty function*, $P\colon \mathbb{R} \to \mathbb{R}$, and we assume throughout this chapter that P is bounded. The net profit value function of the process (7.1) is given by

$$v^{\delta,-c}(x,\bar{x}) = v_{\text{tax}}^{\delta,-c}(x,\bar{x}) - \eta \; v_{\text{inj}}^{\delta,-c}(x,\bar{x}) - \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}\,P(V_{\tau_c^-}^{\delta,\infty})\mathbf{1}_{\{\tau_c^-<\infty\}}\right], \tag{7.2}$$

where $\eta \geq 1$ is a bail-out cost factor,

$$v_{\text{tax}}^{\delta,-c}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^-} e^{-qs}\delta(\overline{V}_s^{\delta,\infty})\ \mathrm{d}\overline{(X+K^\delta)}_s\right],$$

and

$$v_{\text{inj}}^{\delta,-c}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0}^{\tau_c^-} e^{-qs}\ \mathrm{d}K_s^\delta\right].$$

Note that, in this chapter, we derive new fluctuation identities which will be used through the steps of finding the net profit value function for the process $V^{\delta,-c}$, while in Chapter 6, we used the corresponding available results from [10]. Our main results are as follows. For the reflected Lévy process in $[c,\bar{x}]$ and $\bar{x} > 0$, we state and prove Proposition 7.2.1, which we call the two sided exit problem. Further, Proposition 7.2.2 gives the expression of the expected accumulated discounted amount of capital injections before taxation starts in the model. Theorems 7.3.5 and 7.3.8, under the assumption that X has a positive Gaussian coefficient in the unbounded variation case, give explicit expressions for the net present value of taxation and capital injections, respectively, for the process $V^{\delta,-c}$. Also, under the same assumption with the addition that P is a bounded function, Theorem 7.3.11 gives the net present value of penalty for $V^{\delta,-c}$.

This chapter is organised as follows. In Section 7.2, we prove our new results for the reflected Lévy process from below at zero, provided that the crossing-down level is above or equal c. In Section 7.3, we find by our approach each term of (7.2) separately. We also verify that when we take the limit of $c \downarrow -\infty$, they agree with the forced bail-out results in the previous chapter. Moreover, we show in this section through Corollaries 7.3.6 and 7.3.9, the advantage of our approach in finding the value functions. Our approach leads us to prove that our new results in the second section are also valid in the unbounded variation case without the need of an approximation scheme, like for example the work done in [49].

7.2 Reflected Lévy processes

In this section, we give some results for the Lévy process reflected at its infimum, defined as, for $t \geq 0$, $Y_t = X_t - I_t$ where $I_t = \inf_{0\leq s\leq t}(X_s \wedge 0)$. We will use these results in the next section. Define

$$T_c^- := \inf\{t \geq 0 : Y_t < c\},$$

and

$$T_a^+ := \inf\{t \geq 0 : Y_t > a\}.$$

Recall $\rho_c^- := \inf\{t \geq 0 : X_t < c\}$ and $\rho_a^+ := \inf\{t \geq 0 : X_t > a\}$, then [41, Theorem 2] gives the analytic expression of the overshoot for the process X by choosing a suitable extension $\tilde{f}$,

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-} f(X_{\rho_0^-})\mathbf{1}_{\{\rho_0^- < \rho_{\bar{x}}^+\}}\right] \\
&= \tilde{f}(x) - \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,\tilde{f}(\bar{x}) \\
&\quad + \int_0^{\bar{x}} (\mathcal{A}-q)\tilde{f}(z)\left[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-z) - W^{(q)}(x-z)\right]\mathrm{d}z \\
&\quad + \left(f(0) - \tilde{f}(0+)\right)\frac{\sigma^2}{2}\,\left(W^{(q)\prime}(x) - \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,W^{(q)\prime}(\bar{x})\right),
\end{aligned} \tag{7.3}$$

where we recall (4.10),

$$\begin{aligned}
\mathcal{A}f(y) &= \mu f'(y) + \frac{\sigma^2}{2}f''(y) \\
&\quad + \int_{0+}^{\infty}\left[f(y-\theta) - f(y) + \theta f'(y)\mathbf{1}_{\{0<\theta\leq 1\}}\right]\nu(\mathrm{d}\theta).
\end{aligned}$$

Also, we recall the functions l and l' that we introduced in Chapter 4. Let $l : (0,\infty) \to \mathbb{R}$ be given by

$$l(x) = (\mathcal{A}-q)f(x), \text{ where } f(x) = x\mathbf{1}_{\{x\geq 0\}},$$

then we can prove that,

$$l'(x) = (\mathcal{A} - q)g(x), \text{ where } g(x) = \mathbf{1}_{\{x \geq 0\}}. \tag{7.4}$$

Indeed, for $x > 0$,

$$\begin{aligned} l(x) &= (\mathcal{A} - q)f(x) \\ &= \mu \mathbf{1}_{\{x>0\}} + \int_{0^+}^{\infty} \left[(x-\theta)\, \mathbf{1}_{\{x-\theta \geq 0\}} - x\, \mathbf{1}_{\{x \geq 0\}} + \theta\, \mathbf{1}_{\{x>0\}} \mathbf{1}_{\{0<\theta \leq 1\}} \right] \nu(\mathrm{d}\theta) \\ &\quad - q\, x \mathbf{1}_{\{x \geq 0\}} \\ &= \mu + \int_{0^+}^{\infty} \left[(x-\theta)\, \mathbf{1}_{\{x-\theta \geq 0\}} - x + \theta\, \mathbf{1}_{\{0<\theta \leq 1\}} \right] \nu(\mathrm{d}\theta) - q\, x. \end{aligned} \tag{7.5}$$

Then by (7.5), for $x > 0$,

$$\begin{aligned} l'(x) &= \int_{0^+}^{\infty} \left[\mathbf{1}_{\{x>\theta\}} - \mathbf{1}_{\{x>0\}} \right] \nu(\mathrm{d}\theta) - q\, \mathbf{1}_{\{x>0\}} \\ &= -\int_{x}^{\infty} \nu(\mathrm{d}\theta) - q \\ &= -\nu(x, \infty) - q, \end{aligned} \tag{7.6}$$

where we get the last equality because $x > 0$. Now, the right hand side of (7.4), for $x > 0$ is,

$$\begin{aligned} (\mathcal{A} - q)g(x) &= \int_{0^+}^{\infty} \left[\mathbf{1}_{\{x \geq \theta\}} - \mathbf{1}_{\{x \geq 0\}} \right] \nu(\mathrm{d}\theta) - q\, \mathbf{1}_{\{x \geq 0\}} \\ &= -\int_{x}^{\infty} \nu(\mathrm{d}\theta) - q \\ &= -\nu(x, \infty) - q, \end{aligned}$$

hence, (7.4) is satisfied. Also, for $c < 0$, let $f_c(z) = z\, \mathbf{1}_{\{z \geq c\}}$, and $g_c(z) = \mathbf{1}_{\{z \geq c\}}$, then for $z > c$,

$$\begin{aligned} (\mathcal{A} - q)f_c(z) &= (\mathcal{A} - q)f_c(z) - c\, (\mathcal{A} - q)\, g_c(z) + c\, (\mathcal{A} - q)\, g_c(z) \\ &= (\mathcal{A} - q)\, [f_c - c\, g_c]\, (z) + c\, (\mathcal{A} - q)\, g_c(z) \\ &= l(z - c) + c\, l'(z - c), \end{aligned}$$

where we use (7.5) and (7.6) such that the last line follows from spatial homogeneity of $\mathcal{A}$. That is, for $x \in \mathbb{R}$, where $f_x(z) = f(x + z)$, we have that $\mathcal{A}f_x(z) = \mathcal{A}f(x + z)$.

Remark 18 For $z \in \mathbb{R}$ such that $z > c$,

$$\begin{aligned}\lim_{c\downarrow-\infty}\left[l(z-c)+c\,l'(z-c)\right] &= \lim_{c\downarrow-\infty}\Big\{\mu-(z-c)\int_{z-c}^{\infty}\nu(\mathrm{d}\theta)-\int_{1}^{z-c}\theta\nu(\mathrm{d}\theta)-q(z-c)\\ &\qquad -c\int_{z-c}^{\infty}\theta\nu(\mathrm{d}\theta)-q\,c\Big\}\\ &= \lim_{c\downarrow-\infty}\Big\{\mu-z\int_{z-c}^{\infty}\nu(\mathrm{d}\theta)-\int_{1}^{z-c}\theta\nu(\mathrm{d}\theta)-q\,z\Big\}\\ &= \mu-\int_{1}^{\infty}\theta\nu(\mathrm{d}\theta)-q\,z\\ &= \psi'(0^+)-q\,z. \end{aligned} \tag{7.7}$$

Proposition 7.2.1 ***(The two sided exit problem)***

For $c < 0 < a$, $x \le a$, and when X is of bounded variation,

$$\mathbb{E}_x\left[e^{-q\,T_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]=\frac{J_c(x)}{J_c(a)}, \tag{7.8}$$

where $J_c:\mathbb{R}\to\mathbb{R}$ is defined by,

$$J_c(x):=\left[1-\int_0^x l'(z-c)W^{(q)}(x-z)\mathrm{d}z\right]\mathbf{1}_{\{x\ge c\}}. \tag{7.9}$$

Proof Since we look at the reflected process Y when X is in $[c,a]$ for $c<0<a$, we have that

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\,T_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right] &= \mathbb{E}_x\left[e^{-q\rho_0^-}\mathbf{1}_{\left\{X_{\rho_0^-}\ge c\right\}}\mathbf{1}_{\{\rho_0^-<\rho_a^+\}}\right]\;\mathbb{E}_0\left[e^{-q\,T_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]\\ &\quad+\mathbb{E}_x\left[e^{-q\rho_a^+}\mathbf{1}_{\{\rho_a^+<\rho_0^-\}}\right]. \end{aligned} \tag{7.10}$$

For $x \le a$, and by (7.3),

$$\begin{aligned}&\mathbb{E}_x\left[e^{-q\,\rho_0^-}\,\mathbf{1}_{\left\{X_{\rho_0^-}\ge c\right\}}\,\mathbf{1}_{\{\rho_0^-<\rho_a^+\}}\right]\\ &=\mathbb{E}_x\left[e^{-q\,\rho_0^-}\,g_c(X_{\rho_0^-})\,\mathbf{1}_{\{\rho_0^-<\rho_a^+\}}\right]\\ &=g_c(x)-\frac{W^{(q)}(x)}{W^{(q)}(a)}\,g_c(a)\\ &\quad+\int_0^a(\mathcal{A}-q)\,g_c(z)\left[\frac{W^{(q)}(x)}{W^{(q)}(a)}W^{(q)}(a-z)-W^{(q)}(x-z)\right]\mathrm{d}z\\ &=J_c(x)-\frac{W^{(q)}(x)}{W^{(q)}(a)}\,J_c(a). \end{aligned} \tag{7.11}$$

Use (2.7), (7.11) and substitute in (7.10),

$$\mathbb{E}_x\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]=\left[J_c(x)-\frac{W^{(q)}(x)}{W^{(q)}(a)}\,J_c(a)\right]\mathbb{E}_0\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]$$
$$+\frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

Put $x=0$ in both sides and use (2.11), that is $W^{(q)}(0+)=1/d$ in the bounded variation case,

$$\mathbb{E}_0\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]$$
$$=\left\{1-\frac{1/d}{W^{(q)}(a)}+\frac{1/d}{W^{(q)}(a)}\int_0^a l'(z-c)W^{(q)}(a-z)\mathrm{d}z\right\}\mathbb{E}_0\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]$$
$$+\frac{1/d}{W^{(q)}(a)},$$

then,

$$\mathbb{E}_0\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]=\frac{1}{J_c(a)}.$$

Therefore, (7.10) equals,

$$\mathbb{E}_x\left[e^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_c^-\}}\right]=\left[J_c(x)-\frac{W^{(q)}(x)}{W^{(q)}(a)}\,J_c(a)\right]\frac{1}{J_c(a)}+\frac{W^{(q)}(x)}{W^{(q)}(a)}$$
$$=\frac{J_c(x)}{J_c(a)}, \tag{7.12}$$

which completes the proof.□

Remark 19 Note that, by (7.6), taking the limit as $c\downarrow-\infty$ of $l'(z-c)$ gives $-q$. Therefore, if we take the limit as $c\downarrow-\infty$ for (7.12), we get the same result as in (2.2.7)

$$\mathbb{E}_x\left[e^{-qT_a^+}\right]=\frac{Z^{(q)}(x)}{Z^{(q)}(a)}. \tag{7.13}$$

Proposition 7.2.2 *For $c<0<a$, $x\leq a$, and when X is of bounded variation,*

$$\mathbb{E}_x\left[\int_0^{T_a^+\wedge T_c^-}e^{-qs}\,\mathrm{d}I_s\right]=Q_c(x)-\frac{J_c(x)}{J_c(a)}\,Q_c(a), \tag{7.14}$$

where $Q_c:\mathbb{R}\to\mathbb{R}$ is defined by,

$$Q_c(x):=\left[x-\int_0^x[l(z-c)+c\,l'(z-c)]W^{(q)}(x-z)\mathrm{d}z\right]\mathbf{1}_{\{x\geq c\}}. \tag{7.15}$$

PROOF Since we look at the process X when it is in $[c, a]$ for $c < 0 < a$, and as $I_t = \inf_{0 \le s \le t}(X_s \wedge 0)$, we have that

$$\mathbb{E}_x\left[\int_0^{T_a^+ \wedge T_c^-} e^{-qs}\, \mathrm{d}I_s\right] = \mathbb{E}_x\left[e^{-q\rho_0^-} X_{\rho_0^-} \mathbf{1}_{\left\{X_{\rho_0^-} \ge c\right\}} \mathbf{1}_{\{\rho_0^- < \rho_a^+\}}\right] + \mathbb{E}_x\left[e^{-q\rho_0^-} \mathbf{1}_{\left\{X_{\rho_0^-} \ge c\right\}} \mathbf{1}_{\{\rho_0^- < \rho_a^+\}}\right] \mathbb{E}_0\left[\int_{0+}^{T_a^+ \wedge T_c^-} e^{-qs}\, \mathrm{d}I_s\right]. \tag{7.16}$$

To find the value of (7.16), we use (7.3). That is, for $x \le a$,

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\rho_0^-} X_{\rho_0^-} \mathbf{1}_{\left\{X_{\rho_0^-} \ge c\right\}} \mathbf{1}_{\{\rho_0^- < \rho_a^+\}}\right] \\
&= \mathbb{E}_x\left[e^{-q\rho_0^-} f_c(X_{\rho_0^-})\, \mathbf{1}_{\{\rho_0^- < \rho_a^+\}}\right] \\
&= f_c(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\, f_c(a) \\
&\quad + \int_0^a (\mathcal{A} - q) f_c(z) \left[\frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a - z) - W^{(q)}(x - z)\right] \mathrm{d}z \\
&= Q_c(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\, Q_c(a).
\end{aligned} \tag{7.17}$$

Substitute (7.17) and (7.11) in (7.16), put $x = 0$, use that in the bounded variation case $W^{(q)}(0+) = 1/d$, and find by some calculations that,

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^{T_a^+ \wedge T_c^-} e^{-qs}\, \mathrm{d}I_s\right] &= \frac{-a + \int_0^a [l(z - c) + c\, l'(z - c)]\, W^{(q)}(a - z)\, \mathrm{d}z}{1 - \int_0^a l'(z - c) W^{(q)}(a - z) \mathrm{d}z} \\
&= \frac{-Q_c(a)}{J_c(a)}.
\end{aligned} \tag{7.18}$$

Then, put (7.17), (7.11) and (7.18) in (7.16) to get the required statement.□

7.3 Value function

In this section, we find the net present value of profit for the process $V^{\delta,-c}$. Moreover, Propositions 7.2.1 and 7.2.2 are verified for the unbounded variation case in Corollaries 7.3.6 and 7.3.9, respectively.

For convenience, we will first find the value function (7.2) up to some level $a > 0$, and then we can take the limit as $a \uparrow \infty$. That is, we will first find

$$v_a^{\delta,-c}(x, \bar{x}) = v_{\text{tax,a}}^{\delta,-c}(x, \bar{x}) - \eta\, v_{\text{inj,a}}^{\delta,-c}(x, \bar{x}) - \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-} P(V_{\tau_c^-}^{\delta,\infty}) \mathbf{1}_{\left\{\tau_c^- < \tau_a^+\right\}}\right],$$

where

$$v^{\delta,-c}_{\mathrm{tax,a}}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\delta(\overline{V}^{\delta,\infty}_s)\ \mathrm{d}\overline{(X+K^\delta)}_s\right],$$

and

$$v^{\delta,-c}_{\mathrm{inj,a}}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_{0}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\ \mathrm{d}K^\delta_s\right].$$

7.3.1 The tax value function

Since the tax starts when $x = \bar{x}$, then by using the strong Markov property of $(V^{\delta,\infty}, \overline{V}^{\delta,\infty})$, the tax value,

$$\begin{aligned}
v^{\delta,-c}_{\mathrm{tax,a}}(x,\bar{x}) &= \mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\delta(\overline{V}^{\delta,\infty}_s)\ \mathrm{d}\overline{(X+K^\delta)}_s\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\,T^+_{\bar{x}}}\mathbf{1}_{\{T^+_{\bar{x}}<T^-_c\}}\mathbb{E}_{x,\bar{x}}\left[e^{q\,T^+_{\bar{x}}}\int_{\tau^+_{\bar{x}}}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\delta(\overline{V}^{\delta,\infty}_s)\ \mathrm{d}\overline{(X+K^\delta)}_s|\mathcal{F}_{\tau^+_{\bar{x}}}\right]\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\,T^+_{\bar{x}}}\mathbf{1}_{\{T^+_{\bar{x}}<T^-_c\}}\right]\mathbb{E}_{\bar{x},\bar{x}}\left[\int_{0+}^{\tau_c^-\wedge\tau_a^+} e^{-qs}\delta(\overline{V}^{\delta,\infty}_s)\ \mathrm{d}\overline{(X+K^\delta)}_s\right] \\
&= \mathbb{E}_{x,\bar{x}}\left[e^{-q\,T^+_{\bar{x}}}\mathbf{1}_{\{T^+_{\bar{x}}<T^-_c\}}\right]\ v^{\delta,-c}_{\mathrm{tax,a}}(\bar{x},\bar{x}),
\end{aligned} \tag{7.19}$$

where we used that the process $V^{\delta,\infty}$, in the region between level $\bar{x}$ and level c, is just the reflected process Y.

Recall that for $f \in \mathcal{S}_{[b,a]\times[c,d]}$, $y \in [b,a]$ and $z \in [c,d]$,

$$\Gamma^\delta f(y,z) = \frac{\partial}{\partial y}f(y,z)\ \delta(z) - \frac{\partial}{\partial z}f(y,z)(1-\delta(z)). \tag{7.20}$$

Lemma 7.3.1 *Let $V^{\delta,-c}_t := V^{\delta,\infty}_{t\wedge\tau_c^-}$, for all $t \geq 0$, be the natural tax process $V^{\delta,\infty}$ with a limited bail-out at a parameter $c < 0$. For fixed $a > 0$, suppose f is a function with domain $D_f = (-\infty, a] \times (0, a]$, $n \geq 1$, and satisfying the following conditions:*

(I) f is bounded on D_f, and $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \geq 1$, such that it is of the form $f(x,\bar{x}) = g(x)\,h(\bar{x})$, where g and h satisfy the conditions of Definition 4.1.1.

(II)

$$f(x,\bar{x}) = \begin{cases} f(0,\bar{x}), & c \leq x \leq 0, \\ 0, & x < c. \end{cases}$$

(III) $f(a,a)=0$.

(IV) $(\mathcal{A}-q)f(x,\bar{x})=0$ *for* $0<x\leq\bar{x}\leq a$.

(V) $(\mathcal{A}-q)f(0,\bar{x})=0$ *for* $0<\bar{x}\leq a$.

(VI) There exists a locally bounded density for h *such that*

$$\Gamma^{\delta}f(\bar{x},\bar{x})=\delta(\bar{x}),\quad \textit{for all}\ \ 0<\bar{x}\leq a,$$

where Γ^{δ} *is defined in* (7.20).

Then,

$$f(x,\bar{x})=v^{\delta,-c}_{tax,a}(x,\bar{x}),\ \textit{for}\ x\leq\bar{x}\leq a,\ \bar{x}>0. \tag{7.21}$$

Proof Let $(x,\bar{x})$ be fixed, where $x\leq\bar{x}\leq a$ and $\bar{x}>0$. For $t\geq 0$, let $K^{\delta,-c}_t:=K^{\delta}_{t\wedge\tau^-_c}$. Let $\widetilde{V}$ and $\widetilde{K}$ be the right-continuous modifications of $V^{\delta,-c}$ and $K^{\delta,-c}$. Let $\tau^-_c=\inf\{t>0:\widetilde{V}_t<c\}$, $\tau^+_a=\inf\{t>0:\widetilde{V}_t>a\}$, and $\kappa^-_{\frac{1}{n}}=\inf\{t>0:\overline{\widetilde{V}}_t<\frac{1}{n}\}$. Let $T=\tau^-_c\wedge\tau^+_a\wedge\kappa^-_{\frac{1}{n}}$. By (I), $f\in\mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, so we can use Corollary 4.1.3 and follow the same steps of the proof for Lemma 6.2.3 until (6.20),

$$\begin{aligned}
&e^{-q(t\wedge T)}f(\widetilde{V}_{t\wedge T},\overline{\widetilde{V}}_{t\wedge T})-f(x,\bar{x})\\
&\quad=M_{t\wedge T}+\int_{0^+}^{t\wedge T}e^{-qs}\,(\mathcal{A}-q)f(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\,\mathrm{d}s+\int_{0^+}^{t\wedge T}e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c\\
&\qquad+\sum_{0\leq s\leq t\wedge T}e^{-qs}\Big[f(\widetilde{V}_{s-}+\Delta X_s+\Delta\widetilde{K}_s,\overline{\widetilde{V}}_s)-f(\widetilde{V}_{s-}+\Delta X_s,\overline{\widetilde{V}}_s)\Big]\\
&\qquad-\int_{0^+}^{t\wedge T}e^{-qs}\Big[\Gamma^{\delta}f(\overline{\widetilde{V}}_s,\overline{\widetilde{V}}_s)\Big]\,\mathrm{d}\overline{(X+\widetilde{K})}_s,
\end{aligned}$$

where M is a zero mean martingale. Since $\overline{(X+\widetilde{K})}=\overline{(X+K^{\delta})}$, then $\overline{\widetilde{V}}=\overline{V}^{\delta,\infty}$. Also, by conditions (IV), (V) and (VI),

$$\begin{aligned}
&e^{-q(t\wedge T)}f(\widetilde{V}_{t\wedge T},\overline{\widetilde{V}}_{t\wedge T})-f(x,\bar{x})\\
&\quad=M_{t\wedge T}+\int_{0^+}^{t\wedge T}e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c\\
&\qquad+\sum_{0\leq s\leq t\wedge T}e^{-qs}\Big[f(\widetilde{V}_{s-}+\Delta X_s+\Delta\widetilde{K}_s,\overline{\widetilde{V}}_s)-f(\widetilde{V}_{s-}+\Delta X_s,\overline{\widetilde{V}}_s)\Big]\\
&\qquad-\int_{0^+}^{t\wedge T}e^{-qs}\,\delta(\overline{V}^{\delta,\infty}_s)\,\mathrm{d}\overline{(X+K^{\delta})}_s,
\end{aligned}\tag{7.22}$$

By condition (II), $f(x, \bar{x}) = f(0, \bar{x})$ for $c \leq x \leq 0$, then the summation term on the RHS of (7.22) becomes zero as for each $s \leq t \wedge T$,

$$f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) = f(0, \overline{\widetilde{V}}_s) - f(0, \overline{\widetilde{V}}_s) = 0.$$

For the first integral term on the RHS of (7.22), if we are in the bounded variation case, then $(\widetilde{K}_s)^c = 0$. In the unbounded variation case, $(\widetilde{K}_s)^c$ changes when $\widetilde{V}_{s^-} = 0$, but by condition (II) and since $g \in \mathcal{C}^1[c, a]$, $\dfrac{\partial f}{\partial x}(x, \bar{x})|_{x=0} = \lim_{x \uparrow 0} \dfrac{\partial f}{\partial x}(x, \bar{x}) = 0$, as $f(0, \bar{x})$ is constant in x, and hence this term also becomes zero.

After that, we take the expectation and get,

$$\mathbb{E}_{x,\bar{x}} \left[e^{-q(t \wedge T)} f(\widetilde{V}_{t \wedge T}, \overline{\widetilde{V}}_{t \wedge T}) \right] - f(x, \bar{x}) = -\mathbb{E}_{x,\bar{x}} \left[\int_{0+}^{t \wedge T} e^{-qs} \, \delta(\overline{V}_s^{\delta,\infty}) \, \mathrm{d}\overline{(X + K^\delta)}_s \right]. \tag{7.23}$$

Finally, we let t and n go to infinity. By condition (I), f is bounded. So, on the LHS of (7.23), we use the bounded convergence theorem and condition (III) that $f(a, a) = 0$ together with (II) that $f(x, \bar{x}) = 0$ for $x < c$. That is, this term vanishes. The term on the RHS of (7.23), is the tax revenue, which is monotone in t, and therefore, by using the monotone convergence theorem we get,

$$f(x, \bar{x}) = v_{\text{tax},a}^{\delta,-c}(x, \bar{x}).$$

□

Remark 20 We are going to attempt to guess an expression for f, by assuming it satisfies conditions $(I - VI)$ in Lemma 7.3.1, and we will later verify that these conditions hold in Theorem 7.3.5. In order to guess the candidate expression, we use (7.19) and (7.8), for $x \leq \bar{x} \leq a$, so we have that,

$$f(x, \bar{x}) = v_{\text{tax},a}^{\delta,-c}(x, \bar{x}) = \frac{J_c(x)}{J_c(\bar{x})} \, v_{\text{tax},a}^{\delta,-c}(\bar{x}, \bar{x}), \tag{7.24}$$

where J_c as given in (7.9). After that, we use (7.24) and that f satisfies conditions (III) and (VI), so we get the following ODE,

$$\frac{\partial}{\partial \bar{x}} v_{\text{tax},a}^{\delta,-c}(\bar{x}, \bar{x}) - \frac{1}{1 - \delta(\bar{x})} \frac{J_c'(\bar{x})}{J_c(\bar{x})} v_{\text{tax},a}^{\delta,-c}(\bar{x}, \bar{x}) = -\frac{\delta(\bar{x})}{1 - \delta(\bar{x})},$$

such that the boundary condition is

$$v_{\text{tax},a}^{\delta,-c}(a, a) = 0.$$

Thus, solving this ODE by integrating factor method, we can find that

$$
\begin{aligned}
&v^{\delta,-c}_{\mathrm{tax},a}(\bar{x},\bar{x})\\
&\quad = \exp\left\{\int_a^{\bar{x}} \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\left\{-\int_a^{\bar{x}} \exp\left\{-\int_a^{y} \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\frac{\delta(y)}{1-\delta(y)}\mathrm{d}y + C\right\},
\end{aligned}
$$

where C is a constant. To find it, we use the boundary condition and find that $C = 0$. Therefore, we get that the candidate expression should be

$$
v^{\delta,-c}_{\mathrm{tax},a}(\bar{x},\bar{x}) = \int_{\bar{x}}^{a} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\frac{\delta(y)}{1-\delta(y)}\mathrm{d}y.
$$

We give here the meaning for a notation before we present some result,

$$
L^1_{loc}(\mathbb{R}^+) = \left\{f : \mathbb{R}^+ \to \mathbb{R}\,\text{measurable, for any}\, x > 0, \int_0^x |f(y)|\mathrm{d}y < \infty\right\}. \tag{7.25}
$$

Lemma 7.3.2 *[17, Lemma 2.4] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be absolutely continuous on $\mathbb{R}^+$ and $g \in L^1_{loc}(\mathbb{R}^+)$. Assume that $f' \in L^1_{loc}(\mathbb{R}^+)$, where f' denotes a version of the density of f, and further assume that $f(0+) = \lim_{y\downarrow 0} f(y) \in \mathbb{R}$. Then the convolution*

$$
h(y) = \int_0^y f(y-r)\,g(r)\,\mathrm{d}r
$$

has a density on $\mathbb{R}^+$ and a version of it is given, for any $y > 0$, by

$$
h'(y) = \int_0^y f'(y-r)\,g(r)\,\mathrm{d}r + f(0+)\,g(y).
$$

Lemma 7.3.3 *Let $\bar{x} > 0$ and $g(x)$ be a function on $\mathbb{R}$ such that the overshoot expression (7.3) is,*

$$
\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-} f(X_{\rho_0^-})\mathbf{1}_{\{\rho_0^- < \rho_{\bar{x}}^+\}}\right] = g(x), \tag{7.26}
$$

for some function f on $(-\infty, 0]$. Then,

$$
e^{-q(t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-)}\,g(X_{t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-}),\ t \geq 0,
$$

is a martingale.

Proof We use identity (7.26) with the same argument given in [50, p.192] to prove that

$$
e^{-q(t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-)}\,g(X_{t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-})
$$

is a martingale.$\square$

Lemma 7.3.4 *Let g be a function on $\mathbb{R}$ such that $g \in \mathcal{S}_{[0,\bar{x}]}$, for any $\bar{x} > 0$, where the first and the second derivatives of g are right-continuous on $[0, \bar{x}]$. Suppose that*

$$e^{-q(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})}\, g(X_{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}})$$

is a martingale. Then, for $x \in (0,\infty)$,

$$(\mathcal{A} - q)\, g(x) = 0.$$

Proof We use the same argument of proof in [14]. By assumption that $g \in \mathcal{S}_{[0,\bar{x}]}$, then g satisfies the first and second conditions of Definition 4.1.1 on $[0,\bar{x}]$. That implies, we can apply the extant second derivative Meyer-Itô formula [52, Theorem 71] to $e^{-qt}g(X_t)$ in the unbounded variation case and [52, Theorem 78] in the bounded variation case. Therefore, we have the expansion, for $x \in (0, \bar{x})$,

$$\begin{aligned}
&e^{-q(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})}\, g(X_{(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})}) - g(x)\\
&= \int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} -qe^{-qs}\, g(X_{s-})\,\mathrm{d}s + \int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs}\, g'(X_{s-})\,\mathrm{d}X_s\\
&\quad + \frac{\sigma^2}{2}\int_{0}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs} g''(X_{s-})\,\mathrm{d}\,[X,X]_s^c\\
&\quad + \sum_{0<s\le(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})} e^{-qs}\,\left[g(X_s) - g(X_{s-}) - g'(X_{s-})\Delta X_s\right].
\end{aligned}$$

This expansion can be rewritten using (2.3) as

$$\begin{aligned}
&e^{-q(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})}\, g(X_{(t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-})}) - g(x)\\
&\quad = M_t + \int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs}\,(\mathcal{A}-q)\, g(X_{s-})\,\mathrm{d}s,
\end{aligned} \tag{7.27}$$

where

$$\begin{aligned}
M_t = &\int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs}\, g'(X_{s-})\,\mathrm{d}\big[X_s - \mu s - \sum_{0<u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|>1\}}\big]\\
&+\Bigg\{\sum_{0<s\le t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs}\big[g(X_{s-}+\Delta X_s) - g'(X_{s-})\Delta X_s \mathbf{1}_{\{|\Delta X_s|\le 1\}}\big]\\
&\quad - \int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}}\int_{0+}^{\infty} e^{-qs}\big[g(X_{s-}-\theta) - g(X_{s-}) + \theta\, g'(X_{s-})\mathbf{1}_{\{0<\theta\le 1\}}\big]\nu(\mathrm{d}\theta)\,\mathrm{d}s\Bigg\}
\end{aligned}$$

is a martingale. Since the left hand side of (7.27) is a martingale by assumptions, then

$$\int_{0+}^{t\wedge\rho_{\bar{x}}^{+}\wedge\rho_0^{-}} e^{-qs}\,(\mathcal{A}-q)\, g(X_{s-})\,\mathrm{d}s = 0.$$

Therefore, by the right-continuity assumptions, for $x \in (0, \bar{x})$,

$$(\mathcal{A} - q)\, g(x) = 0,$$

which is valid also for any $x > 0$ as $\bar{x} > 0$ is arbitrary here.□

Theorem 7.3.5 *For $\bar{x} > 0$, let $\delta : [\bar{x}, \infty) \to [0, 1)$ be a natural tax rate function such that the function $1/(1-\delta(s))$ is locally bounded. Suppose that X has positive Gaussian coefficient in the unbounded variation case. Then, the net present value of taxation for the process $V^{\delta,-c}$, for $x \leq \bar{x}$ and $\bar{x} > 0$, is given by*

$$v_{tax}^{\delta,-c}(x, \bar{x}) = \frac{J_c(x)}{J_c(\bar{x})} \int_{\bar{x}}^{\infty} \exp\left\{ - \int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y, \tag{7.28}$$

where J_c as given in (7.9).

PROOF For fixed $a > 0$, let $f : (-\infty, a] \times (0, a] \to \mathbb{R}$ be given by,

$$f(x, \bar{x}) = g(x)\ h(\bar{x}),$$

where the function $g : (-\infty, a] \to \mathbb{R}$ is,

$$g(x) = J_c(x) = \left[1 - \int_0^x l'(z - c) W^{(q)}(x - z) \mathrm{d}z \right] \mathbf{1}_{\{x \geq c\}}. \tag{7.29}$$

The function $h : (0, a] \to \mathbb{R}$ is given by,

$$h(\bar{x}) = \frac{1}{J_c(\bar{x})} \int_{\bar{x}}^{a} \exp\left\{ - \int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \frac{\delta(y)}{1-\delta(y)}\, \mathrm{d}y,$$

which can be written as

$$h(\bar{x}) = \frac{1}{J_c(\bar{x})} \exp\left\{ - \int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \int_{\bar{x}}^{a} \exp\left\{ - \int_{a}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \frac{\delta(y)}{1-\delta(y)} \mathrm{d}y.$$

We only need to show that f satisfies the conditions in Lemma 7.3.1, then by (7.21) and taking the limit as a goes to infinity, the statement is proved. Note first that, from (7.6), $l'(z - c)$ is finite and the integral term in (7.29) is finite. Also, as $l'(z - c)$ is negative, then $J_c(x) \geq 1$ and bounded on $(-\infty, a]$. As by assumption $1/(1-\delta(s))$ is bounded, then this implies that f is bounded on $(-\infty, a] \times (0, a]$. Next, we prove that $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \geq 1$. For that, we verify that g and h satisfy the conditions in Definition 4.1.1 on $[c, a]$, and on $[\frac{1}{n}, a]$, for each $n \geq 1$, respectively. By definition, g

equals 0 on $(-\infty, c)$ and 1 on $[c, 0]$. Accordingly, g' is zero on $(-\infty, 0)$, and on $(0, \infty)$ is

$$g'(x) = -l'(x-c)\, W^{(q)}(0+) - \int_0^x l'(z-c)\, W^{(q)\prime}(x-z)\,\mathrm{d}z. \tag{7.30}$$

Therefore, in the bounded variation case, g is absolutely continuous with locally bounded density g' on $[c, a]$. This is clear on $(-\infty, c)$ or $[c, 0]$, as explained above. On $(0, \infty)$, since $W^{(q)}$ is absolutely continuous with density $W^{(q)\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $l'(z-c) \in L^1_{loc}(\mathbb{R}^+)$, then we use Lemma (7.3.2) and see that g has a density on $\mathbb{R}^+$ given by (7.30), which is bounded on $(0, \infty)$. In the unbounded variation case, on $(0, \infty)$, clearly g is continuous, and as $W^{(q)}(0+) = 0$, $g'(x) = -\int_0^x l'(z-c)\, W^{(q)\prime}(x-z)\,\mathrm{d}z$. Since X has positive Gaussian component, then by Theorem 2.2.4, $W^{(q)}$ is twice continuously differentiable. Since $W^{(q)\prime}$ is absolutely continuous with density $W^{(q)\prime\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $l'(z-c) \in L^1_{loc}(\mathbb{R}^+)$, then again we use Lemma (7.3.2), and see that g' is an absolutely continuous with a density g'',

$$g''(x) = -l'(x-c)\, W^{(q)\prime}(0+) - \int_0^x l'(z-c)\, W^{(q)\prime\prime}(x-z)\,\mathrm{d}z,$$

which is clearly bounded on $(0, \infty)$. For the function h, it is absolutely continuous with locally bounded density on $[\frac{1}{n}, a]$, for each $n \geq 1$, since we have that

$$h(\bar{x}) - h(\frac{1}{n}) = \int_{\frac{1}{n}}^{s} h'(r)\,\mathrm{d}r,$$

for all $s \in [\frac{1}{n}, a]$, for each $n \geq 1$, where h' is given by

$$\begin{aligned} h'(\bar{x}) = {} & \frac{-J_c'(\bar{x})}{J_c(\bar{x})^2}\, A(\bar{x}) \exp\left\{-\int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \\ & + \frac{1}{J_c(\bar{x})} \exp\left\{-\int_{\bar{x}}^{a} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} A(\bar{x}) \frac{J_c'(\bar{x})}{J_c(\bar{x})} \frac{1}{1-\delta(\bar{x})} - \frac{1}{J_c(\bar{x})} \frac{\delta(\bar{x})}{1-\delta(\bar{x})}, \end{aligned} \tag{7.31}$$

where

$$A(\bar{x}) = \int_{\bar{x}}^{a} \exp\left\{-\int_{a}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \frac{\delta(y)}{1-\delta(y)}\mathrm{d}y.$$

For the second condition of Definition 4.1.1, we need to show that there exists $\lambda > 0$ such that $s \mapsto \int_\lambda^\infty g(s-\theta)\; \nu(\mathrm{d}\theta)$ is bounded on (c, a). Since we have that

$$g(s-\theta) = \left[1 - \int_0^{s-\theta} l'(z-c) W^{(q)}(s-\theta-z)\mathrm{d}z\right] \mathbf{1}_{\{s-\theta \geq c\}},$$

so if we choose $\lambda = a$, then $\theta \geq \lambda = a \geq s$ which implies that $s - \theta \leq 0$. Therefore, either $s - \theta < c$ and thus, $g(s-\theta) = 0$, or $s - \theta \geq c$ which implies that $g(s-\theta) = 1$, and thus, by the definition of Lévy measure (see [9, p.29]), $\nu(\lambda,\infty)<\infty$. So, in either cases, the condition is satisfied. Hence, f satisfies condition (I).

Since $h(a)=0$, then $f(a,a)=0$. From (7.29), it is clear that $g(x)=g(0)$ for $c\le x\le 0$, and $g(x)=0$ for $x<c$. Hence, conditions (II) and (III) are satisfied.

By (7.11),

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\,\rho_0^-}\,\mathbf{1}_{\left\{X_{\rho_0^-}\ge c\right\}}\,\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right]=g(x)-\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\,g(\bar{x}),$$

then by Lemma 7.3.3, (2.13), and taking linear combination,

$$e^{-q(t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-)}\,g(X_{t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-})$$

is a martingale. Since g satisfies the assumptions of Lemma 7.3.4, then

$$(\mathcal{A}-q)\,g(x)=0,\quad x>0,$$

which is condition (IV). By continuity of g, right-continuity of g' and g'',

$$(\mathcal{A}-q)\,g(0)=0,$$

which is condition (V).

For condition (VI), we have,

$$\frac{\partial f(x,\bar{x})}{\partial x}|_{x=\bar{x}}=J_c'(\bar{x})\,h(\bar{x}),\tag{7.32}$$

where J' is given by (7.30) and

$$\frac{\partial f(x,\bar{x})}{\partial \bar{x}}|_{x=\bar{x}}=J_c(\bar{x})\ h'(\bar{x}),\tag{7.33}$$

where h' is given by (7.31). By (7.32), (7.33) and (7.31) we find that $\Gamma^\delta f(\bar{x},\bar{x})=\delta(\bar{x})$ for all $0<\bar{x}\le a$.□

Remark 21 Note that, when we take limit of $c\downarrow-\infty$ in (7.28), and using that $\lim_{c\downarrow-\infty} l'(z-c)=-q$, we get the same result as in (6.28).

Corollary 7.3.6 *For $x\le\bar{x}$, the two sided exit formula given in* (7.8) *is correct even if X is in the unbounded variation case.*

Proof By Lemma 7.3.5, we verified that the expression (7.28) is the correct one in all cases of X. Hence, by (7.19) the statement is proved.□

7.3.2 The injection value function

We have that,

$$v_{\text{inj}}^{\delta,-c}(x,\bar{x}) = \mathbb{E}_{x,\bar{x}}\left[\int_0^{\tau_{\bar{x}}^+\wedge\tau_c^-} e^{-qs}\, \mathrm{d}K_s^\delta\right] + \mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+\wedge\tau_c^-}^{\tau_c^-} e^{-qs}\, \mathrm{d}K_s^\delta\right]. \tag{7.34}$$

For the first term in (7.34), we use Proposition 7.2.2,

$$\mathbb{E}_{x,\bar{x}}\left[\int_0^{\tau_{\bar{x}}^+\wedge\tau_c^-} e^{-qs}\, \mathrm{d}K_s^\delta\right] = -\mathbb{E}_{x,\bar{x}}\left[\int_0^{T_{\bar{x}}^+\wedge T_c^-} e^{-qs}\, \mathrm{d}I_s\right] \quad = -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})}\, Q_c(\bar{x}), \tag{7.35}$$

where Q_c is given by (7.15) and J_c is given by (7.9).

Remark 22 By (7.7) and Remark 19, one can show that when $c \downarrow -\infty$, (7.35) becomes,

$$\mathbb{E}_{x,\bar{x}}\left[\int_{0+}^{T_{\bar{x}}^+} e^{-qs}\, \mathrm{d}K_s^\delta\right] = -\left[\overline{Z}^{(q)}(x) + \frac{\psi'(0^+)}{q}\right] + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})}\left[\overline{Z}^{(q)}(\bar{x}) + \frac{\psi'(0^+)}{q}\right],$$

which agrees with the previous expression in the forced bail-out case, (6.33).

For the second term in (7.34), by the strong Markov property of $(V^{\delta,\infty}, \overline{V}^{\delta,\infty})$,

$$\mathbb{E}_{x,\bar{x}}\left[\int_{\tau_{\bar{x}}^+\wedge\tau_c^-}^{\tau_c^-} e^{-qs}\, \mathrm{d}K_s^\delta\right] = \mathbb{E}_{x,\bar{x}}\left[e^{-q\,T_{\bar{x}}^+}\mathbf{1}_{\{T_{\bar{x}}^+ < T_c^-\}}\right]\, v_{\text{inj}}^{\delta,-c}(\bar{x},\bar{x}).$$

Therefore, the injection value function for the process (7.1), for $x \le \bar{x} \le a$,

$$v_{\text{inj},a}^{\delta,-c}(x,\bar{x}) = -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})}\left[Q_c(\bar{x}) + v_{\text{inj},a}^{\delta,-c}(\bar{x},\bar{x})\right]. \tag{7.36}$$

Lemma 7.3.7 *Let $V_t^{\delta,-c} := V_{t\wedge\tau_c^-}^{\delta,\infty}$, for all $t \ge 0$, be the natural tax process $V^{\delta,\infty}$ with a limited bail-out at a parameter $c < 0$. For fixed $a > 0$, suppose f is a function with domain $D_f = (-\infty, a] \times (0, a]$ and satisfying the following conditions:*

(I) f is bounded on D_f and $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \ge 1$, such that it is of the form $f(x,\bar{x}) = f_1(x,\bar{x}) + f_2(x,\bar{x})$, where for each $i = 1, 2$, $f_i(x,\bar{x}) = g_i(x)\, h_i(\bar{x})$, and each g_i and h_i satisfies the conditions of Definition 4.1.1.

(II)

$$f(x,\bar{x}) = \begin{cases} -x + f(0,\bar{x}), & c \le x \le 0, \\ 0, & x < c, \end{cases}$$

(III) $f(a,a) = 0$.

(IV) $(\mathcal{A}-q)f(x,\bar{x}) = 0$ *for* $0 < x \le \bar{x} \le a$.

(V) $(\mathcal{A}-q)f(0,\bar{x}) = 0$ *for* $0 < \bar{x} \le a$.

(VI) There exists a locally bounded density for each h_i *such that*

$$\Gamma^\delta f(\bar{x},\bar{x}) = 0 \quad \textit{for all} \quad 0 < \bar{x} \le a,$$

where Γ^δ *is defined in* (7.20).

Then,

$$f(x,\bar{x}) = v^{\delta,-c}_{inj,a}(x,\bar{x}), \ \textit{for } x \le \bar{x} \le a, \ \bar{x} > 0. \tag{7.37}$$

PROOF Let $(x,\bar{x})$ be fixed, where $x \le \bar{x} \le a$ and $\bar{x} > 0$. For $t \ge 0$, let $K^{\delta,-c}_t := K^\delta_{t\wedge\tau_c^-}$. Let $\widetilde{V}$ and $\widetilde{K}$ be the right-continuous modifications of $V^{\delta,-c}$ and $K^{\delta,-c}$. Let $\tau_c^- = \inf\{t > 0 : \widetilde{V}_t < c\}$, $\tau_a^+ = \inf\{t > 0 : \widetilde{V}_t > a\}$, and $\kappa^-_{\frac{1}{n}} = \inf\{t > 0 : \overline{\widetilde{V}}_t < \frac{1}{n}\}$. Let $T = \tau_c^- \wedge \tau_a^+ \wedge \kappa^-_{\frac{1}{n}}$. By condition (I), $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \ge 1$, so we can use Corollary 4.1.3 and follow the same steps of the proof for Lemma 6.2.3 until (6.20),

$$\begin{aligned} &e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x,\bar{x}) \\ &\quad = M_{t\wedge T} + \int_{0^+}^{t\wedge T} e^{-qs}\,(\mathcal{A}-q)f(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\,\mathrm{d}s + \int_{0^+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c \\ &\qquad + \sum_{0\le s\le t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s)\right] \\ &\qquad - \int_{0^+}^{t\wedge T} e^{-qs}\left[\Gamma^\delta f(\overline{\widetilde{V}}_s, \overline{\widetilde{V}}_s)\right]\mathrm{d}\overline{(X+\widetilde{K})}_s, \end{aligned}$$

where M is a zero mean martingale, and $\Delta\widetilde{K}_0 = \widetilde{K}_0 - \widetilde{K}_{0^-}$ such that $\widetilde{K}_{0^-} := 0$. Since $\overline{(X+\widetilde{K})} = \overline{(X+K^\delta)}$, then $\overline{\widetilde{V}} = \overline{V}^{\delta,\infty}$. Also, by conditions (IV), (V), and (VI),

$$\begin{aligned} &e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x,\bar{x}) \\ &\quad = M_{t\wedge T} + \int_{0^+}^{t\wedge T} e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c \\ &\qquad + \sum_{0\le s\le t\wedge T} e^{-qs}\left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta\widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s)\right]. \end{aligned} \tag{7.38}$$

By condition (II), since for any $s \le t \wedge T$, $\widetilde{V}_{s^-} + \Delta X_s, \widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s \in [c, 0]$, then

$$f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) = -\Delta \widetilde{K}_s.$$

This implies that the summation term of (7.38) becomes

$$-\sum_{0<s\le t\wedge T} e^{-qs}\Delta \widetilde{K}_s - \widetilde{K}_0. \tag{7.39}$$

For the integral term on the RHS of (7.38), if we are in the bounded variation case, then $(\widetilde{K}_s)^c = 0$, that is, $\widetilde{K}_s = \sum_{0<u\le s}\Delta \widetilde{K}_u$, and hence, by (7.39) this integral term becomes,

$$-\int_{0^+}^{t\wedge T} e^{-qs}\mathrm{d}\widetilde{K}_s - \widetilde{K}_0. \tag{7.40}$$

If we are in the unbounded variation case, for the first integral term on the RHS of (7.38), $(\widetilde{K}_s)^c$ changes when $\widetilde{V}_{s^-} = 0$. Since $g_i \in \mathcal{C}^1[c, a]$, for $i = 1, 2$, then $\dfrac{\partial f}{\partial x}(x, \bar{x})|_{x=0} = \lim_{x\uparrow 0}\dfrac{\partial f}{\partial x}(x, \bar{x}) = -1$, and hence, by using (7.39), also this integral term becomes (7.40). Since

$$\int_{0^+}^{t\wedge T} e^{-qs}\mathrm{d}\widetilde{K}_s = \int_0^{t\wedge T} e^{-qs}\mathrm{d}K^\delta_s - K^\delta_{0+},$$

where $K^\delta_{0+} = \widetilde{K}_0$. Therefore, (7.38) becomes,

$$e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x, \bar{x}) = M_{t\wedge T} - \int_0^{t\wedge T} e^{-qs}\mathrm{d}K^\delta_s.$$

Next, we take expectations,

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T})\right] - f(x, \bar{x}) = -\mathbb{E}_{x,\bar{x}}\left[\int_0^{t\wedge T} e^{-qs}\mathrm{d}K^\delta_s\right]. \tag{7.41}$$

Finally, we let t and n go to infinity. For the LHS of (7.41), since f is bounded by condition (I), we use the bounded convergence theorem together with conditions (II) that $f(a, a) = 0$, and condition (III) that $f(x, \bar{x}) = 0$ for $x < c$, hence, in either cases the first term on the LHS vanishes. The term on the RHS is the accumulate of capital injections K^δ, which is monotone in t, and therefore, by using the monotone convergence theorem, we get,

$$f(x, \bar{x}) = v^{\delta,-c}_{\mathrm{inj},a}(x, \bar{x}).$$

□

Remark 23 We are going to attempt to guess an expression for f, by assuming it satisfies conditions $(I - VI)$ in Lemma 7.3.7, and we will later verify that these conditions hold in Theorem 7.3.8. In order to guess the candidate expression for $v^{\delta,-c}_{\mathrm{inj},a}$, we use (7.36) and (7.37) for $x \le \bar{x} \le a$,

$$f(x,\bar{x}) = v^{\delta,-c}_{\mathrm{inj},a}(x,\bar{x}) = -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})}\left[Q_c(\bar{x}) + v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x})\right]. \tag{7.42}$$

We use (7.42) and that f satisfies conditions (III) and (VI), so we get the following ODE,

$$\frac{\partial}{\partial \bar{x}} v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x}) - \frac{1}{1-\delta(\bar{x})}\frac{J_c'(\bar{x})}{J_c(\bar{x})} v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x}) = -\frac{1}{1-\delta(\bar{x})}\left[Q_c'(\bar{x}) - \frac{J_c'(\bar{x})}{J_c(\bar{x})} Q_c(\bar{x})\right],$$

with the boundary condition

$$v^{\delta,-c}_{\mathrm{inj},a}(a,a) = 0.$$

Thus, solving this ODE by integrating factor method, we can find that

$$\begin{aligned}
&v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x})\\
&= \left\{-\int_a^{\bar{x}} \frac{1}{1-\delta(y)} \exp\left\{-\int_a^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y)\right]\mathrm{d}y\right\}\times\\
&\quad \exp\left\{\int_a^{\bar{x}} \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}.
\end{aligned}$$

Therefore, we get the expression,

$$\begin{aligned}
&v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x})\\
&= \int_{\bar{x}}^a \frac{1}{1-\delta(y)} \exp\left\{-\int_{\bar{x}}^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y)\right]\mathrm{d}y.
\end{aligned}$$

This implies that the final candidate expression for $v^{\delta,-c}_{\mathrm{inj},a}(x,\bar{x})$, $x \le \bar{x}$, is given by,

$$\begin{aligned}
v^{\delta,-c}_{\mathrm{inj},a}(x,\bar{x}) &= -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})}\left[Q_c(\bar{x}) + v^{\delta,-c}_{\mathrm{inj},a}(\bar{x},\bar{x})\right]\\
&= -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})}\left\{Q_c(\bar{x})\right.\\
&\quad \left. + \int_{\bar{x}}^a \frac{1}{1-\delta(y)} \exp\left\{-\int_{\bar{x}}^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\}\left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y)\right]\mathrm{d}y\right\}.
\end{aligned} \tag{7.43}$$

Remark 24 Note that, when we take limit of (7.43) as $c \downarrow -\infty$ by using Remark 18, and by following the steps starting from (6.42), we get expression (6.44). Then by taking the limit as $a \uparrow \infty$, we get the same value of injections as in the forced bail-out case (6.45).

Theorem 7.3.8 *For $\bar{x} > 0$, let $\delta : [\bar{x}, \infty) \to [0, 1)$ be a natural tax rate function such that the function $1/(1 - \delta(s))$ is locally bounded. Suppose that X has positive Gaussian coefficient in the unbounded variation case. Then, the net present value of capital injections for the process $V^{\delta,-c}$, for $x \leq \bar{x}$ and $\bar{x} > 0$, is given by*

$$\begin{aligned} v_{inj}^{\delta,-c}(x, \bar{x}) \\ &= -Q_c(x) + \frac{J_c(x)}{J_c(\bar{x})} \Bigg\{ Q_c(\bar{x}) \\ &\quad + \int_{\bar{x}}^{\infty} \frac{1}{1 - \delta(y)} \exp\left\{ -\int_{\bar{x}}^{y} \frac{1}{1 - \delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y) \right] \mathrm{d}y \Bigg\}, \end{aligned} \tag{7.44}$$

where J_c and Q_c are given by (7.9) and (7.15), respectively.

Proof For fixed $a > 0$, let $f : (-\infty, a] \times (0, a] \to \mathbb{R}$ be given by,

$$f(x, \bar{x}) = g_1(x)\ h_1(\bar{x}) + g_2(x)\ h_2(\bar{x}).$$

The functions $g_i : (-\infty, a] \to \mathbb{R}$, for $i = 1, 2$, are

$$g_1(x) = Q_c(x) = \left[x - \int_0^x [l(z - c) + c\, l'(z - c)] W^{(q)}(x - z) \mathrm{d}z \right] \mathbf{1}_{\{x \geq c\}}, \tag{7.45}$$

and

$$g_2(x) = J_c(x) = \left[1 - \int_0^x l'(z - c) W^{(q)}(x - z) \mathrm{d}z \right] \mathbf{1}_{\{x \geq c\}}. \tag{7.46}$$

The functions $h_i : (0, a] \to \mathbb{R}$, for $i = 1, 2$, are $h_1(\bar{x}) = -1$ and

$$\begin{aligned} h_2(\bar{x}) = \frac{1}{J_c(\bar{x})} \Bigg\{ & Q_c(\bar{x}) \\ & + \int_{\bar{x}}^{a} \frac{1}{1 - \delta(y)} \exp\left\{ -\int_{\bar{x}}^{y} \frac{1}{1 - \delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y) \right] \mathrm{d}y \Bigg\}, \end{aligned}$$

which can be written as

$$\begin{aligned} h_2(\bar{x}) = \frac{1}{J_c(\bar{x})} \Bigg\{ & Q_c(\bar{x}) + \exp\left\{ -\int_{\bar{x}}^{a} \frac{1}{1 - \delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \times \\ & \int_{\bar{x}}^{a} \frac{1}{1 - \delta(y)} \exp\left\{ -\int_{a}^{y} \frac{1}{1 - \delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \left[Q_c'(y) - \frac{J_c'(y)}{J_c(y)} Q_c(y) \right] \mathrm{d}y \Bigg\}. \end{aligned}$$

We only need to show that f satisfies the conditions in Lemma 7.3.7, then by (7.37) and taking the limit as a goes to infinity, the statement is proved. From the proof of Theorem (7.3.5), $J_c(x) \geq 1$ is bounded on $(-\infty, a]$. Moreover, by the expansion in Remark 18, $l(z-c) + c\, l'(z-c)$ is finite and the integral term in (7.45) is finite, and x is bounded by a, that is, $Q_c(x)$ is bounded on its domain. Also, as by assumption $1/(1-\delta(s))$ is bounded, then this implies that f is bounded on $(-\infty, a] \times (0, a]$. Next, we prove that $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \geq 1$. For that, we should verify that each g_i and h_i, for $i = 1, 2$, satisfy the conditions in Definition 4.1.1 on $[c, a]$, and on $[\frac{1}{n}, a]$, for each $n \geq 1$, respectively. For condition (I), by definition, $g_1(x)$ equals 0 on $(-\infty, c)$ and x on $[c, 0]$. Accordingly, g_1' is zero on $(-\infty, c)$, and 1 on $(c, 0)$. On $(0, \infty)$, it is given by

$$g_1'(x) = 1 - [l(x-c) + c\, l'(x-c)]\, W^{(q)}(0+) - \int_0^x [l(z-c) + c\, l'(z-c)]\, W^{(q)\prime}(x-z)\, \mathrm{d}z. \tag{7.47}$$

Therefore, in the bounded variation case, g_1 is absolutely continuous with locally bounded density g_1' on $[c, a]$. This is clear on $(-\infty, c)$ or $[c, 0]$, as explained above. On $(0, \infty)$, since $W^{(q)}$ is absolutely continuous with density $W^{(q)\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $[l(z-c)+c\, l'(z-c)] \in L^1_{loc}(\mathbb{R}^+)$, then we use Lemma 7.3.2 and see that g_1 has a density on $\mathbb{R}^+$ given by (7.47), which is bounded on $(0, \infty)$. In the unbounded variation case, on $(0, \infty)$, clearly g_1 is continuous, and as $W^{(q)}(0+) = 0$, $g_1'(x) = 1 - \int_0^x [l(z-c) + c\, l'(z-c)]\, W^{(q)\prime}(x-z)\, \mathrm{d}z$. Since X has positive Gaussian component, then by Theorem 2.2.4, $W^{(q)}$ is twice continuously differentiable. Since $W^{(q)\prime}$ is absolutely continuous with density $W^{(q)\prime\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $[l(z-c) + c\, l'(z-c)] \in L^1_{loc}(\mathbb{R}^+)$, then again we use Lemma 7.3.2, and see that g_1' is an absolutely continuous with a density g_1'',

$$g_1''(x) = -[l(x-c) + c\, l'(x-c)]\, W^{(q)\prime}(0+) - \int_0^x [l(z-c) + c\, l'(z-c)]\, W^{(q)\prime\prime}(x-z)\, \mathrm{d}z,$$

which is clearly bounded on $(0, \infty)$. For the functions h_i, $i = 1, 2$, each one is absolutely continuous with locally bounded density on $[\frac{1}{n}, a]$, for each $n \geq 1$. This is because h_1 is constant, and for h_2 since we have that,

$$h_2(\bar{x}) - h_2(\frac{1}{n}) = \int_{\frac{1}{n}}^s h_2'(r)\, \mathrm{d}r,$$

for all $s \in [\frac{1}{n}, a]$, for each $n \geq 1$, where h_2' is given by

$$\begin{aligned}
h_2'(\bar{x}) = &-\frac{J_c'(\bar{x})}{J_c(\bar{x})}\, h_2(\bar{x}) + \frac{1}{J_c(\bar{x})}\, Q_c'(\bar{x}) \\
&+ \frac{J_c'(\bar{x})}{J_c(\bar{x})^2}\, \frac{1}{1-\delta(\bar{x})} \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)}\, \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)}\, \frac{J_c'(s)}{J_c(s)} \mathrm{d}s\right\} A(y)\, \mathrm{d}y \\
&- \frac{1}{J_c(\bar{x})}\, \frac{1}{1-\delta(\bar{x})}\, A(\bar{x}),
\end{aligned} \tag{7.48}$$

such that

$$A(y) = Q_c'(y) - \frac{J_c'(y)}{J_c(y)}\, Q_c(y).$$

For the second condition of Definition 4.1.1, we need to show that there exists $\lambda > 0$ such that $s \mapsto \int_{\lambda}^{\infty} g_1(s-\theta)\, \nu(\mathrm{d}\theta)$ is bounded on (c, a). Now, we have that

$$g_1(s-\theta) = \left[(s-\theta) - \int_0^{s-\theta} [l(z-c) + c\, l'(z-c)]\, W^{(q)}(s-\theta-z)\mathrm{d}z\right] \mathbf{1}_{\{s-\theta \geq c\}}, \tag{7.49}$$

so if we choose $\lambda = a$, then $\theta \geq \lambda = a \geq s$ and hence $s - \theta \leq 0$. Therefore, either $s - \theta < c$ which implies that $g_1(s-\theta) = 0$, or $0 \geq s - \theta \geq c$ which implies that $g_1(s-\theta) = s - \theta$, but then g_1 is bounded by a and thus

$$\left| \int_{\lambda}^{\infty} g_1(s-\theta)\, \nu(\mathrm{d}\theta)\right| \leq a \int_{\lambda}^{\infty} \nu(\mathrm{d}\theta),$$

where the last integral is a bounded integral for all $s \in (c, a)$. So, in either cases, the condition is satisfied. For g_2, see the proof of Lemma 7.3.5. Hence, f satisfies condition (I).

Clearly, $f(a, a) = -Q_c(a) + Q_c(a) = 0$. From (7.45) and (7.46), it is clear that $g_1(x) = x$ and $g_2(x) = 1$ for $c \leq x \leq 0$. Also, $g_1(x) = 0$ and $g_2(x) = 0$ for $x < c$. Thus, $f(x, \bar{x}) = -x + f(0, \bar{x})$ for $c \leq x \leq 0$ and $f(x, \bar{x}) = 0$ for $x < c$. Hence, conditions (II) and (III) are satisfied.

By (7.17),

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}\left(X_{\rho_0^-} \mathbf{1}_{\left\{X_{\rho_0^-} \geq c\right\}}\right) \mathbf{1}_{\{\rho_0^- < \rho_{\bar{x}}^+\}}\right] = g_1(x) - \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\, g_1(\bar{x}),$$

then by Lemma 7.3.3, (2.13), and taking linear combination,

$$e^{-q(t \wedge \rho_{\bar{x}}^+ \wedge \rho_0^-)}\, g_1(X_{t \wedge \rho_{\bar{x}}^+ \wedge \rho_0^-})$$

is a martingale. Since g_1 satisfies the assumptions of Lemma 7.3.4, then

$$(\mathcal{A} - q)\, g_1(x) = 0, \quad x > 0,$$

which is condition (IV). By continuity of g_1, right-continuity of g_1' and g_1'',

$$(\mathcal{A} - q)\, g_1(0) = 0,$$

which is condition (V).

For condition (VI),

Now,

$$\frac{\partial f(x,\bar{x})}{\partial x}|_{x=\bar{x}} = -Q_c'(\bar{x}) + J_c'(\bar{x})\, h_2(\bar{x}), \tag{7.50}$$

and

$$\frac{\partial f(x,\bar{x})}{\partial \bar{x}}|_{x=\bar{x}} = J_c(\bar{x})\, h_2'(\bar{x}), \tag{7.51}$$

where h_2' is given by (7.48)

Then, by (7.50), (7.51) and (7.48), we find that $\Gamma^{\delta} f(\bar{x},\bar{x}) = 0.\Box$

Corollary 7.3.9 *For $x \leq \bar{x}$, the identity in (7.14) is correct even if X is in the unbounded variation case.*

Proof By Lemma 7.3.8, we verified that the expression (7.44) is the correct one in all cases of X. Hence, by (7.36) the statement is proved.$\Box$

7.3.3 The penalty value function

Recall that $P\colon \mathbb{R} \to \mathbb{R}$ is a penalty function which is a bounded function.

Lemma 7.3.10 *Let $V_t^{\delta,-c} := V_{t\wedge\tau_c^-}^{\delta,\infty}$, for all $t \geq 0$, be the natural tax process $V^{\delta,\infty}$ with a limited bail-out at a parameter $c < 0$. For fixed $a > 0$, suppose f is a function with domain $D_f = (-\infty, a] \times (0, a]$ and satisfying the following conditions:*

(I) f is bounded on D_f, and $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \geq 1$, such that it is of the form $f(x,\bar{x}) = f_1(x,\bar{x}) + f_2(x,\bar{x})$, where for each $i = 1, 2$, $f_i(x,\bar{x}) = g_i(x)\, h_i(\bar{x})$, and each g_i and h_i satisfies the conditions of Definition 4.1.1.

(II)

$$f(x,\bar{x}) = \begin{cases} f(0,\bar{x}), & c \leq x \leq 0, \\ P(x), & x < c, \end{cases}$$

(III) $f(a,a)=0$.

(IV) $(\mathcal{A}-q)f(x,\bar{x})=0$ *for* $0<x\le\bar{x}\le a$.

(V) $(\mathcal{A}-q)f(0,\bar{x})=0$ *for* $0<\bar{x}\le a$.

(VI) There exists a locally bounded density for each h_i *such that*

$$\Gamma^\delta f(\bar{x},\bar{x})=0 \ \text{ for all } \ 0<\bar{x}\le a,$$

where Γ^δ *is defined in* (7.20).

Then,

$$f(x,\bar{x})=\mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-}P(V^{\delta,\infty}_{\tau_c^-})\mathbf{1}_{\{\tau_c^-<\tau_a^+\}}\right], \quad \text{for } x\le\bar{x}\le a,\ \bar{x}>0. \tag{7.52}$$

Proof Let $(x,\bar{x})$ be fixed, where $x\le\bar{x}\le a$ and $\bar{x}>0$. For $t\ge 0$, let $K_t^{\delta,-c}:=K^\delta_{t\wedge\tau_c^-}$. Let $\widetilde{V}$ and $\widetilde{K}$ be the right-continuous modifications of $V^{\delta,-c}$ and $K^{\delta,-c}$. Let $\tau_c^-=\inf\{t>0:\widetilde{V}_t<c\}$, $\tau_a^+=\inf\{t>0:\widetilde{V}_t>a\}$, and $\kappa^-_{\frac{1}{n}}=\inf\{t>0:\overline{\widetilde{V}}_t<\frac{1}{n}\}$. Let $T=\tau_c^-\wedge\tau_a^+\wedge\kappa^-_{\frac{1}{n}}=\tau_c^-$. By condition (I), $f\in\mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n\ge1$, so we can use Corollary 4.1.3 and follow the same steps of the proof for Lemma 6.2.3 until (6.20),

$$\begin{aligned}
&e^{-q(t\wedge T)}f(\widetilde{V}_{t\wedge T},\overline{\widetilde{V}}_{t\wedge T})-f(x,\bar{x})\\
&\quad=M_{t\wedge T}+\int_{0+}^{t\wedge T}e^{-qs}\,(\mathcal{A}-q)f(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\,\mathrm{d}s+\int_{0+}^{t\wedge T}e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c\\
&\qquad+\sum_{0\le s\le t\wedge T}e^{-qs}\left[f(\widetilde{V}_{s^-}+\Delta X_s+\Delta\widetilde{K}_s,\overline{\widetilde{V}}_s)-f(\widetilde{V}_{s^-}+\Delta X_s,\overline{\widetilde{V}}_s)\right]\\
&\qquad-\int_{0+}^{t\wedge T}e^{-qs}\left[\Gamma^\delta f(\overline{\widetilde{V}}_s,\overline{\widetilde{V}}_s)\right]\mathrm{d}\overline{(X+\widetilde{K})}_s,
\end{aligned}$$

where M is a zero mean martingale. By conditions (IV), (V), and (VI),

$$\begin{aligned}
&e^{-q(t\wedge T)}f(\widetilde{V}_{t\wedge T},\overline{\widetilde{V}}_{t\wedge T})-f(x,\bar{x})\\
&\quad=M_{t\wedge T}+\int_{0+}^{t\wedge T}e^{-qs}\frac{\partial f}{\partial x}(\widetilde{V}_{s-},\overline{\widetilde{V}}_s)\mathrm{d}(\widetilde{K}_s)^c\\
&\qquad+\sum_{0\le s\le t\wedge T}e^{-qs}\left[f(\widetilde{V}_{s^-}+\Delta X_s+\Delta\widetilde{K}_s,\overline{\widetilde{V}}_s)-f(\widetilde{V}_{s^-}+\Delta X_s,\overline{\widetilde{V}}_s)\right],
\end{aligned}$$

which can be written as,

$$\begin{aligned}
&e^{-q(t\wedge T)} f(\widetilde{V}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x,\bar{x}) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\le s<t\wedge T} e^{-qs} \left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right] \\
&\qquad + e^{-q(t\wedge T)} \left[f(\widetilde{V}_{(t\wedge T)^-} + \Delta X_{t\wedge T} + \Delta \widetilde{K}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(\widetilde{V}_{(t\wedge T)^-} + \Delta X_{t\wedge T}, \overline{\widetilde{V}}_s) \right].
\end{aligned} \tag{7.53}$$

Since $\widetilde{V}_{(t\wedge T)^-} + \Delta X_{t\wedge T} = V^{\delta,\infty}_{t\wedge T}$ and $\widetilde{V}_{(t\wedge T)^-} + \Delta X_{t\wedge T} + \Delta \widetilde{K}_{t\wedge T} = \widetilde{V}_{t\wedge T}$, then (7.53) becomes,

$$\begin{aligned}
&e^{-q(t\wedge T)} f(V^{\delta,\infty}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) - f(x,\bar{x}) \\
&\quad = M_{t\wedge T} + \int_{0+}^{t\wedge T} e^{-qs} \frac{\partial f}{\partial x}(\widetilde{V}_{s-}, \overline{\widetilde{V}}_s) \mathrm{d}(\widetilde{K}_s)^c \\
&\qquad + \sum_{0\le s<t\wedge T} e^{-qs} \left[f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) \right].
\end{aligned} \tag{7.54}$$

By condition (II), $f(x,\bar{x}) = f(0,\bar{x})$ for $c \le x \le 0$, then the summation term on the RHS of (7.54) becomes zero as for each $s < t \wedge T$,

$$f(\widetilde{V}_{s^-} + \Delta X_s + \Delta \widetilde{K}_s, \overline{\widetilde{V}}_s) - f(\widetilde{V}_{s^-} + \Delta X_s, \overline{\widetilde{V}}_s) = f(0, \overline{\widetilde{V}}_s) - f(0, \overline{\widetilde{V}}_s) = 0.$$

For the integral term on the RHS of (7.54), if we are in the bounded variation case, $(\widetilde{K}_s)^c = 0$. If we are in the unbounded variation case, $(\widetilde{K}_s)^c$ changes only when $\widetilde{V}_{s^-} = 0$, but as for each i, $g_i \in \mathcal{C}^1[c,a]$, $\frac{\partial f}{\partial x}(x,\bar{x})|_{x=0} = \lim_{x\uparrow 0} \frac{\partial f}{\partial x}(x,\bar{x}) = 0$. Therefore, all terms on the RHS of (7.54) becomes zero except the martingale. After that, we take the expectation on both sides and get,

$$f(x,\bar{x}) = \mathbb{E}_{x,\bar{x}} \left[e^{-q(t\wedge T)} f(V^{\delta,\infty}_{t\wedge T}, \overline{\widetilde{V}}_{t\wedge T}) \right]. \tag{7.55}$$

Finally, we n go to infinity and t go to infinity. By condition (I), f is bounded. So, on the RHS of (7.55), we use the bounded convergence theorem and get,

$$\begin{aligned}
f(x,\bar{x}) &= \mathbb{E}_{x,\bar{x}} \left[e^{-q(\tau_c^-\wedge\tau_a^+)} f(V^{\delta,\infty}_{\tau_c^-\wedge\tau_a^+}, \overline{\widetilde{V}}_{\tau_c^-\wedge\tau_a^+}) \right] \\
&= \mathbb{E}_{x,\bar{x}} \left[e^{-q\tau_c^-} f(V^{\delta,\infty}_{\tau_c^-}, \overline{\widetilde{V}}_{\tau_c^-}) \mathbf{1}_{\{\tau_c^- < \tau_a^+\}} \right] + \mathbb{E}_{x,\bar{x}} \left[e^{-q\tau_a^+} f(V^{\delta,\infty}_{\tau_a^+}, \overline{\widetilde{V}}_{\tau_a^+}) \mathbf{1}_{\{\tau_a^+ < \tau_c^-\}} \right] \\
&= \mathbb{E}_{x,\bar{x}} \left[e^{-q\tau_c^-} P(V^{\delta,\infty}_{\tau_c^-}) \mathbf{1}_{\{\tau_c^- < \tau_a^+\}} \right],
\end{aligned}$$

where the last equality comes from condition (III) that $f(a,a)=0$ together with (II) that $f(x,\bar{x})=P(x)$ for $x<c$.$\Box$

Remark 25 In order to guess the candidate expression of the penalty function, we have to consider some facts. First we know that for $x \le \bar{x} \le a$,

$$\begin{aligned} f(x,\bar{x}) &= \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_c^-} P(V_{\tau_c^-}^{\delta,\infty})\mathbf{1}_{\{\tau_c^-<\tau_a^+\}}\right] \\ &= P(x)\,\mathbf{1}_{\{x<c\}} + R(x,\bar{x})\,\mathbf{1}_{\{x\ge c\}}, \end{aligned} \tag{7.56}$$

where

$$R(x,\bar{x}) = g(x,\bar{x}) + \mathbb{E}_{x,\bar{x}}\left[e^{-q\tau_{\bar{x}}^+}\mathbf{1}_{\{\tau_{\bar{x}}^+<\tau_c^-\}}\right] f(\bar{x},\bar{x}),$$

such that, for $c \le x \le 0$,

$$R(x,\bar{x}) = R(0,\bar{x}).$$

So, we find $R(x,\bar{x})$ for $0<x\le\bar{x}$. To do so, we need first to find $g(x,\bar{x})$ and then we continue to find the guessing candidate of $f(x,\bar{x})$.

$$\begin{aligned} g(x,\bar{x}) = {} & \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-} P(X_{\rho_0^-})\mathbf{1}_{\left\{X_{\rho_0^-}<c\right\}}\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] \\ & + \mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}\mathbf{1}_{\left\{X_{\rho_0^-}\ge c\right\}}\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] g(0,\bar{x}). \end{aligned} \tag{7.57}$$

We are going to use the overshoot formula, (7.3), for each term separately in (7.57). So, for the first term, our $f:(-\infty,0]\to\mathbb{R}$ is given by,

$$f(x) = P(x)\,\mathbf{1}_{\{x<c\}},$$

and the extension is $\tilde{f}:(-\infty,\bar{x}]\to\mathbb{R}$ is given by,

$$\tilde{f}(x) = P(x)\,\mathbf{1}_{\{x<c\}}.$$

By definition of $\tilde{f}$, the creeping term in (7.3) vanishes and the first term in (7.57), for $0 < x \leq \bar{x}$ is given by,

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}P(X_{\rho_0^-})\mathbf{1}_{\left\{X_{\rho_0^-}<c\right\}}\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] \\
&= \int_0^{\bar{x}}(\mathcal{A}-q)\tilde{f}(z)\left[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-z)-W^{(q)}(x-z)\right]\mathrm{d}z \\
&= \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}}\mathcal{A}\tilde{f}(z)W^{(q)}(\bar{x}-z)\mathrm{d}z - \int_0^x \mathcal{A}\tilde{f}(z)W^{(q)}(x-z)\mathrm{d}z \\
&= \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z \\
&- \int_0^x\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(x-z)\mathrm{d}z, \qquad (7.58)
\end{aligned}$$

where as $\tilde{f}(z-\theta) = P(z-\theta)\mathbf{1}_{\{z-\theta<c\}}$ and P is a bounded function, then

$$\mathcal{A}\tilde{f}(z) = \int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta) < \infty.$$

The second term in (7.57) has been found before in (7.11) and we recall it here, so for $0 < x \leq \bar{x}$,

$$\begin{aligned}
&\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-}\mathbf{1}_{\left\{X_{\rho_0^-}\geq c\right\}}\mathbf{1}_{\{\rho_0^-<\rho_{\bar{x}}^+\}}\right] \\
&= 1-\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}+\int_0^{\bar{x}}l'(z-c)\left[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-z)-W^{(q)}(x-z)\right]\mathrm{d}z.
\end{aligned}$$

So,

$$\begin{aligned}
g(x,\bar{x}) = &\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z \\
&- \int_0^x\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(x-z)\mathrm{d}z \\
&+ g(0,\bar{x})\left[1-\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}+\int_0^{\bar{x}}l'(z-c)\left[\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}W^{(q)}(\bar{x}-z)-W^{(q)}(x-z)\right]\mathrm{d}z\right].
\end{aligned} \qquad (7.59)$$

Now, put $x = 0$ in (7.59) and use that in the bounded variation case $W^{(q)}(0+) = \frac{1}{d}$ to find

$$g(0,\bar{x}) = \frac{\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z}{1-\int_0^{\bar{x}}l'(z-c)W^{(q)}(\bar{x}-z)\mathrm{d}z}. \qquad (7.60)$$

Substitute (7.60) in (7.59) and then use that in (7.56) to get,

$$
\begin{aligned}
&f(x,\bar{x})\\
&= P(x)\,\mathbf{1}_{\{x<c\}} + \mathbf{1}_{\{x\geq c\}}\Bigg\{\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z\\
&\quad-\int_0^{x}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(x-z)\mathrm{d}z\\
&\quad+\frac{\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z}{1-\int_0^{\bar{x}}l'(z-c)W^{(q)}(\bar{x}-z)\mathrm{d}z}\times\\
&\quad\left[1-\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}+\frac{W^{(q)}(x)}{W^{(q)}(\bar{x})}\int_0^{\bar{x}}l'(z-c)W^{(q)}(\bar{x}-z)\mathrm{d}z-\int_0^{x}l'(z-c)W^{(q)}(x-z)\mathrm{d}z\right]\\
&\quad+\frac{1-\int_0^{x}l'(z-c)W^{(q)}(x-z)\mathrm{d}z}{1-\int_0^{\bar{x}}l'(z-c)W^{(q)}(\bar{x}-z)\mathrm{d}z}\,f(\bar{x},\bar{x})\Bigg\},
\end{aligned}
$$

which is,

$$
\begin{aligned}
&f(x,\bar{x})\\
&\quad= P(x)\,\mathbf{1}_{\{x<c\}} - \mathbf{1}_{\{x\geq c\}}\int_0^{x}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(x-z)\mathrm{d}z\\
&\qquad+\frac{J_c(x)}{J_c(\bar{x})}\left[\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z+f(\bar{x},\bar{x})\right].
\end{aligned}
$$

Now, to derive the ODE for $f(\bar{x},\bar{x})$ we use that $\Gamma^{\delta}f(\bar{x},\bar{x})=0$, that is,

$$
\frac{\partial f(x,\bar{x})}{\partial x}|_{x=\bar{x}}\delta(\bar{x})-\frac{\partial f(x,\bar{x})}{\partial \bar{x}}|_{x=\bar{x}}(1-\delta(\bar{x}))=0,
$$

which after some simple calculations, we get the following ODE,

$$
\begin{aligned}
&\frac{\partial f(\bar{x},\bar{x})}{\partial \bar{x}}-\frac{1}{1-\delta(\bar{x})}\frac{J_c'(\bar{x})}{J_c(\bar{x})}f(\bar{x},\bar{x})\\
&=\frac{1}{1-\delta(\bar{x})}\Bigg\{\frac{J_c'(\bar{x})}{J_c(\bar{x})}\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)}(\bar{x}-z)\mathrm{d}z\\
&\quad-W^{(q)}(0+)\int_{\bar{x}-c}^{\infty}P(\bar{x}-\theta)\nu(\mathrm{d}\theta)\\
&\quad-\int_0^{\bar{x}}\left[\int_{z-c}^{\infty}P(z-\theta)\nu(\mathrm{d}\theta)\right]W^{(q)\prime}(\bar{x}-z)\mathrm{d}z\Bigg\},
\end{aligned}
\tag{7.61}
$$

with the boundary condition

$$
f(a,a)=0.
$$

The solution of the ODE (7.61) is given by

$$
\begin{aligned}
f(\bar{x},\bar{x}) = \exp\left\{\int_a^{\bar{x}} \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \int_a^{\bar{x}} \frac{1}{1-\delta(y)} \exp\left\{-\int_a^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \times \\
\left\{\frac{J_c'(y)}{J_c(y)}\int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right.\\
- W^{(q)}(0+)\int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta)\\
\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z\right\}\mathrm{d}y.
\end{aligned}
$$

That is,

$$
\begin{aligned}
f(\bar{x},\bar{x}) = -\int_{\bar{x}}^a \frac{1}{1-\delta(y)} \exp\left\{-\int_{\bar{x}}^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \times \\
\left\{\frac{J_c'(y)}{J_c(y)}\int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right.\\
- W^{(q)}(0+)\int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta)\\
\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z\right\}\mathrm{d}y.
\end{aligned}
$$

Therefore, the candidate expression for the penalty term equals

$$
\begin{aligned}
f(x,\bar{x}) = P(x)\,\mathbf{1}_{\{x<c\}} - \mathbf{1}_{\{x\geq c\}}\int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(x-z)\mathrm{d}z \\
+ \frac{J_c(x)}{J_c(\bar{x})}\left\{\int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z \right.\\
- \int_{\bar{x}}^a \frac{1}{1-\delta(y)} \exp\left\{-\int_{\bar{x}}^y \frac{1}{1-\delta(s)}\frac{J_c'(s)}{J_c(s)}\mathrm{d}s\right\} \times \\
\left\{\frac{J_c'(y)}{J_c(y)}\int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right.\\
- W^{(q)}(0+)\int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta)\\
\left.\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z\right\}\mathrm{d}y\right\}.
\end{aligned} \tag{7.62}
$$

Remark 26 It is clear that when $c \downarrow -\infty$, the penalty term (7.62) equals zero.

Theorem 7.3.11 *For $\bar{x} > 0$, let $\delta : [\bar{x}, \infty) \to [0, 1)$ be a natural tax rate function such that the function $1/(1-\delta(s))$ is locally bounded. Suppose that X has positive Gaussian coefficient in the unbounded variation case, and $P\colon \mathbb{R} \to \mathbb{R}$ is a bounded* *function. Then, the net present value of penalty for the process $V^{\delta,-c}$, for $x \leq \bar{x}$ and $\bar{x} > 0$, is given by*

$$
\begin{aligned}
f(x,\bar{x}) = {} & P(x)\,\mathbf{1}_{\{x<c\}} - \mathbf{1}_{\{x\geq c\}} \int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(x-z)\mathrm{d}z \\
& + \frac{J_c(x)}{J_c(\bar{x})} \left\{ \int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z \right. \\
& - \int_{\bar{x}}^{\infty} \frac{1}{1-\delta(y)} \exp\left\{ - \int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \times \\
& \left\{ \frac{J_c'(y)}{J_c(y)} \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right. \\
& - W^{(q)}(0+) \int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta) \\
& \left.\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z \right\} \mathrm{d}y \right\}.
\end{aligned}
$$

Proof For fixed $a > 0$, let $f : (-\infty, a] \times (0, a] \to \mathbb{R}$ be given by, $f(x,\bar{x}) = g_1(x)\,h_1(\bar{x}) + g_2(x)\,h_2(\bar{x})$, where

$$
g_1(x) = \begin{cases} P(x), & x < c, \\ -\int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(x-z)\mathrm{d}z, & x \geq c, \end{cases} \tag{7.63}
$$

$h_1(\bar{x}) = 1$, $g_2(x) = J_c(x)$, and

$$
\begin{aligned}
h_2(\bar{x}) = {} & \frac{1}{J_c(\bar{x})} \left\{ \int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z \right. \\
& - \int_{\bar{x}}^{a} \frac{1}{1-\delta(y)} \exp\left\{ - \int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s \right\} \times \\
& \left\{ \frac{J_c'(y)}{J_c(y)} \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right. \\
& - W^{(q)}(0+) \int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta) \\
& \left.\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z \right\} \mathrm{d}y \right\}.
\end{aligned} \tag{7.64}
$$

We only need to show that f satisfies the conditions in Lemma 7.3.10, then by (7.52) and taking the limit as a goes to infinity, the statement is proved. Recall that, since P is bounded, $\int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(x-z)\mathrm{d}z$ is finite. So, g_1 is bounded on $(-\infty, a]$. Also, as $J_c(\bar{x}) \geq 1$, and as by assumption $1/(1-\delta(s))$ is bounded, then this implies that f is bounded on $(-\infty, a] \times (0, a]$.

Next, we prove that $f \in \mathcal{S}_{[c,a]\times[\frac{1}{n},a]}$, for each $n \geq 1$. For that, we should verify that each g_i and h_i, for $i = 1, 2$, satisfy the conditions in Definition 4.1.1 on $[c, a]$, and on $[\frac{1}{n}, a]$, for each $n \geq 1$, respectively. We prove first condition (I) for g_i, $i = 1, 2$. By definition, g_1 equals 0 on $[c, 0]$, and on $(0, a]$, g_1' is

$$g_1'(x) = -W^{(q)}(0+) \int_{x-c}^{\infty} P(x-\theta)\nu(\mathrm{d}\theta) - \int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(x-z)\mathrm{d}z. \tag{7.65}$$

In the bounded variation case, we need to prove that g_1 is absolutely continuous with locally bounded density g_1' on $[c, a]$. This is clear on $[c, 0]$, as g_1 is zero. On $(0, a]$, since $W^{(q)}$ is absolutely continuous with density $W^{(q)\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta) \in L^1_{loc}(\mathbb{R}^+)$, then we use Lemma 7.3.2 and see that g_1 has a density on $\mathbb{R}^+$ given by (7.65), which is bounded on $(0, a]$. In the unbounded variation case, on $(0, a]$, clearly g_1 is continuous on $[c, a]$, and as $W^{(q)}(0+) = 0$, $g_1'(x) = -\int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(x-z)\mathrm{d}z$. Since X has positive Gaussian component, then by Theorem 2.2.4, $W^{(q)}$ is twice continuously differentiable. Since $W^{(q)\prime}$ is absolutely continuous with density $W^{(q)\prime\prime} \in L^1_{loc}(\mathbb{R}^+)$, and as $\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta) \in L^1_{loc}(\mathbb{R}^+)$, then again we use Lemma 7.3.2, and see that g_1' is an absolutely continuous with a density g_1'',

$$g_1''(x) = -W^{(q)\prime}(0+) \int_{x-c}^{\infty} P(x-\theta)\nu(\mathrm{d}\theta) - \int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime\prime}(x-z)\mathrm{d}z,$$

which is clearly bounded on $(0, a]$. For the functions h_i, $i = 1, 2$, it is clear that each one is absolutely continuous with locally bounded density on $[\frac{1}{n}, a]$, for each $n \geq 1$. This is because h_1 is constant, and for h_2 since we have that,

$$h_2(\bar{x}) - h_2(\frac{1}{n}) = \int_{\frac{1}{n}}^{s} h_2'(r)\,\mathrm{d}r,$$

for all $s \in [\frac{1}{n}, a]$, for each $n \geq 1$, where h_2' is given by

$$\begin{aligned} h_2'(\bar{x}) = &-\frac{J_c'(\bar{x})}{J_c^2(\bar{x})} \left[\int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z + A(\bar{x})\right] \\ &+ \frac{1}{J_c(\bar{x})}\left[-g_1'(\bar{x}) + A'(\bar{x})\right], \end{aligned} \tag{7.66}$$

such that,

$$A(\bar{x}) = -\int_{\bar{x}}^{a} \frac{1}{1-\delta(y)} \exp\left\{-\int_{\bar{x}}^{y} \frac{1}{1-\delta(s)} \frac{J_c'(s)}{J_c(s)} \mathrm{d}s\right\} \times$$
$$\left\{\frac{J_c'(y)}{J_c(y)} \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(y-z)\mathrm{d}z \right.$$
$$- W^{(q)}(0+) \int_{y-c}^{\infty} P(y-\theta)\nu(\mathrm{d}\theta)$$
$$\left. - \int_0^y \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)\prime}(y-z)\mathrm{d}z\right\}\mathrm{d}y, \tag{7.67}$$

and

$$A'(\bar{x}) = \frac{1}{1-\delta(\bar{x})} \frac{J_c'(\bar{x})}{J_c(\bar{x})} A(\bar{x})$$
$$+ \frac{1}{1-\delta(\bar{x})} \left[\frac{J_c'(\bar{x})}{J_c(\bar{x})} \int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z - g_1'(\bar{x})\right]. \tag{7.68}$$

For the second condition of Definition 4.1.1, we need to show that there exists $\lambda > 0$ such that $s \mapsto \int_\lambda^\infty g_1(s-\theta)\ \nu(\mathrm{d}\theta)$ is bounded on (c, a). For that, we choose $\lambda = a$, so that $\theta \geq \lambda = a \geq s$ which implies $s - \theta \leq 0$. Therefore, either $c \leq s - \theta \leq 0$ which implies that $g_1(s-\theta) = 0$, or $s - \theta < c$ which implies that $g_1(s-\theta) = P(s-\theta)$, which is bounded by assumptions. Therefore, in either cases, the condition is satisfied. For g_2, see the proof of Lemma 7.3.5. Hence, f satisfies condition (I).

Clearly, from (7.63) and (7.64), $f(a, a) = 0$. By definitions, it is clear that $g_1(x) = g_1(0)$ and $g_2(x) = g_2(0) = 1$ for $c \leq x \leq 0$. Also, $g_1(x) = P(x)$ and $g_2(x) = 0$ for $x < c$. Thus, by the construction of f above, we can see that $f(x, \bar{x}) = f(0, \bar{x})$ for $c \leq x \leq 0$ and $f(x, \bar{x}) = P(x)$ for $x < c$. Hence, conditions (II) and (III) are satisfied.

On $[0, \bar{x}]$, by (7.58),

$$\mathbb{E}_{x,\bar{x}}\left[e^{-q\rho_0^-} P(X_{\rho_0^-})\mathbf{1}_{\left\{X_{\rho_0^-}\geq c\right\}}\mathbf{1}_{\{\rho_0^- < \rho_{\bar{x}}^+\}}\right]$$
$$= -\int_0^x \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(x-z)\mathrm{d}z$$
$$+ \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})} \int_0^{\bar{x}} \left[\int_{z-c}^{\infty} P(z-\theta)\nu(\mathrm{d}\theta)\right] W^{(q)}(\bar{x}-z)\mathrm{d}z$$
$$= g_1(x) + \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})} g_1(\bar{x}),$$

then by Lemma 7.3.3, (2.13), and taking linear combination,

$$e^{-q(t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-)}\, g_1(X_{t\wedge\rho_{\bar{x}}^+\wedge\rho_0^-})$$

is a martingale. Since g_1 satisfies the assumptions of Lemma 7.3.4, then

$$(\mathcal{A} - q)\, g_1(x) = 0, \quad x > 0,$$

which is condition (IV). By continuity of g_1, right-continuity of g_1' and g_1'',

$$(\mathcal{A} - q)\, g_1(0) = 0,$$

which is condition (V).

For the last condition in Lemma 7.3.10, we have that

$$\frac{\partial f(x, \bar{x})}{\partial x}|_{x=\bar{x}} = g_1'(\bar{x}) + J_c'(\bar{x})\, h_2(\bar{x}), \tag{7.69}$$

where g_1 as given by (7.63) and h_2 by (7.64). Also,

$$\frac{\partial f(x, \bar{x})}{\partial \bar{x}}|_{x=\bar{x}} = J_c(\bar{x})\, h_2'(\bar{x}), \tag{7.70}$$

where h_2' is given by (7.66).

By (7.69), (7.70), (7.66), (7.67) and (7.68), we see that $\Gamma^\delta f(\bar{x}, \bar{x}) = 0$ is verified for all $0 < \bar{x} \leq a$. Hence, the proof is complete.$\square$

Chapter 8

Further literature review

We recall that dividends of an insurance company can be modelled by reflecting the paths of the risk process X at a given barrier. In a series of papers, many authors studied dividend barrier models and the problem of finding the optimal dividend strategy that maximises the net present value of dividends paid out until ruin time. For example, [38, 70, 23] in the Cramér-Lundberg setting, and [10, 42] for a general spectrally negative Lévy process. Further, we recall that also X can be modified to model the case where dividends are paid out at a certain fixed rate, whenever the capital is above the level by 'refracting' the paths of X at a given level and with a given angle, as given in [31]. The class of modified risk processes between the reflected and refracted processes, we call it loss-carry-forward tax processes. As we mentioned in the introduction of this thesis, it was introduced in [2], in the case where X is a Cramér–Lundberg process and with a constant tax rate $\gamma \in (0, 1)$. In that study, authors studied the ruin probabilities, proving a strikingly simple relation between ruin probabilities with and without tax, the so-called tax identity. Moreover, they derived the expression for the expected accumulated discounted tax payments, and obtained the optimal surplus threshold for starting taxation such that the tax value is maximized. It was also commented in [2], that when the tax rate $\gamma = 1$, the risk process with tax payments is the same as the risk process with horizontal dividend barrier strategy, with the barrier equals to the initial surplus. In [7], the tax identity is generalised to arbitrary surplus-dependent tax rate in the Cramér–Lundberg model. After that, Wei in [68] derived the tax identity for the Cramér–Lundberg risk model with a surplus-dependant premium rate, surplus-dependant tax rate and in the presence of a constant force of interest.

The same model is studied in [56], such that an explicit expression for the tax value is obtained and the optimal surplus threshold for starting taxation is characterized when the tax rate is constant. In the context of the latter model, [18] extended the tax identity to a generalised Gerber-Shiu function (the quantity that relates ruin time, surplus prior to ruin and the deficit at ruin) in which the maximum surplus preceding ruin is involved. Authors in [47] considered the Cramér–Lundberg risk model with debit interest and tax payments. That is, when the insurer is in deficit and borrows some amount of money with a debit interest say $\rho > 0$ to continue his business and meanwhile repay continuously the debt from his premium income, say the premium rate is $c > 0$. In the setting of this model, when the surplus reaches or below the level $-c/\rho$, then the surplus is no longer to recover and the time at that moment is called the absolute ruin time. In that article, some important quantities are obtained, such as the expected discounted total sum of tax payments until the absolute time and the Laplace-Stieltjes transform of the total duration of negative surplus. In [61], considering the generalised Cramér–Lundberg risk model with tax payments, the expression of the expected discounted penalty due at ruin is derived together with some other ruin-related quantities, such as the discounted joint probability density function of the surplus immediately before ruin and the deficit at ruin. In [66], the tax value function is found for the the Cramér-Lundberg risk model, including a constant force of interest and with surplus-dependent tax rate. For the same model, a recent research [46] is made regarding the problem of finding an optimal policy that maximises the tax value function.

The previous results are extended to general spectrally negative Lévy process. For instance, the work in [2] was extended in [6] to the case where X is a general spectrally negative Lévy process, with the tax rate still constant. In [33], authors extend the tax rate γ to be a function, and studied problems related to the two-sided exit problem and the net present value of the taxes paid before ruin. In the same setting in [53], a formula for arbitrary moments of the discounted tax payments until ruin time is derived. Also, [57] studied an optimal control problem for the tax process given in [33], where one seeks to maximise the net present value of the taxes paid before ruin. Lately, [63] obtained representations of joint Laplace transforms of occupation times of intervals for this tax process. We point out that, a study of this tax process where the tax rate function could exceeds the value 1 can be found in [32].

In the same loss-carry-forward tax structure, specific cases are investigated in some articles such as the following. In [67], the results in [2] are extended into the Markov-modulated Lundberg risk model, which is an extension of the Lundberg process such that the claim inter-arrivals and claim sizes are influenced by an external environment process, assumed to be a homogenous continuous-time Markov chain. It was mentioned in [67], that this type of model could be useful in a way to capture the advantage that insurance policies may need to change when the political or economical environment changes. This Markov-modulated Lundberg risk model is a special case of spectrally negative Markov additive risk processes (spectrally negative MAPs) that is defined in [8]. The tax identity is obtained in both [67] and [8]. On the other hand, the results of [2] were also extended in [5] to a more complicated model called the dual risk model. This model is relevant for companies whose business requires a constant outflow of expenses whilst arrival of revenues happens at random as a result of some probable events such as sales or discoveries; e.g. petroleum or pharmaceutical companies. Note that, in the dual risk model without tax, the aggregate revenue process is added to the initial surplus, while the expenses are subtracted (paid out) at a constant rate. In the setting of a time-homogenous diffusion risk process, [36] addressed the two sided exit problem and obtained an expression of the expected present value of taxation. In [12], a Brownian motion risk model with interest rate collection and tax payments is investigated and an approximation for the ruin probability in the model when the initial capital is very large is obtained. For a spectrally negative Lévy process X, some research like [60] and [**?**] studied problems related to a draw-down time. In loss-carry-forward taxation, some recent articles studied the case of stopping the tax process at its draw-down time instead of the classical ruin time such as [11] and [59]. In [11], for a constant tax rate, the two sided exit problem is solved, the expression of the tax value is derived and choosing the optimal delay point to start taxation is investigated. For a general tax rate function, authors in [59] obtained a solution to the two sided exit problem and found the expression of tax value function. Moreover, they studied the optimal control problem introduced in [57] but until the draw-down time. The interesting point in the draw-down time is that, studying optimality until this time could give some balance between the taxation optimisation for the tax authority and solvency of the insurance company. Authors in [65, Section 4.1] explained how draw-down problems in models without tax is related to loss-carry-forward tax ruin problems. This relation has been used in [64], where an implementation delay of tax is investigated for two cases, one when there is a termination value incurred to the insurance company at the termination time of business, and the other case when adding capital injections to the risk process. The loss-carry-forward tax with capital injections is studied first in [3], then in [4] and in Chapters 6 and 7 of this thesis. While [64] obtained the net present tax value for the tax process with capital injections for a constant tax rate, our work in Chapter 6 found it for general tax rate function. Moreover, in Chapter 6 we found the optimal solution that maximises the net present value of tax payments over larger class of strategies than the one considered in [64].

In the literature, some curious topics were also discussed such as the periodic taxation. For a spectrally negative Lévy process with loss-carry-forward tax, [69] studied the case when the observation of the insurer's surplus level is made only at a sequence of Poisson arrival times. This comes from the idea that tax payments are collected periodically(e.g. monthly, quarterly or annually) and which was commented first in [25]. The analytic expression for the tax value function and the expected discounted penalty function are obtained. Furthermore, in the literature, the idea of considering tax payments and dividends together is studied in some recent articles such as [62, 39, 58].

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