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A QUASI-STATIONARY APPROACH TO THE LONG-TERM ASYMPTOTICS OF THE GROWTH-FRAGMENTATION EQUATION

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In a growth-fragmentation system, cells grow in size slowly and split apart at random. Typically, the number of cells in the system grows exponentially and the distribution of the sizes of cells settles into an equilibrium "asymptotic profile". In this work we introduce a new method to prove this asymptotic behaviour for the growth-fragmentation equation, and show that the convergence to the asymptotic profile occurs at exponential rate. We do this by identifying an associated sub-Markov process and studying its quasi-stationary behaviour via a Lyapunov function condition. By doing so, we are able to simplify and generalise results in a number of common cases and offer a unified framework for their study. In the course of this work we are also able to prove the existence and uniqueness of solutions to the growth-fragmentation equation in a wide range of situations.

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1. Introduction. Growth-fragmentation describes a system of objects which grow slowly and deterministically, and split apart suddenly at random. It arises in biophysical models of cell division (see [37], §4), cellular aggregates [2] and protein polymerisation [41]. We are concerned in this work with a mathematical model of a growth-fragmentation system which describes its average behaviour over time. We will give general conditions for such a model to make sense, and characterise its long-term behaviour, by showing that cell numbers grow exponentially and the cell size distribution settles into an equilibrium, and that this occurs at exponential rate.

In a growth-fragmentation system, each cell has a trait associated with it, called its size. As time progresses, the size of the cell increases in a deterministic way, mathematically modelled by an ordinary differential equation. At some random time, it undergoes fragmentation, and splits its size, again at random, into a collection of descendant cells.

A common starting point for the study of these phenomena is the equation

$$\partial_t u_t(x) + \partial_x\big(c(x)u_t(x)\big) = \int_x^\infty u_t(y)k(y,x)\,\mathrm{d}y - K(x)u_t(x), \tag{1}$$

where $u_t(x)$ represents the density of cells of size x at time t, c and K are growth and fragmentation rates respectively, and k represents the repartition of size between parent and descendant cells.

This equation can be expressed in a more general form, without requiring densities, by considering a semigroup T which solves the following equation:

$$\partial_t T_t f(x) = T_t \mathcal{A} f(x), \mathcal{A} f(x) = \frac{\partial f}{\partial s}(x) + \int_{(0,x)} f(y)k(x,\,\mathrm{d}y) - K(x)f(x), \tag{2}$$

for suitable functions f. Here, s represents the growth term, $K(x)$ is again the rate at which a cell of size x experiences fragmentation, and $k(x,\,\mathrm{d}y)$ is the rate at which a cell of size y appears as the result of the fragmentation of a cell of size x. We call (2) the *growth-fragmentation equation*.

Speaking formally, if u_t solves (1), then $T_t f(u_0) := \int u_t(x)f(x)\,\mathrm{d}x$ solves a version of (2) integrated against $u_0(x)\,\mathrm{d}x$, with $s(x) = \int_1^x \frac{\mathrm{d}y}{c(y)}$. On the other hand, if T solves (2) and $T_t f(x) = \int u_t(y)f(y)\,\mathrm{d}y$, and once again $s(x) = \int_1^x \frac{\mathrm{d}y}{c(y)}$, then u_t satisfies (1) with $u_0 = \delta_x$, the Dirac delta measure. However, it is not straightforward to make this connection rigorous (see [23] for one possible approach for growth-fragmentation, and [21], Theorem 8.1, for a related model). We take (2) as our main object of study.

Our standing assumptions on the coefficients of (2) will be given at the beginning of Section 2. For the moment, we note that s should be continuous and strictly increasing, we define (see, e.g., [39] and references therein)

$$\frac{\partial f}{\partial s}(x) = \lim_{\delta\to 0,\delta>0} \frac{f(x+\delta)-f(x)}{s(x+\delta)-s(x)},$$

and $C^{(s)}$ to be the set of continuous functions $f\colon (0,+\infty)\to(0,+\infty)$ such that $\partial f/\partial s$ is well defined on $(0,+\infty)$. We also write $C_c^{(s)}$ for functions $f \in C^{(s)}$ with compact support and with $\partial f/\partial s$ bounded; and $C_{\mathrm{loc}}^{(s)}$ the set of functions $f \in C^{(s)}$ with $\partial f/\partial s$ locally bounded.

The purpose of this work is to give general conditions for the existence and uniqueness of the semigroup solving the growth-fragmentation (2), and to describe its long-term behaviour precisely.

For the first of our results, we require the following assumption on the existence of a Lyapunov type function for $\mathcal{A}$.

ASSUMPTION 1. There exists a positive function $h \in C^{(s)}_{\text{loc}}$ such that, for all $M > 0$,

$$\sup_{x\in(0,M)} \int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) < +\infty \tag{3}$$

and such that $(0, \infty) \ni x \mapsto \frac{\mathcal{A}h(x)}{h(x)}$ is bounded from above and locally bounded.

This assumption is quite abstract, but we will show shortly that it is verified for a wide class of coefficients, covering many commonly studied cases in the literature. The main result of this paper is the following statement on existence and uniqueness of solutions to the growth-fragmentation equation. We consider semigroups acting on the Banach space

$$B = \{ f : (0, \infty) \to \mathbb{R} : f \text{ is Borel and } f/h \text{ is bounded} \}$$

with associated norm $\|f\|_B = \|f/h\|_{L^\infty((0,\infty))}$. We caution that our definition of semigroup (which we defer to page 1239) does not impose any strong continuity requirement.

THEOREM 1. *Assume that Assumption 1 holds true. There exists a unique semigroup $(T_t)_{t\ge0}$ on B such that, for all $f \in \mathcal{D}(\mathcal{A}) := C^{(s)}_c \cup \{h\}$,*

$$\int_0^t T_u|\mathcal{A}f|(x)\,\mathrm{d}u < \infty \quad \textit{and} \quad T_t f(x) = f(x) + \int_0^t T_u \mathcal{A}f(x)\,\mathrm{d}u. \tag{4}$$

We study the semigroup T by connecting it to that of a Markov process via an h-transform, and this is a feature shared by other recent work such as [5, 11, 15, 17]. However, whereas these previous works have been concerned with finding either a specific superhamonic function (in the first two cases) or an eigenfunction (in the latter two) connected to $\mathcal{A}$, we are quite free in our choice of the function h, provided that we verify Assumption 1. In turn, we make use of the theory of sub-Markov processes and their quasi-stationary distributions. This gives us a great deal of freedom and accounts for the flexibility of our approach. In particular, we do not require conservation of size at splitting events (i.e., $K(x) = \int \frac{y}{x} k(x, \mathrm{d}y)$), and both $K(x) \le \int \frac{y}{x} k(x, \mathrm{d}y)$ and $K(x) \ge \int \frac{y}{x} k(x, \mathrm{d}y)$ are possible in our framework, modelling respectively size creation and destruction; see Section 3.1.1 for a representative example.

Other approaches, which do not adapt well to our situation, have been proposed. An approach via Hille–Yosida theory may be found in [4, 7, 8], and further references therein; a method using strongly continuous semigroups in L^1 spaces is contained in [17, 30, 33]; [35] discusses perturbation results for C_0-semigroups in well chosen function spaces; an approach from martingale theory can be found in [11]; and [6] uses a fixed point argument.

The second part of our work consists of describing the long-term behaviour of T, the unique solution of the growth-fragmentation equation. In order to do this, we leverage a representation of T in terms of a sub-Markov process which is developed in the proof of Theorem 1, and make use of the theory of quasi-stationary limits in weighted total variation distance.

The following additional assumptions are required. These are discussed in more detail in Section 3. The first can be regarded as a sufficient condition for irreducibility of the auxiliary Markov process referred to above; see Proposition 4 in Section 3 for a proof of this.

ASSUMPTION 2. For all $x \in (0, +\infty)$, the Lebesgue measure of $s(\{y \in (x, +\infty) : k(y, (0, x)) > 0\})$ is positive.

The second assumption implies a certain Doeblin condition for said Markov process, as shown in Proposition 5 in Section 3.

ASSUMPTION 3. One of the following holds true:

(a) There exist a positive constant $a > 0$, a nonempty, open, compactly contained interval $I \subset (0, +\infty)$, a function $T : [0, 1] \times (0, +\infty) \to (0, +\infty)$ and a probability measure μ on $[0, 1]$ such that, for all positive measurable function f on $(0, +\infty)$,

$$\int_{(0,+\infty)} f(y)k(x, \mathrm{d}y) \geq a \int_{[0,1]} f\big(T(\theta, x)\big)\mu(\mathrm{d}\theta), \quad x \in I,$$

and such that for all $\theta \in [0, 1]$, $s \circ T(\theta, \cdot)$ is continuously differentiable with respect to s on I, with

$$\frac{\partial s \circ T(\theta, \cdot)}{\partial s}(x) \neq 1, \quad x \in I. \tag{5}$$

(b) There exists a nonnegative nonzero kernel β from $(0, +\infty)$ to $(0, +\infty)$ such that, for all $x \in (0, +\infty)$, $k(x, \mathrm{d}y) \geq \beta(x, \mathrm{d}y)$ and such that, for all measurable $A \subset (0, +\infty)$ and $x \in (0, +\infty)$ with $\beta(x, A) > 0$,

$$\liminf_{y \to x, y < x} \beta(y, A) > 0. \tag{6}$$

With these in place, we can state the second main result. A more general version of this appears, with proof, as Theorem 3 and Proposition 7 in Section 3.

THEOREM 2. *Assume that Assumptions 1, 2 and 3 hold, and that there exist positive functions* $\psi, \xi \in C^{(s)}_{loc}$, *constants* $\lambda_1 \geq \lambda_2$ *and* $c_1, c_2, C > 0$, *and a compact interval* $L \subset (0, +\infty)$, *such that* $c_1\xi \leq h \leq c_2\psi$, $\lim_{x\to 0,+\infty} \frac{\xi(x)}{\psi(x)} = 0$, $\sup_{x\in(0,M)} \int_{(0,x)} \frac{\xi(y)}{\xi(x)} k(x, \mathrm{d}y) < +\infty$ *for all* $M > 0$, *and*

$$\mathcal{A}\psi(x) \leq -\lambda_1\psi(x) + C\mathbf{1}_L(x), \quad x \in (0, +\infty),$$
$$\mathcal{A}\xi(x) \geq -\lambda_2\xi(x), \quad x \in (0, +\infty).$$

Then, there exist $\lambda_0 \leq \lambda_2$, *a unique positive measure* m *on* $(0, +\infty)$ *and a unique function* $\varphi : (0, +\infty) \to (0, +\infty)$ *such that* $m(\psi) = 1$ *and* $\|\varphi/\psi\|_\infty = 1$ *and such that, for all* $t \geq 0$, $mT_t = e^{\lambda_0 t}m$ *and* $T_t\varphi = e^{\lambda_0 t}\varphi$. *Moreover, for all* $f : (0, +\infty) \to \mathbb{R}$ *such that* $|f| \leq \psi$, *we have*

$$\big|e^{\lambda_0 t} T_t f(x) - \varphi(x)m(f)\big| \leq ce^{-\gamma t}\psi(x),$$

for some constants $c, \gamma > 0$. *If moreover* $\frac{\mathcal{A}\xi}{\xi}$ *is not constant, then* $\lambda_0 < \lambda_2$.

This result is exactly what one hopes for from a Lyapunov function approach, but the reader may still wonder whether these conditions and assumptions can be verified in practice. A representative case is the following, which appears later, with proof, as Proposition 9.

Consider the operator $\mathcal{A}$ given in the form

$$\mathcal{A}f = c(x)f'(x) + K(x)\bigg(\int_{(0,1)} f(ux)p(\mathrm{d}u) - f(x)\bigg),$$

where p is a finite measure on $(0, 1)$ such that $\int_{(0,1)} up(\mathrm{d}u) = 1$, K is right-continuous and $c : (0, +\infty) \to (0, +\infty)$ is right-continuous and locally bounded. This means that $p(\mathrm{d}u)$ describes the rate of seeing children of relative size $\mathrm{d}u$ at splitting, regardless of the size of the parent (we say that k is "self-similar"); there is conservation of size at splitting events; and that prior to splitting, the size x_t of a cell follows the ordinary differential equation $\dot{x}_t = c(x_t)$. To put this into the framework of (2), we may take $s(x) = \int_1^x \frac{\mathrm{d}y}{c(y)}$ and $k(x, \cdot) = K(x)p \circ m_x^{-1}$, where $m_x(u) = xu$.

PROPOSITION 1. *Assume that* $\sup_{x\in(0,M)} K(x) < +\infty$ *for each* $M > 0$, *that Assumptions* 2 *and* 3 *hold true*, *that*

$$\int_{(0,1)} \frac{K(x)}{c(x)} \,\mathrm{d}x < +\infty$$

and that there exists $\alpha > 1$ *such that*, *for all* $u \in (0, 1)$,

$$\liminf_{x\to+\infty} \int_{ux}^{x} \frac{K(x)}{c(x)} \,\mathrm{d}x > \frac{-\alpha \ln u}{1 - \int_{(0,1)} v^{\alpha} p(\mathrm{d}v)}. \tag{7}$$

Then, *Assumption* 1 *holds*, *and the conclusions of Theorem* 2 *are valid*, *with* $\lambda_0 < 0$.

In the case where $p(\mathrm{d}u) = 2\,\mathrm{d}u$, which represents splitting into an average of two children with uniform size repartition, Assumption 3(a) is satisfied provided K has some positive lower bound on a compact interval (see Remark 4(i) in Section 3.1), and the inequality (7) holds if

$$\liminf_{x\to+\infty} \frac{xK(x)}{c(x)} > 3 + 2\sqrt{2}. \tag{8}$$

On the contrary, when $p(\mathrm{d}u) = 2\delta_{1/2}(\mathrm{d}u)$, representing equal mitosis, Assumption 3(a) holds provided that K has some positive lower bound on a compact interval I and that $c(x) \neq 2c(x/2)$ for $x \in I$ (see Remark 4(ii) in Section 3.1). Moreover, the right-hand side of (7) has minimum approximately $-3.86 \ln u$ (with the exact expression involving an implicit function). This implies that (7) holds if

$$\liminf_{x\to+\infty} \frac{xK(x)}{c(x)} > 3.86. \tag{9}$$

Proposition 1 and inequalities (8) and (9) give very concrete conditions for checking the long-term behaviour in these common cases.

Comparison with the literature. We give here a brief overview of the literature and describe how our results compare with the state of the art, without pretending to be comprehensive.

The situation in Proposition 1 was considered in Theorem 1.3 of [15] where, as discussed earlier, the authors begin by finding an eigenfunction φ for $\mathcal{A}$, using functional analysis techniques, and then use Lyapunov function criteria for the convergence of the resulting (conservative) semigroup. Proposition 1 improves upon this by reducing the regularity assumptions on c and K and the requirements on the relative growth rates of these functions. Proposition 1 also recovers Theorem 4.3 in [6] while making slightly weaker assumptions on the growth rate and balance between K and c. Both of these cases are discussed in more detail in Section 3.1.2.

Bertoin [9], Section 3.6, considers the setting where c is a positive, continuous, approximately linear function and k is self-similar, and develops moment conditions for convergence at geometric rate. The method used in that work bears some similarity with our own: the authors identify a particular superharmonic function and study an associated Markov process. Section 6 of [12] applies the same idea in a more specific setting. Our approach is based on the study of a sub-Markov process, which allows us to extend the geometric convergence to a broader class of models. In particular, Proposition 11, whose conditions are similar to those in the model studied in [9], gives a similar, but more general moment conditions for this to occur. Cavalli [17] uses methods similar to those of [9] in a situation in which the fragmentation rate is bounded and cells can grow to positive size starting from size 0. We reach the same conclusion in Section 3.1.1, making weaker regularity assumptions than [17].

In [1, 14, 27], K is comparable to a power law, c is either constant or linear, and $\frac{k(x,\cdot)}{xK(x)}$ is bounded both below and above. The authors prove geometric convergence in a weighted L^2 norm. In the case of self-similar k, Propositions 9 and 10 offer the convergence in [14] and [27] respectively, albeit in a weighted L^∞ space; the bounds on $\frac{k(x,\cdot)}{xK(x)}$ can be replaced with the weaker Assumption 3. When k is not self-similar, our main result also applies under weaker requirements, as detailed in Remark 5. We also note in this remark that our geometric convergence result covers, under weaker assumptions, the setting of Dębiec et al. [23], who use a relative entropy method to prove convergence without geometric rate, and of Bernard and Gabriel [8], where the authors prove geometric convergence rates by means of quasi-compactness.

Maillard and Paquette [30] consider the particular case where $c(x) = x$, $K(x) = xR(x)$ and $k(x, \mathrm{d}y) = xR(x)\frac{2y}{x^2}\,\mathrm{d}y$ for some positive function R, so that there is no conservation of mass, but rather conservation of the number of fragments. They establish a concise necessary and sufficient condition for existence of a stationary distribution and convergence without geometric rates. Our results can be applied to this particular model to provide sufficient conditions for convergence with a geometric rate. This is discussed in Remark 5, where we emphasize the additional requirements of our conditions (which ensure geometric rates of convergence) to the ones from [30]. Bouguet [13] also studies the situation where the solution to the growth fragmentation equation is the semigroup of a (conservative) Markov process, under moment conditions on the fragmentation kernel and asymptotic conditions on the growth and fragmentation rate, which also enter in our setting.

Finally, we give some additional pointers to the (wide) literature. In [3] the authors consider a growth-fragmentation model in a discrete state space; in [7], the authors study the non-convergence of the growth fragmentation equation; in [16], the author obtains a sharp bound for the coefficients of a critical growth-fragmentation equation (we actually recover this sharp bound in Section 3.1.4); the two papers [12, 31] study the branching process representation of the growth-fragmentation phenomenon and corresponding laws of large numbers; in [29], the authors prove an explicit geometric rate of convergence under a specific monotonicity condition on integrands of the kernel in the case where $s(x) = x - 1$ and we do recover under weaker assumptions geometric convergence in the examples they consider, however it does not seem that their assumptions imply the Doeblin condition required to apply our result; in [35], the author studies a situation in which loss of mass occurs, either at division events or directly by cell "death" and proves quasi-compactness properties. We also refer the reader to the seminal works [24, 28] where dynamics in (0, 1] are studied; see also [5, 43] for an extension of this model.

Discussion of the growth term. Besides giving general Lyapunov function criteria for solutions of the growth-fragmentation and their long-term behaviour, the present work also makes it possible to consider more general growth dynamics, since the growth term in $\mathcal{A}$ is given by the general differential $\partial f/\partial s$. As intimated in the previous example, the classical situation, where $\partial f/\partial s$ is replaced by cf' for some continuous positive function c, can be recovered by setting $s(x) = \int_1^x \frac{\mathrm{d}y}{c(y)}$. However, our setting allows us to handle, in particular, situations where the drift c vanishes and is not Lipschitz. Indeed, consider the case where $c(x) = \sqrt{|x-1|}$. Then the flow directed by the generator $f \mapsto cf'$, acting on continuously differentiable functions, has multiple solutions, whereas the flow directed by the generator $f \mapsto \partial f/\partial s$, acting on functions with bounded s derivatives, admits only one solution. It also covers seamlessly the situation where the drift c is not locally bounded. The fact that the generator is not restricted to continuously differentiable functions is of course a central component.

Outline of the paper. In Section 2, we prove that the growth-fragmentation equation admits a unique solution, by representing it as an h-transform of the semigroup of a sub-Markov process. In Section 3, we state and prove a general result which implies Theorem 2, and we provide several applications to different families of growth fragmentation equations, with a comparison to the state of the art. Finally, in the Appendix, we give some extensions of Davis' work [22] on piecewise-deterministic Markov processes (PDMP) which are required in Section 2, proving in particular that the martingale problem is well posed.

2. Existence of a unique solution to the growth-fragmentation equation. This section is devoted to the proof of Theorem 1, which is to say, the existence and uniqueness of a semigroup T solving the growth-fragmentation (2). Before discussing this in detail, we should clarify our standing assumptions, notation and definitions.

The coefficients of (2) have the following standing assumptions in place. Let k be a positive kernel from $(0,+\infty)$ to itself such that $k(x,[x,+\infty)) = 0$ for all $x \in (0,+\infty)$, let $s\colon (0,+\infty) \to \mathbb{R}$ be a strictly increasing continuous function such that $s(1) = 0$ and $\lim_{x\to+\infty} s(x) = +\infty$, and $K : (0,+\infty) \to \mathbb{R}$ be a measurable locally bounded function.

Recall the definition given earlier of the derivative of f with respect to s,

$$\frac{\partial f}{\partial s}(x) = \lim_{\delta\to 0,\delta>0} \frac{f(x+\delta) - f(x)}{s(x+\delta) - s(x)},$$

and the function spaces $C^{(s)}$, $C_c^{(s)}$ and $C_{\text{loc}}^{(s)}$ of s-differentiable functions. It is also useful at this point to observe that, if a function f is s-differentiable on the right with locally bounded derivatives in the above sense, then f is s-absolutely continuous (as defined in the Appendix) and $\partial f/\partial s$ is its Radon–Nikodym derivative. On the other hand, if f is s-absolutely continuous, then the right-hand side above is equal to its Radon–Nikodym derivative almost everywhere.

We say that $T = (T_t)_{t\geq 0}$ is a *semigroup* on a measurable space E if:

(i) for each $t \geq 0$, T_t is a kernel from E to itself,
(ii) for each $t, u \geq 0$, $x \in E$ and measurable $A \subset E$, $T_{t+u}(x, A) = \int_E T_t(x, \mathrm{d}y)T_u(y, A)$,
(iii) $T_0(x,\cdot) = \delta_x$.

As is usual for kernels, we can regard T_t as acting on a measurable function $f\colon E \to \mathbb{R}_+$ by the definition $T_t f(x) = \int_E T_t(x, \mathrm{d}y) f(y)$, and if μ is a measure on E, we can also define a measure $\mu T_t = \int_E \mu(\mathrm{d}x)T_t(x,\cdot)$. If B is some space of functions on E with the property that $T_t(B) \subset B$, we will refer to T_t as a semigroup on B. Crucially, we do not make the requirement that T is strongly continuous. In addition (see Corollary 2 below) the semigroup T does not depend on the choice of h made in Assumption 1.

The proof of our first theorem is based on the study of an auxiliary sub-Markov process. More precisely, setting

$$b := \sup_{x\in(0,+\infty)} \frac{\mathcal{A}h(x)}{h(x)} \tag{10}$$

which is finite by assumption, we show that the action $\mathcal{L}f(x) = \frac{\mathcal{A}(hf)(x)}{h(x)} - bf(x)$ on suitable f uniquely characterises a sub-Markov process X, whose killing rate is given by

$$q(x) := b - \frac{\mathcal{A}h(x)}{h(x)} \geq 0, \quad \forall x \in (0,+\infty).$$

The auxiliary sub-Markov process X can be described informally as follows. The growth-fragmentation equation (2) can be seen as characterising the expected behaviour of a system of growing and dividing cells (see [12] for a precise description). We assign each cell a time-dependent weight, with the property that at each time, the sum of weights of cells is less than 1. At time t, we treat the weights as a probability distribution and select a cell based on this (with positive probability of selecting no cell). The size of the selected cell is equal in law to X_t, and this procedure can be iterated to obtain the law of a sub-Markov process X. This process is characterised in the following result, which is proved in Section 2.1; however, for our purposes, we do not need any system of cells, and work solely with semigroups and their associated operators.

PROPOSITION 2. *Assume that Assumption* 1 *holds true. Let* $E = (0,\infty) \cup \{\partial\}$, *where* ∂ *is an isolated point. Consider the operator* $\mathcal{L}$ *given by* $\mathcal{L}f(\partial) = 0$ *and*

$$\begin{aligned}\mathcal{L}f(x) &= \frac{\mathcal{A}(hf)(x)}{h(x)} - bf(x) + q(x)f(\partial),\\ &= \frac{\partial f}{\partial s}(x)\\ &\quad + \int_{(0,x)} \big(f(y) - f(x)\big)\frac{h(y)}{h(x)} k(x, \mathrm{d}y) + q(x)\big(f(\partial) - f(x)\big), \quad x \in (0,+\infty)\end{aligned}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \{f : E \to \mathbb{R} : f(\partial) \in \mathbb{R} \text{ and } f|_{(0,\infty)} \in C_c^{(s)}\}.$$

There exists a unique càdlàg solution to the martingale problem $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ *for any initial measure* δ_x. *Moreover, its semigroup* Q *satisfies*: *for all* $t \geq 0$, *all* $x \in E$ *and all* $f \in \mathcal{D}(\mathcal{L})$,

$$\int_0^t Q_u|\mathcal{L}f|(x)\,\mathrm{d}u < +\infty \quad \text{and} \quad Q_t f(x) = f(x) + \int_0^t Q_u \mathcal{L}f(x)\,\mathrm{d}u. \tag{11}$$

Then we show that there is at most one Markov semigroup Q on $L^\infty(E)$ and satisfying (11). A semigroup Q on E is called *Markov* if $Q_t \mathbf{1}_E = \mathbf{1}_E$ for all $t \geq 0$. The following result is proved in Section 2.2.

PROPOSITION 3. *Assume that Assumption* 1 *holds true. Then there is at most one Markov semigroup* Q *on* $L^\infty(E)$ *satisfying*: *for all* $t \geq 0$, *all* $x \in E$ *and all* $f \in \mathcal{D}(\mathcal{L})$,

$$\int_0^t Q_u|\mathcal{L}f|(x)\,\mathrm{d}u < +\infty \quad \text{and} \quad Q_t f(x) = f(x) + \int_0^t Q_u \mathcal{L}f(x)\,\mathrm{d}u. \tag{12}$$

With these results on the auxiliary semigroup Q in place, we can conclude the proof of Theorem 1 in Section 2.3, by representing Q as an h-transform of the semigroup T solving the growth-fragmentation equation (4).

Representing T in terms of Q is very useful, and not only for existence and uniqueness: our results on the spectral gap in Section 3 also rely on this technique.

We conclude with a few useful properties of T. First, we note a simple bound on the support of $\delta_x T$, and observe that (4) holds true on an extension of the domain of $\mathcal{A}$. This result is proved in Section 2.4.

COROLLARY 1. *Assume that Assumption* 1 *holds true. Then, for all* $x \in (0,+\infty)$ *and all* $t \geq 0$, *the support of* $\delta_x T_t$ *is included in* $[0, s^{-1}(s(x)+t)]$. *Let* $f \in C_{loc}^{(s)}$ *such that* $|f|/|h|$ *is bounded and such that* $\inf \frac{\mathcal{A}f}{h} > -\infty$ *or* $\sup \frac{\mathcal{A}f}{h} < +\infty$. *Then equality* (4) *holds true.*

Second, we observe that the solution to (4) does not depend on the choice of h. This is proved in Section 2.5, ensuring that;

COROLLARY 2. *Let h_1 and h_2 satisfy Assumption 1. Then, the solution T^1 to (4) with h_1 instead of h, and the solution T^2 to (4) with h_2 instead of h, are identical.*

REMARK 1. In this paper, we assume that sizes take values in $(0, +\infty)$. However, when 0 is an entrance boundary for the growth component, that is, when $s(0+) > -\infty$, it is straightforward to adapt the method and results of this paper to the case where the space $(0, +\infty)$ is replaced by $[0, +\infty)$, with $k(x, \{0\}) \geq 0$ for all $x \in [0, +\infty)$.

REMARK 2. The primary difficulty in proving the results from this section is that we wish to characterise T (and Q) in terms of analytic, rather than probabilistic conditions; that is, the semigroup condition and the equations (4) and (12). Moreover, we want to avoid assumptions such as the Feller property, which may not hold in the absence of similar conditions on the kernel k.

Indeed, Proposition 2, concerning the probabilistic question of the well-posedness of the martingale problem in the space of càdlàg paths, is fairly straightforward to prove, and we rely primarily on the work of Davis [22] (slightly extended in the Appendix) and standard techniques of Ethier and Kurtz [26].

Proposition 3 is substantially more involved, but the techinque is essentially to show that any semigroup solving (12) can be represented in terms of a càdlàg Markov process, which in turn is the unique solution of the martingale problem already addressed. The ideas in this part, such as the study of upcrossings and regularisation of paths, are familiar, but typical references (for instance, [42] under Feller-type (§III.7) or Ray (§III.36) conditions) have sufficiently different hypotheses that we were not able to reduce the situation to one covered by these results.

2.1. *An auxiliary Markov process*. This section is devoted to the proof of Proposition 2. From now on, we set

$$k_h(x, \mathrm{d}y) = \frac{h(y)}{h(x)} k(x, \mathrm{d}y),$$

so that, by Assumption 1, $x \mapsto k_h(x, (0, x))$ is bounded on $(0, M)$, for all $M > 0$. Before proving Proposition 2, we start with a useful technical lemma. We define $f_-(x) = \max\{-f(x), 0\}$.

LEMMA 1. *Assume $f \in \mathcal{D}(\mathcal{L})$, meaning that $f|_{(0,+\infty)} \in C_c^{(s)}$, and that Assumption 1 holds true. Then*:

(i) *$\mathcal{L}f$ is locally bounded*;
(ii) *if f is nonnegative, then $\mathcal{L}f$ is bounded below*;
(iii) *if f is nonnegative and $f(\partial) = 0$, then, for all $M > 0$, $\sup_{x \in (0,M)} \mathcal{L}f(x) < +\infty$.*

PROOF. Since $f \in \mathcal{D}(\mathcal{L})$, $f|_{(0,\infty)} \in C_c^{(s)}$. Define $F = \operatorname{supp} f|_{(0,\infty)}$, a compact subset of $(0, \infty)$. We first note the following: for all $x \in (0, +\infty)$,

$$|\mathcal{L}f(x)| \leq \left\| \frac{\partial f}{\partial s} \right\|_\infty + 2\|f\|_\infty k_h\big(x, (0, x)\big) + 2\|f\|_\infty q(x),$$

where $q(x) = b - \frac{Ah(x)}{h(x)} \geq 0$ and $k_h(x, (0, x))$ are locally bounded by Assumption 1. This proves the first point.

If f is nonnegative, then

$$\mathcal{L}f(x) \geq -\left\|\frac{\partial f}{\partial s}\right\|_\infty - f(x)k_h\big(x,(0,x)\big) - q(x)f(x)$$
$$\geq -\left\|\frac{\partial f}{\partial s}\right\|_\infty - \mathbf{1}_{x\in F}\|f\|_\infty\big(k_h\big(x,(0,x)\big)+q(x)\big) \tag{13}$$

which is bounded below since F is compact and $q(x)$ and $k_h(x,(0,x))$ are locally bounded. This proves the second point of Lemma 1.

If f is nonnegative and $f(\partial)=0$, then

$$\mathcal{L}f(x) \leq \left\|\frac{\partial f}{\partial s}\right\|_\infty + \int_{(0,x)} f(y)k_h(x,\mathrm{d}y) \leq \left\|\frac{\partial f}{\partial s}\right\|_\infty + \|f\|_\infty k_h\big(x,(0,x)\big)$$

which is bounded over $x\in(0,M)$, for all $M>0$, according to Assumption 1. □

We can now proceed to the proof of Proposition 2.

PROOF OF PROPOSITION 2. We first show that there exists a càdlàg solution of the $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ martingale problem, and then prove that this solution is unique.

(1) There exists a càdlàg solution of the $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ *martingale problem.*

Since s is continuous and strictly increasing, there exists a unique flow $\phi:(0,+\infty)\times[0,+\infty)\to(0,+\infty)$ such that $\phi(x,0)=x$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}s\big(\phi(x,t)\big)=1,\quad \forall x,t, \tag{14}$$

which is given by $\phi(x,t)=s^{-1}(s(x)+t)$ for all $x\in(0,+\infty)$ and $t\geq 0$. We also set $\phi(\partial,t)=\partial$ for all $t\geq 0$. We observe that ϕ is not explosive since it satisfies $s(\phi(x,t))=s(x)+t$ for all $t\geq 0$ and $x\in(0,+\infty)$, while $s(y)\to+\infty$ when $y\to+\infty$. Moreover, for all $f\in\mathcal{D}(\mathcal{L})$, we have

$$\frac{\mathrm{d}_+}{\mathrm{d}t}f\big(\phi(x,t)\big) := \lim_{h\to 0,h>0}\frac{f(\phi(x,t+h))-f(\phi(x,t))}{h}$$
$$= \lim_{h\to 0,h>0}\frac{f(s^{-1}(s(x)+t+h))-f(s^{-1}(s(x)+t))}{h} \tag{15}$$
$$= \lim_{y\to\phi(x,t),y>\phi(x,t)}\frac{f(y)-f(\phi(x,t))}{s(y)-s(\phi(x,t))} = \frac{\partial f}{\partial s}\big(\phi(x,t)\big).$$

Let us consider the piecewise-deterministic Markov process (PDMP) X directed by the flow ϕ between its jumps and with jump kernel k_h and killing rate q, constructed jump after jump, similarly as in [22], with values on $(0,+\infty)\cup\{\infty,\partial\}$ and up to the time of explosion of the number of jumps. Here ∞ is the point to which the process is sent after explosion of the number of jumps and ∂ is the cemetery point.

We prove now that the process X is nonexplosive, so that it defines a càdlàg Markov process on E. For all $k\geq 2$, we set $\tau_k^+=\inf\{t\geq 0, X_t\geq k \text{ or } X_{t-}\geq k\}$ and $\tau_{\frac{1}{k}}^-=\inf\{t\geq 0, X_t\leq\frac{1}{k} \text{ or } X_{t-}\leq\frac{1}{k}\}$. As pointed out above, we know that the flow ϕ does not explode. Since the process only admits negative jumps, $X_t\leq\phi(X_0,t)$ almost surely, so that, for all $x\in(0,+\infty)$ and all $t\geq 0$, there exists $k_{x,t}\geq 2$ such that

$$\mathbb{P}_x\big(\tau_{k_{x,t}}^+\leq t\big)=0, \tag{16}$$

where $\mathbb{P}_x$ denotes the law of X with initial distribution δ_x (as usual, we extend this notation to initial distribution μ by $\mathbb{P}_\mu$ and denote $\mathbb{E}_x$ and $\mathbb{E}_\mu$ the associated expectations).

According to (3), the jump rate of X from $(0,+\infty)$ to $(0,+\infty)$, that is, $y \mapsto k_h(y,(0,y))$, is uniformly bounded on $(0,k_{x,t}]$. Since in addition ∂ is an absorbing point, the process does not undergo an infinity of negative jumps before time $t\wedge\tau^+_{k_{x,t}}$, $\mathbb{P}_x$-almost surely for all $x\in E$. Using the fact that the flow ϕ is increasing, we deduce that the process does not converge to 0 before time $t\wedge\tau^+_{k_{x,t}}$ $\mathbb{P}_x$-almost surely for all $x\in E$, that is,

$$\lim_{k\to+\infty}\mathbb{P}_x\big(\tau^-_{\frac{1}{k}}\le t\wedge\tau^+_{k_{x,t}}\big)=0,\qquad\forall x\in E. \tag{17}$$

Combining both (16) and (17), we deduce that, for all initial distribution ν on $(0,+\infty)\cup\partial$,

$$\lim_{k\to+\infty}\mathbb{P}_\nu\big(\tau^-_{\frac{1}{k}}\wedge\tau^+_k\le t\big)=0. \tag{18}$$

This concludes the proof that X defines a nonexplosive càdlàg Markov process on E.

Let us now remark that it satisfies the $(\mathcal{L},\mathcal{D}(\mathcal{L}))$-local martingale problem. Indeed, for all $f\in\mathcal{D}(\mathcal{L})$, f belongs to the domain of the extended generator of X, as proved in Theorem 26.14 in [22], with the only difference being that, in our case, the flow ϕ is not determined by a locally Lipschitz continuous vector field χ, but instead by s. The only adaptation to be made in the proof of Theorem 26.14 in [22] to obtain that f is an element of the domain of the extended generator of X is as follows. Denote by J_{i-1} and J_i the $i-1$th and ith jump times of X; then, $f(X_{J_i-})-f(X_{J_{i-1}})=0$ when $X_{J_{i-1}}=\partial$, and otherwise,

$$\begin{aligned}
f(X_{J_i-})-f(X_{J_{i-1}})&=\int_0^{J_i-J_{i-1}}\frac{\mathrm{d}_+}{\mathrm{d}t}f\big(\phi(X_{J_{i-1}},t)\big)\,\mathrm{d}t\\
&=\int_0^{J_i-J_{i-1}}\frac{\partial f}{\partial s}\big(\phi(X_{J_{i-1}},t)\big)\,\mathrm{d}t=\int_{J_{i-1}}^{J_i}\frac{\partial f}{\partial s}(X_t)\,\mathrm{d}t.
\end{aligned}$$

This replaces the expression $\int_{J_{i-1}}^{J_i}\mathcal{X}(X_t)\,\mathrm{d}t$ in [22]. The rest of the proof is identical.

Let us now prove that X satisfies the $(\mathcal{L},\mathcal{D}(\mathcal{L}))$-martingale problem. We have that, for all $x\in E$ and under $\mathbb{P}_x$, $M^f_t:=f(X_t)-f(x)-\int_0^t\mathcal{L}f(X_u)\,\mathrm{d}u$ is a càdlàg local martingale. Moreover, since f and $\mathcal{L}f$ are locally bounded by Lemma 1 point (i), the sequence $\tau_k=\tau^-_{\frac{1}{k}}\wedge\tau^+_k$ is a localization sequence.

We initially focus on the case where $f\in\mathcal{D}(\mathcal{L})$ is nonnegative, and set $a=\inf_E\mathcal{L}f$, which is finite by Lemma 1 point (ii). We have, for any fixed $t>0$ and any $k\ge 2$,

$$\big|M^f_{t\wedge\tau_k}\big|\le 2\|f\|_\infty+\int_0^t\big|\mathcal{L}f(X_u)\big|\,\mathrm{d}u\le 2\|f\|_\infty+|a|t+\int_0^t\big|\mathcal{L}f(X_u)-a\big|\,\mathrm{d}u,$$

where, by the monotone convergence theorem and the local martingale property for M^f,

$$\begin{aligned}
\mathbb{E}_x\bigg(\int_0^t\big|\mathcal{L}f(X_u)-a\big|\,\mathrm{d}u\bigg)&=\mathbb{E}_x\bigg(\liminf_{k\to+\infty}\int_0^{t\wedge\tau_k}\big|\mathcal{L}f(X_u)-a\big|\,\mathrm{d}u\bigg)\\
&=\liminf_{k\to+\infty}\mathbb{E}_x\bigg(\int_0^{t\wedge\tau_k}\big|\mathcal{L}f(X_u)-a\big|\,\mathrm{d}u\bigg)\\
&=\liminf_{k\to+\infty}\mathbb{E}_x\bigg(\int_0^{t\wedge\tau_k}\big(\mathcal{L}f(X_u)-a\big)\,\mathrm{d}u\bigg)\\
&=\liminf_{k\to+\infty}\mathbb{E}_x\big(f(X_{t\wedge\tau_k})-f(x)-M^f_{t\wedge\tau_k}\big)+|a|t.\\
&\le 2\|f\|_\infty+|a|t.
\end{aligned}$$

Hence, for all $T\ge 0$, $\{|M^f_{t\wedge\tau_k}|:t\le T,k\ge 2\}$ is dominated by an integrable random variable. We conclude by [40], Theorem 51, that, for all $x\in E$, under $\mathbb{P}_x$, M^f is a martingale.

Next, we remove the assumption that f is nonnegative, and permit any $f \in \mathcal{D}(\mathcal{L})$. Let $\varphi \in \mathcal{D}(\mathcal{L})$ such that $\varphi \geq f_+$, where $f_+(x) = \max\{f(x), 0\}$. Then, according to the above result, M^φ is a martingale. Setting $\psi = \varphi - f$, we have $\psi \geq 0$ and $\psi \in \mathcal{D}(\mathcal{L})$ and hence M^ψ is also a martingale. Since $M^f = M^\varphi - M^\psi$, we deduce that M^f is a martingale.

Finally, we conclude that X defines a nonexplosive càdlàg Markov process on E, which satisfies the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem. In particular, we have shown that $\int_0^t \mathbb{E}_x(|\mathcal{L}f|(X_u))\,\mathrm{d}u < +\infty$, and we observe that the semigroup of X satisfies (11).

(2) X is the unique càdlàg solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ martingale problem. For all $n \geq 2$, we consider the operator $\mathcal{L}_n$ on $\mathcal{D}(\mathcal{L})$ defined, for all $x \in E$ and $g \in \mathcal{D}(\mathcal{L})$, by

$$\mathcal{L}_n g(x) = \mathbf{1}_{x \in (0,+\infty)} \left[\frac{\partial g}{\partial s_n}(x) + \int_{(0,x)\cup\partial} [g(y) - g(x)] Q_n(x, \mathrm{d}y) \right],$$

where s_n is a continuous increasing function on $(0, +\infty)$ and Q_n a kernel such that

$$\begin{cases} s_n(x) = s(x), \quad x > 1/n, \\ \lim_{x\downarrow 0} s_n(x) = -\infty, \\ Q_n(x, \mathrm{d}y) = \mathbf{1}_{x \in (\frac{1}{n}, n)} [k_h(x, \mathrm{d}y) + q(x)\delta_\partial(\mathrm{d}y)]. \end{cases}$$

According to Proposition 17 in the Appendix, the solution of the martingale problem for $(\mathcal{L}_n, \mathcal{D}(\mathcal{L}))$ is unique. In particular, any two solutions of the $D_E[0, +\infty)$ martingale problem for $\mathcal{L}_n$ have the same distribution on $D_E[0, +\infty)$ (see Corollary 4.4.3 in [26]). This and Theorem 4.6.1 in [26] imply that, for each $n \geq 2$ and all probability measures ν on E, the stopped martingale problem for $(\mathcal{L}_n, \nu, (\frac{1}{n}, n) \cup \{\partial\})$ admits a unique solution with sample paths in $D_E[0, +\infty)$. Since, for all $g \in \mathcal{D}(\mathcal{L})$, we have $\mathcal{L}_n g(x) = \mathcal{L}g(x)$ for all $x \in (\frac{1}{n}, n) \cup \{\partial\}$, we deduce that the stopped martingale problem for $(\mathcal{L}, \nu, (\frac{1}{n}, n) \cup \{\partial\})$ also admits a unique solution with sample paths in $D_E[0, +\infty)$. Since X stopped at time $\tau_n := \inf\{t \geq 0, X_t \text{ or } X_{t-} \notin (\frac{1}{n}, n) \cup \{\partial\}\}$ is a càdlàg solution to this stopped martingale problem, it gives its unique solution in $D_E[0, +\infty)$. Since it satisfies in addition

$$\lim_{n\to+\infty} \mathbb{P}_\nu(\tau_n \leq t) = 0,$$

we deduce from Theorem 4.6.3 in [26] that there is a unique solution to the $D_E[0, +\infty)$-martingale problem associated to $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$. □

2.2. *Uniqueness of a Markov semigroup generated by $\mathcal{L}$.* This section is devoted to the proof of Proposition 3, that is, to the uniqueness of a Markov semigroup Q satisfying (12).

In order to do so, we first prove useful technical lemmas. Then, we show that, given such a Markov semigroup Q, one can construct a càdlàg solution to the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem with semigroup Q. The uniqueness of the solution to (12) then derives from the uniqueness of this martingale problem, proved in Proposition 2.

2.2.1. *Technical lemmas.* Let Q be a Markov semigroup solution to (12). The following lemmas will be useful to prove the nonexplosion of a process with semigroup Q.

Throughout this section, we will apply the operator $\mathcal{L}$ to functions (such as W and f_m^A in the following Lemma) which do not lie in $\mathcal{D}(\mathcal{L})$. We note that, under Assumption 1, $\mathcal{L}f$ is well defined by its integro-differential action for any $f \in C^{(s)}$ with f and $\partial f/\partial s$ locally bounded, whereas $\mathcal{D}(\mathcal{L})$ represents the space on which the equation (12), defining Q, is valid.

LEMMA 2. *Assume that Assumption 1 holds true. Let $W\colon (0, +\infty) \cup \{\partial\} \to [0, +\infty)$ be such that $W|_{(0,+\infty)}$ is a nondecreasing function in $C^{(s)}_{loc}$ with $W(\partial) = 0$ and such that $\sup_{z>0} \frac{\partial W}{\partial s}(z) < +\infty$. Then, $\int_0^t Q_u|\mathcal{L}W(x)|\,\mathrm{d}u < \infty$ and*

$$Q_t W(x) \leq W(x) + \int_0^t Q_u \mathcal{L}W(x)\,\mathrm{d}u, \quad t \geq 0 \text{ and } x \in (0, +\infty).$$

PROOF OF LEMMA 2. For any $A \geq 1$, we consider the $C^{(s)}_{\text{loc}}$ function $W_A : (0, +\infty) \to [0, +\infty)$ defined by

$$W_A(x) = \begin{cases} W(x) & \text{if } x \leq A + 1, \\ W(A+1) & \text{if } x \geq A + 1, \end{cases}$$

and also set $W_A(\partial) = 0$. For any $m \geq 3$, we consider a $C^{(s)}_{\text{loc}}$ function $f^A_m : (0, +\infty) \to [0, +\infty)$ such that

$$f^A_m(x) = \begin{cases} W(A+1) & \text{if } x \leq 1/m, \\ W_A(x) & \text{if } x \geq 2/m, \end{cases}$$

such that f^A_m which is nonincreasing on $(1/m, 2/m)$. In particular, $\frac{\partial f^A_m}{\partial s}(x) \leq \frac{\partial W_A}{\partial s}(x) \leq \frac{\partial W}{\partial s}(x)$ for all $x \in (0, +\infty)$. We also set $f^A_m(\partial) = 0$.

Since $g^A_m := W(A+1)\mathbf{1}_E - f^A_m \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}\mathbf{1}_E = 0$, we deduce from (12) that, for all $t \geq 0$ and all $x \in (0, +\infty)$,

$$Q_t g^A_m(x) = g^A_m(x) + \int_0^t Q_u \mathcal{L} g^A_m(x)\,\mathrm{d}u = g^A_m(x) - \int_0^t Q_u \mathcal{L} f^A_m(x)\,\mathrm{d}u.$$

But $Q_t \mathbf{1}_E = \mathbf{1}_E$, and hence, subtracting $W(A+1)\mathbf{1}_E(x)$ on both sides of the equation, we deduce that

$$Q_t f^A_m(x) = f^A_m(x) + \int_0^t Q_u \mathcal{L} f^A_m(x)\,\mathrm{d}u. \tag{19}$$

Set $h^A_m(z) = \int_{(0,z)} [f^A_m(y) - f^A_m(z)] k_h(z, \mathrm{d}y)$ for all $z > 0$. We observe that $h^A_m(z) \leq 0$ for all $z \geq A+1$ and that $h^A_m(z) \leq W(A+1)\sup_{y \in (0, A+1)} k_h(y, (0, y])$ for all $z \leq A+1$, which is finite according to Assumption 1. Using the fact that, for all $z > 0$,

$$\frac{\partial f^A_m}{\partial s}(z) + q(x)\big(f^A_m(\partial) - f^A_m(z)\big) = \frac{\partial f^A_m}{\partial s}(z) - q(x) f^A_m(z) \leq C_W := \sup_{y > 0} \frac{\partial W}{\partial s}(y),$$

we deduce that

$$\sup_{m \geq 1} \sup_{z \in (0, +\infty)} \mathcal{L} f^A_m(z) < +\infty.$$

Hence, applying Fatou's lemma in the integral part of (19), we deduce that

$$\limsup_{m \to +\infty} Q_t f^A_m(x) \leq \limsup_{m \to +\infty} f^A_m(x) + \int_0^t Q_u \Big(\limsup_{m \to +\infty} \mathcal{L} f^A_m\Big)(x)\,\mathrm{d}u. \tag{20}$$

We have $\limsup_{m \to +\infty} f^A_m(x) = W_A(x)$ and the left-hand side is equal to $Q_t W_A(x)$ by dominated convergence (recall that $f_A \leq W(A+1)$). Moreover, for any fixed $z > 0$, we deduce from Fatou's lemma (recall that, when $m \to +\infty$, $f^A_m(y) - f^A_m(z)$ is uniformly bounded from above in y and converges pointwise to $W_A(y) - W_A(z)$, while $k_h(z, \mathrm{d}y)$ has finite mass) that

$$\limsup_{m \to +\infty} \int_{(0,z)} [f^A_m(y) - f^A_m(z)] k_h(z, \mathrm{d}y) \leq \int_{(0,z)} [W_A(y) - W_A(z)] k_h(z, \mathrm{d}y),$$

while $\partial f^A_m / \partial s(z)$ converges pointwisely toward $\partial W^A / \partial s(z)$, so that

$$\limsup_{m \to +\infty} \mathcal{L} f^A_m(z) \leq \mathcal{L} W_A(z).$$

This and (20) thus entail that, for all $A \geq 2$,

$$Q_t W_A(x) \leq W_A(x) + \int_0^t Q_u \mathcal{L} W_A(x)\,\mathrm{d}u.$$

Since $\mathcal{L}W_A \le C_W$, we can use again Fatou's lemma, and deduce

$$\limsup_{A\to+\infty} Q_t W_A(x) \le \limsup_{A\to+\infty} W_A(x) + \int_0^t Q_u\Big(\limsup_{A\to+\infty} \mathcal{L}W_A\Big)(x)\,\mathrm{d}u.$$

On the one hand, $\limsup_{A\to+\infty} W_A(x) = W(x)$ and, by monotone convergence, we obtain that $\limsup_{A\to+\infty} Q_t W_A(x) = Q_t W(x)$. On the other hand, using the monotone convergence theorem (note that $W_A(y)$ is increasing in A, for any fixed y), we deduce that, for all $z > 0$,

$$\limsup_{A\to+\infty} \int_{(0,z)} [W_A(y) - W_A(z)] k_h(z, \mathrm{d}y) = \int_{(0,z)} [W(y) - W(z)] k_h(z, \mathrm{d}y)$$

and hence that $\limsup_{A\to+\infty} \mathcal{L}W_A(z) = \mathcal{L}W(z)$. This implies that

$$Q_t W(x) \le W(x) + \int_0^t Q_u \mathcal{L}W(x)\,\mathrm{d}u.$$

Since $\mathcal{L}W$ is bounded from above by C_W, this implies that $\int_0^t Q_u(\mathcal{L}W)_-(x)\,\mathrm{d}u < +\infty$ and hence that $\int_0^t Q_u|\mathcal{L}W|(x)\,\mathrm{d}u < +\infty$. This concludes the proof of Lemma 2. □

LEMMA 3. *We define the function* $p\colon E \to [0, +\infty]$ *by* $p(z) = k_h(z, (0, 1))$ *and* $p(\partial) = 0$. *If Assumption 1 holds true, then* $p(E) \subset [0, +\infty)$ *and, for all* $x \in (0, +\infty)$,

$$\int_0^t Q_u p(x)\,\mathrm{d}u < +\infty.$$

PROOF OF LEMMA 3. We first observe that $p(z) < +\infty$ for all $z \in (0, +\infty)$ according to (3). Let W be a $C^{(s)}_{\mathrm{loc}}$ nondecreasing function with $W(\partial) = 0$, such that $C_W := \sup_{z>0} \frac{\partial W}{\partial s}(z) < +\infty$ and

$$W(x) = \begin{cases} 0 & \text{if } x \le 1, \\ 1 & \text{if } x \ge 2. \end{cases}$$

For all $z > 0$, we have

$$\begin{aligned} \mathcal{L}W(z) &\le C_W + \int_{(0,z)} [W(y) - W(z)] k_h(z, dy) \\ &\le C_W + \begin{cases} 0 & \text{if } z \le 2 \\ -\int_{(0,z)} \mathbf{1}_{y\le 1} k_h(z, \mathrm{d}y) & \text{if } z \ge 2. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} (\mathcal{L}W)_-(z) &\ge \mathbf{1}_{z\ge 2} \int_{(0,z)} \mathbf{1}_{y\le 1} k_h(z, \mathrm{d}y) - C_W \\ &= p(z)\mathbf{1}_{z\ge 2} - C_W \ge p(z) - \sup_{r\in(0,2)} k_h\big(r, (0, r)\big) - C_W, \end{aligned}$$

where $\sup_{r\in(0,2)} k_h(r, (0, r)) < +\infty$ by Assumption 1. Hence

$$\int_0^t Q_u p(x)\,\mathrm{d}u \le \int_0^t Q_u(\mathcal{L}W)_-(x)\,\mathrm{d}u + t\Big(\sup_{r\in(0,2)} k_h\big(r, (0, r)\big) + C_W\Big).$$

According to Lemma 2, we have $\int_0^t Q_u(\mathcal{L}W)_-(x)\,\mathrm{d}u < +\infty$. Hence we obtain

$$\int_0^t Q_u p(x)\,\mathrm{d}u < +\infty.$$

□

LEMMA 4. *Assume that Assumption 1 holds true. Let $W: E \to [0, +\infty)$ be a $C^{(s)}_{loc}$ nonincreasing function such that $W(x) = 0$ for all $x \geq 1$ and $W(\partial) = 0$. Assume that $p_W(x) < +\infty$ and $\int_0^t Q_u p_W(x)\,\mathrm{d}u < +\infty$ for all $t > 0$ and $x \in (0, +\infty)$, where $p_W(x) = \int_{(0,x)} W(y) k_h(x, \mathrm{d}y)$. Then $\int_0^t Q_u |\mathcal{L}| W(x)\,\mathrm{d}u < +\infty$ and*

$$Q_t W(x) \leq W(x) + \int_0^t Q_u \mathcal{L} W(x)\,\mathrm{d}u, \quad \forall x \in (0, +\infty) \text{ and } t \geq 0.$$

PROOF. For all $A \geq 2$, let $W_A : (0, +\infty) \to [0, +\infty)$ be the nonincreasing $C^{(s)}_{\mathrm{loc}}$ function defined as

$$W_A(x) = \begin{cases} W(1/A) & \text{if } x \leq 1/A, \\ W(x) & \text{if } x \geq 1/A. \end{cases}$$

We also set $W_A(\partial) = 0$. For all $m \geq 2$, let $m' > 0$ be such that $s(m') = s(m) + W(1/A)$ and let $f_m^A : (0, +\infty) \to [0, +\infty)$ be a $C^{(s)}_{\mathrm{loc}}$ function such that

$$f_m^A(x) = \begin{cases} W_A(x) & \text{if } x \leq m, \\ W(1/A) & \text{if } x \geq m', \end{cases}$$

such that f_m^A is nondecreasing on $(1, +\infty)$ and such that $\frac{\partial f_m^A}{\partial s}(x) \leq 1$ for all $x \in (0, +\infty)$. We set $f_m^A(\partial) = 0$. Proceeding as in the proof of Lemma 2, we have

$$Q_t f_m^A(x) = f_m^A(x) + \int_0^t Q_u \mathcal{L} f_m^A(x)\,\mathrm{d}u, \quad \forall t \geq 0 \text{ and } x \in (0, +\infty). \tag{21}$$

Set $h_m^A(z) = \int_{(0,z)} [f_m^A(y) - f_m^A(z)] k_h(z, \mathrm{d}y)$ for all $z > 0$. We have, for all $0 < y \leq z$,

$$f_m^A(y) - f_m^A(z) \leq W(1/A) \mathbf{1}_{y<1}$$

and hence

$$h_m^A(z) \leq W(1/A) \int_{(0,z)} \mathbf{1}_{y<1} k_h(z, \mathrm{d}y) \leq W(1/A) p(z),$$

where p is defined in the previous lemma. Since $\frac{\partial f_m^A}{\partial s}(z) \leq 1$ for all $z > 0$, we deduce that $\mathcal{L} f_m^A(z) \leq 1 + W(1/A) p(z)$. Since $\int_0^t Q_u(1 + W(1/A)p)(x)\,\mathrm{d}u < +\infty$ according to Lemma 3, we deduce using Fatou's lemma in (21), that

$$\limsup_{m \to +\infty} Q_t f_m^A(x) \leq \limsup_{m \to +\infty} f_m^A(x) + \int_0^t Q_u \Big(\limsup_{m \to +\infty} \mathcal{L} f_m^A \Big)(x)\,\mathrm{d}u.$$

As in the proof of Lemma 2, this entails that

$$Q_t W_A(x) \leq W_A(x) + \int_0^t Q_u \mathcal{L} W_A(x)\,\mathrm{d}u.$$

Now we observe that, for all $z > 0$, for all $A \geq 2$,

$$\mathcal{L} W_A(z) \leq \int_{(0,z)} W(y) k_h(z, \mathrm{d}y) = p_W(z).$$

Since p_W is integrable by assumption, we can apply again Fatou's lemma to deduce that

$$\limsup_{A \to +\infty} Q_t W_A(x) \leq \limsup_{A \to +\infty} W_A(x) + \int_0^t Q_u \Big(\limsup_{A \to +\infty} \mathcal{L} W_A \Big)(x)\,\mathrm{d}u.$$

As in the proof of Lemma 2, this entails that

$$Q_t W(x) \le W(x) + \int_0^t Q_u \mathcal{L} W(x)\,\mathrm{d}u.$$

In addition, $\int_0^t Q_u(\mathcal{L}W)_+(x)\,\mathrm{d}u \le \int_0^t Q_u p_W(x)\,\mathrm{d}u < +\infty$, and hence $\int_0^t Q_u(\mathcal{L}W)_-(x)\,\mathrm{d}u < +\infty$, which concludes the proof. □

2.2.2. *Representation of the semigroup Q by a càdlàg Markov process.* In this section, Q is a Markov semigroup satisfying (12). In Lemma 5, we prove the continuity and the nonexplosion of any process $(Z_t)_{t\in F}$ with semigroup Q, where $F \subset [0,+\infty)$ contains $\mathbb{Q}_+ = [0,+\infty)\cap\mathbb{Q}$ and is countable.

LEMMA 5. *Assume that Assumption 1 holds true. Let $F \supset \mathbb{Q}_+$ be a countable subset of $[0,+\infty)$ and let $(Z_t)_{t\in F}$ be a Markov process on E with semigroup Q, defined on the probability space $\Omega = E^F$. Then, almost surely (for any starting distribution), the process $(Z_t)_{t\in F}$ is continuous at any time $t \in F$ and, for all $T > 0$, $\sup_{t\in F\cap[0,T]} \mathbf{1}_{Z_t\neq\partial}/Z_t < +\infty$ and $\sup_{t\in F\cap[0,T]} \mathbf{1}_{Z_t\neq\partial} Z_t < +\infty$.*

PROOF. First note that the existence of $(Z_t)_{t\in F}$ is guaranteed by the Kolmogorov extension theorem. In order to simplify the expressions, we consider the case $F = \mathbb{Q}_+$. We denote by $\mathbb{P}_x^Z$ (resp. $\mathbb{P}_\mu^Z$) the law of Z with initial measure δ_x (resp. μ), with the associated expectations $\mathbb{E}_x^Z$ and $\mathbb{E}_\mu^Z$. We first prove that Z is right-continuous almost surely, then that it is left-continuous almost surely, and conclude by proving that, on any finite time horizon, the trajectories of the process are almost surely bounded away from 0 and $+\infty$.

(1) *The process $(Z_t)_{t\in\mathbb{Q}_+}$ is right-continuous almost surely.*

Let $x \in (0,+\infty)$ and $f : E \to [0,+\infty)$ such that $f|_{(0,+\infty)} \in C_c^{(s)}$ with $f(\partial) = 0$ and such that f is maximal at x. Fix $\delta > 0$ a positive rational number. For all $n \ge 1$, let $M_0^{(n)} = 0$ and, for all $k \ge 0$,

$$M_{k+1}^{(n)} - M_k^{(n)} = f(Z_{\delta(k+1)/n}) - f(Z_{\delta k/n}) - \int_0^{\delta/n} Q_u \mathcal{L} f(Z_{\delta k/n})\,\mathrm{d}u.$$

The process $M^{(n)}$ is a discrete time martingale and, using Doob's inequality, we deduce that, for all $\varepsilon > 0$,

$$\mathbb{P}_x^Z\Big(\sup_{k\in\{0,\dots,n\}} |M_k^{(n)}| > \varepsilon\Big) \le \frac{\mathbb{E}_x^Z(|M_n^{(n)}|)}{\varepsilon}. \tag{22}$$

But $M_k^{(n)} = f(Z_{\delta k/n}) - f(x) - \sum_{l=0}^{k-1}\int_0^{\delta/n} Q_u\mathcal{L}f(Z_{\delta l/n})\,\mathrm{d}u$, so that

$$|M_n^{(n)}| \le f(x) - f(Z_\delta) + \sum_{l=0}^{n-1}\int_0^{\delta/n} Q_u|\mathcal{L}f|(Z_{\delta l/n})\,\mathrm{d}u,$$

since the maximum of f is attained at x. Taking the expectation on both sides of the inequality, we obtain

$$\begin{aligned}\mathbb{E}_x^Z(|M_n^{(n)}|) &\le f(x) - Q_\delta f(x) + \sum_{l=0}^{n-1}\int_0^{\delta/n} Q_{u+\delta l/n}|\mathcal{L}f|(x)\,\mathrm{d}u \\ &\le Q_0 f(x) - Q_\delta f(x) + \int_0^\delta Q_u|\mathcal{L}f|(x)\,\mathrm{d}u.\end{aligned}$$

We also obtain that

$$|M_k^{(n)}| \geq |f(Z_{\delta k/n}) - f(x)| - \sum_{l=0}^{n-1} \int_0^{\delta/n} Q_u|\mathcal{L}f|(Z_{\delta l/n})\,\mathrm{d}u,$$

where

$$\begin{aligned}\mathbb{P}\left(\sum_{l=0}^{n-1} \int_0^{\delta/n} Q_u|\mathcal{L}f|(Z_{\delta l/n})\,\mathrm{d}u > \varepsilon\right) &\leq \frac{1}{\varepsilon}\mathbb{E}\left(\sum_{l=0}^{n-1} \int_0^{\delta/n} Q_u|\mathcal{L}f|(Z_{\delta l/n})\,\mathrm{d}u\right) \\ &\leq \frac{1}{\varepsilon}\int_0^{\delta} Q_u|\mathcal{L}f|(x)\,\mathrm{d}u.\end{aligned}$$

Hence (22) implies that

$$\begin{aligned}&\mathbb{P}_x^Z\Big(\sup_{k\in\{0,\dots,n\}} |f(Z_{\delta k/n}) - f(x)| > 2\varepsilon\Big) \\ &\quad\leq \mathbb{P}_x^Z\Big(\sup_{k\in\{0,\dots,n\}} |M_k^n| > \varepsilon\Big) + \mathbb{P}_x^Z\left(\sum_{l=0}^{n-1} \int_0^{\delta/n} Q_u|\mathcal{L}f|(Z_{\delta l/n})\,\mathrm{d}u > \varepsilon\right) \\ &\quad\leq \frac{Q_0 f(x) - Q_\delta f(x) + 2\int_0^\delta Q_u|\mathcal{L}f|(x)\,\mathrm{d}u}{\varepsilon}.\end{aligned}$$

Setting $h_x(\delta) = Q_0 f(x) - Q_\delta f(x) + 2\int_0^\delta Q_u|\mathcal{L}f|(x)\,\mathrm{d}u$, this implies in particular that, for all $n \geq 1$,

$$\mathbb{P}_x^Z\Big(\sup_{k\in\{0,\dots,n!\}} |f(Z_{\delta k/n!}) - f(x)| > 2\varepsilon\Big) \leq \frac{h_x(\delta)}{\varepsilon}.$$

But, almost surely,

$$\sup_{k\in\{0,\dots,n!\}} |f(Z_{\delta k/n!}) - f(x)| \leq \sup_{k\in\{0,\dots,(n+1)!\}} |f(Z_{\delta k/(n+1)!}) - f(x)|$$

and hence we can take the limit when $n \to +\infty$ in the penultimate inequality, which leads to

$$\begin{aligned}\mathbb{P}_x^Z\Big(\sup_{\substack{n\geq 1,\\ k\in\{0,\dots,n!\}}} |f(Z_{\delta k/n!}) - f(x)| > 2\varepsilon\Big) &= \mathbb{P}_x^Z\left(\bigcup_{n\geq 1}\Big\{\sup_{k\in\{0,\dots,n!\}} |f(Z_{\delta k/n!}) - f(x)| > 2\varepsilon\Big\}\right) \\ &\leq 1 \wedge \frac{h_x(\delta)}{\varepsilon}.\end{aligned}$$

Since $\{k/n! : n \geq 1, 0 \leq k \leq n!\} = [0,1] \cap \mathbb{Q}$, we deduce that

$$\mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |f(Z_q) - f(x)| > 2\varepsilon\Big) \leq 1 \wedge \frac{h_x(\delta)}{\varepsilon}. \tag{23}$$

Note that $h_x(\delta) \to 0$ when $\delta \to 0$, since $Q_t f(x)$ is continuous in t by (12) and $Q_u|\mathcal{L}f|(x)$ is integrable over $[0, t]$. We deduce that

$$\mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |f(Z_q) - f(x)| > 2\varepsilon\Big) \xrightarrow[\delta\to 0]{} 0. \tag{24}$$

Since this is true for all functions $f \in C_c^{(s)}$ such that f is maximal at x, this implies that $(Z_t)_{t\in\mathbb{Q}}$ is (right)-continuous at time $t = 0$, $\mathbb{P}_x$-almost surely. In particular

$$\mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_q - x| > \varepsilon\Big) \xrightarrow[\delta\to 0]{} 0, \quad \forall x \in (0, +\infty).$$

For $x = \partial$, we have, for all $t \ge 0$, $Q_t \mathbf{1}_\partial(x) = Q_0 \mathbf{1}_\partial(x) = 1$, so that $Z_t = \partial$ $\mathbb{P}_\partial$-almost surely, which of course implies the right-continuity of $(Z)_{t\in\mathbb{Q}_+}$ $\mathbb{P}_\partial$-almost surely. Hence the last convergence also holds true under $\mathbb{P}_\partial$ (taking for instance $|y - \partial| = +\infty$ for all $y \in (0, +\infty)$).

Now, for any probability measure μ on E, integrating with respect to $\mu(\mathrm{d}x)$ the last convergence and using the dominated convergence theorem, we deduce that

$$\mathbb{P}^Z_\mu\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_q - Z_0| > \varepsilon\Big) \xrightarrow[\delta\to 0]{} 0,$$

which implies that Z is continuous at time 0, $\mathbb{P}_\mu$-almost surely.

Finally, fixing $t \in \mathbb{Q}_+$ and using the Markov property at time t, we deduce that the process is right continuous at time $t \in \mathbb{Q}_+$ almost surely. This implies that Z is right-continuous at any time $t \in \mathbb{Q}_+$, $\mathbb{P}^Z_x$-almost surely for all $x \in E$.

(2) *The process $(Z_t)_{t\in\mathbb{Q}_+}$ is left-continuous almost surely.*

Fix $\varepsilon > 0$. For each $x \in (2\varepsilon, 1/\varepsilon)$, let $f_{x,\varepsilon} \in C_c^{(s)}$ be a function with support in $(\varepsilon/2, 1/\varepsilon + 2\varepsilon)$ such that $f_{x,\varepsilon}(y) \le \mathbf{1}_{|y-x|<\varepsilon}$ and $0 \le f_{x,\varepsilon}(y) \le f_{x,\varepsilon}(x) = 1$ for $y \in (0, +\infty) \cup \{\partial\}$. The collection of functions $f_{x,\varepsilon}$ can be chosen such that $f_{x,\varepsilon}$ and $\partial f_{x,\varepsilon}/\partial s$ are bounded uniformly in x. Define $h_{x,\varepsilon}(\delta) = Q_0 f_{x,\varepsilon}(x) - Q_\delta f_{x,\varepsilon}(x) + 2\int_0^\delta Q_u|\mathcal{L} f_{x,\varepsilon}|(x)\,\mathrm{d}u$. By applying (23), we obtain

$$\begin{aligned}\mathbb{P}^Z_x\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_q - x| > \varepsilon\Big) &\le \mathbb{P}^Z_x\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} \big|f_{x,\varepsilon}(Z_q) - f_{x,\varepsilon}(x)\big| \ge 1\Big)\\ &\le 2 \sup_{x\in(2\varepsilon,1/\varepsilon)} h_{x,\varepsilon}(\delta),\end{aligned} \tag{25}$$

for all $x \in (2\varepsilon, 1/\varepsilon) \cup \{\partial\}$. (The case $x = \partial$ is immediate, since we observed in Step 1 that ∂ is absorbing.)

Using the fact that $f_{x,\varepsilon}$ is maximal at x, we deduce that $Q_0 f_{x,\varepsilon}(x) - Q_\delta f_{x,\varepsilon}(x)$ is nonnegative, hence

$$\begin{aligned}h_{x,\varepsilon}(\delta) &= Q_0 f_{x,\varepsilon}(x) - Q_\delta f_{x,\varepsilon}(x) + 2\int_0^\delta Q_u|\mathcal{L} f_{x,\varepsilon}|(x)\,\mathrm{d}u\\ &\le 2\big(Q_0 f_{x,\varepsilon}(x) - Q_\delta f_{x,\varepsilon}(x)\big) + 2\int_0^\delta Q_u|\mathcal{L} f_{x,\varepsilon}|(x)\,\mathrm{d}u\\ &= -2\int_0^\delta Q_u \mathcal{L} f_{x,\varepsilon}(x)\,\mathrm{d}u + 2\int_0^\delta Q_u|\mathcal{L} f_{x,\varepsilon}|(x)\,\mathrm{d}u\\ &= 4\int_0^\delta Q_u(\mathcal{L} f_{x,\varepsilon})_-(x)\,\mathrm{d}u,\end{aligned}$$

where we used (12) for the penultimate equality. We observe that $(\mathcal{L} f_{x,\varepsilon})_-(z)$ is bounded in $z \in (0, +\infty) \cup \{\partial\}$ according to Lemma 1 point (ii), uniformly in $x \in (2\varepsilon, 1/\varepsilon)$ according to (13) in its proof (for this last claim, we simply observe that $\|f_{x,\varepsilon}\|_\infty$ and $\|\partial f_{x,\varepsilon}/\partial s\|_\infty$ are bounded in x by assumption and that the union of the supports of these functions is included in a compact subset of $(0, +\infty)$). Hence $C_\varepsilon(\delta) := 2\sup_{x\in(2\varepsilon,1/\varepsilon)} h_{x,\varepsilon}(\delta)$ goes to 0 when $\delta \to 0$.

Fix $x \in E$, a positive time $t \in \mathbb{Q}_+$ and $\delta \in [0, t] \cap \mathbb{Q}$. Note that for $q \in [0, \delta] \cap \mathbb{Q}$, $|Z_t - Z_{t-q}| \le |Z_t - Z_{t-\delta}| + |Z_{t-\delta} - Z_{t-q}|$. Taking we can conclude (using the preceding remark in the first line, the Markov property in the second and (25) in the third) that, for $x \in (0, +\infty)$

and $\varepsilon' \in (0, \varepsilon/2]$,

$$\begin{aligned}\mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_t - Z_{t-q}| > \varepsilon\Big) &\leq \mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_{t-q} - Z_{t-\delta}| > \varepsilon'\Big)\\ &= \mathbb{E}_x^Z\Big(\mathbb{P}_{Z_{t-\delta}}^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_{\delta-q} - Z_0| > \varepsilon'\Big)\Big)\\ &\leq C_{\varepsilon'}(\delta) + \mathbb{P}_x^Z\big(Z_{t-\delta} \notin (2\varepsilon', 1/\varepsilon') \cup \{\partial\}\big).\end{aligned} \tag{26}$$

But

$$\mathbb{P}_x^Z\big(Z_{t-\delta} \notin (2\varepsilon', 1/\varepsilon') \cup \{\partial\}\big) = 1 - Q_{t-\delta}(\mathbf{1}_{(2\varepsilon',1/\varepsilon')\cup\{\partial\}})(x) \leq 1 - Q_{t-\delta}g_{\varepsilon'}(x),$$

where $g_{\varepsilon'}$ is any nonnegative function in $\mathcal{D}(\mathcal{L})$ bounded by 1, equal to 1 on $(3\varepsilon', \frac{1}{2\varepsilon'}) \cup \{\partial\}$ and vanishing outside $(2\varepsilon', 1/\varepsilon') \cup \{\partial\}$. Now, for all $\eta > 0$, there exists $\varepsilon' > 0$ such that $1 - Q_t g_{\varepsilon'}(x) \leq \eta/2$ (by dominated convergence theorem and the fact that $\mathbf{1}_E \geq g_{\varepsilon'} \to \mathbf{1}_E$ pointwisely, with $Q_t\mathbf{1}_E = \mathbf{1}_E$) and $\delta' > 0$ such that, for all $\delta \in (0, \delta')$, $|Q_t g_{\varepsilon'}(x) - Q_{t-\delta}g_{\varepsilon'}(x)| \leq \eta/2$ (by continuity of $u \mapsto Q_u g_{\varepsilon'}$ at time t). In particular, for all $\delta \in (0, \delta')$,

$$\mathbb{P}_x^Z\big(Z_{t-\delta} \notin (2\varepsilon', 1/\varepsilon') \cup \{\partial\}\big) \leq \eta.$$

Hence, we deduce from (26) that

$$\mathbb{P}_x^Z\Big(\sup_{q\in[0,\delta]\cap\mathbb{Q}} |Z_t - Z_{t-q}| > \varepsilon\Big) \xrightarrow[\delta\to 0]{} 0, \tag{27}$$

so that Z is $\mathbb{P}_x^Z$-almost surely left continuous at time t.

The extension to non-Dirac initial distribution can be done as in Step 1, and this concludes the proof of the first part of Lemma 5.

(3) *The trajectories of the process $(Z_t)_{t\in[0,T]\cap\mathbb{Q}_+}$ are bounded away from 0 and $+\infty$.*

Fix $T > 0$. We first show that, for all $x \in (0, +\infty) \cup \{\partial\}$, Z is $\mathbb{P}_x^Z$-almost surely bounded from above. In order to do so, fix $x \in (0, +\infty)$ (the result is trivial for $x = \partial$). Let W_1 be a $C^{(s)}_{\mathrm{loc}}$ nondecreasing function such that $C_1 := \sup_{z>0} \partial W_1/\partial s(z) < +\infty$ and $\lim_{m\to+\infty} W_1(m) = +\infty$ (such a function exists since $\lim_{z\to+\infty} s(z) = +\infty$ by assumption) and set $W_1(\partial) = 0$. According to Lemma 2 and using the fact that $\mathcal{L}W_1 \leq C_1$, we obtain that, for all $n \geq 1$,

$$M_k^{(n)} = W_1(Z_{Tk/n}) - C_1 Tk/n$$

defines a super-martingale. Hence, for any $m > 0$, defining the stopping time $\sigma_m^n = \inf\{lT/n, l \in \mathbb{Z}_+, Z_{lT/n} > m\}$ and using the optional sampling theorem, we deduce that

$$\mathbb{E}_x^Z\big(W_1(Z_{\sigma_m^n \wedge T})\big) \leq W_1(x) + C_1 T.$$

Since $W_1(Z_{\sigma_m^n\wedge T}) \geq W_1(m)$ on the event $\sigma_m^n \leq T$, we deduce that

$$\mathbb{P}_x^Z\big(\sigma_m^n \leq T\big) \leq \frac{W_1(x) + C_1 T}{W_1(m)}.$$

Since $(\sigma_m^{n!})_n$ is almost surely nonincreasing and converges toward $\sigma_m = \inf\{u \in \mathbb{Q}_+, Z_u > m\}$, we deduce that

$$\mathbb{P}_x^Z(\sigma_m \leq T) \leq \frac{W_1(x) + C_1 T}{W_1(m)}.$$

Using now that $(\sigma_m)_m$ is almost surely nondecreasing, we deduce that

$$\mathbb{P}_x^Z\Big(\sup_{u\in[0,T]\cap\mathbb{Q}_+} \mathbf{1}_{Z_u\neq\partial} Z_u = +\infty\Big) = \mathbb{P}_x^Z\Big(\lim_{m\to+\infty} \sigma_m \leq T\Big) = 0. \tag{28}$$

We prove now that Z is almost surely bounded away from 0, starting from $x \in (0, +\infty)$. We consider the nonnegative measure ν on $(0, 1)$ defined by

$$\nu(A) := \int_0^t Q_u p_A(x)\,\mathrm{d}u,$$

where $p_A(z) = \int_0^z \mathbf{1}_A(y) k_h(z, \mathrm{d}y)$ for all measurable A subset of $(0, 1)$. This is a finite measure according to Lemma 3. Hence there exists a nonincreasing $C^{(s)}_{\mathrm{loc}}$ function $W_2 : (0, +\infty) \to (0, +\infty)$ such that $W_2(z) \to +\infty$ when $z \to 0$ and $W_2(z) = 0$ for all $z \geq 1$, and such that $\nu(W_2) < +\infty$; see Lemma 6 below.

According to Lemma 4 and using the fact that $\int_0^t Q_u \mathcal{L} W_2\,\mathrm{d}u \leq \nu(W_2)$ (with $W_2(\partial) := 0$), we have that, for all $n \geq 1$,

$$N_k^{(n)} = W_2(Z_{Tk/n}) - \nu(W_2) Tk/n$$

defines a super-martingale. Defining the stopping time $\sigma^n_{1/m} = \inf\{lT/n : l \in \mathbb{Z}_+, Z_{lT/n} < 1/m\}$ and using the same method used to obtain (28), we deduce that

$$\mathbb{P}^Z_x\Bigl(\sup_{u \in [0,T] \cap \mathbb{Q}_+} \mathbf{1}_{Z_u \neq \partial} / Z_u = +\infty\Bigr) = 0.$$

This and equation (28) concludes the proof of Lemma 5. $\square$

LEMMA 6. *Let ν be a finite measure on $(0, 1)$. Then, there exists a nonincreasing $C^{(s)}_{loc}$ function W_2 such that $W_2(x) \to \infty$ when $x \to 0$, $W_2(x) = 0$ for $x > 1$ and $\nu(W_2) < +\infty$.*

PROOF. Let $y_n = 2^{n-1} - 1$ for $n \geq 1$. Let (x_n) be a decreasing sequence of numbers in $(0, 1)$ such that $\nu(0, x_n) < 3^{-n}$ for $n \geq 1$, which exists because $\nu((0, 1)) < +\infty$. Then

$$A := \sum_{n \geq 1} y_{n+1} \nu[x_{n+1}, x_n) \leq \sum_{n \geq 1} 2^n 3^{-n} < \infty.$$

Now let W_2 be defined by

$$W_2(x) = y_{n+1} + \frac{s(x) - s(x_{n+1})}{s(x_n) - s(x_{n+1})}(y_n - y_{n+1}), \qquad x \in [x_{n+1}, x_n),$$

so that $W_2(x) \in (y_n, y_{n+1}]$ when $x \in [x_{n+1}, x_n)$. Let $W_2(x) = 0$ for $x \geq 1$. Then W_2 is a positive, nonincreasing, continuous and admits a right derivative with respect to s given by

$$\frac{\partial W_2}{\partial s}(x) = \frac{y_n - y_{n+1}}{s(x_n) - s(x_{n+1})} \leq 0, \qquad x \in [x_{n+1}, x_n),$$

and, for all $x \geq 1$, by $\frac{\partial W_2}{\partial s}(x) = 0$. Moreover, we have

$$\int_{(0,1)} W_2(x) \nu(\mathrm{d}x) \leq \sum_{n \in \mathbb{N}} y_{n+1} \nu\bigl([x_{n+1}, x_n)\bigr) < +\infty,$$

which proves the lemma. $\square$

We state now the uniqueness of the Markov semigroup, so that the proof of the following lemma concludes the proof of Proposition 3. In order to do so, we show that $(Z_t)_{t \in \mathbb{Q}_+}$ (as in the proof of the preceeding lemma) can be extended to a càdlàg process $(Y_t)_{t \in [0,+\infty)}$ with values in E, which appears to be the solution to the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem. The conclusion is then obtained from Proposition 2.

LEMMA 7. *Assume that Assumption 1 holds true and that Q is a semigroup satisfying (12). Then $Q_t f(x) = \mathbb{E}_x(f(X_t))$ for all bounded measurable functions f on E, where X is the unique càdlàg solution to the martingale problem $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. Moreover, $Q_t \mathbf{1}_{(0,\infty)}(x) = \mathbf{1}_{(0,\infty)}(x) - \int_0^t Q_u q(x)\,\mathrm{d}u$, for all $x \in E$.*

PROOF. Let $(Z_t)_{t\in\mathbb{Q}_+}$ be as in the proof of Lemma 5. In a first step, we show that, for any sufficiently regular function f, $(f(Z_t))_{t\in\mathbb{Q}_+}$ admits only finitely many upcrossings over nonempty open intervals. In a second step, we use this to deduce that Z can be extended to a càdlàg Markov process $(Y_t)_{t\in[0,+\infty)}$ with semigroup Q and taking its values in the one point compactification of E. Finally, we prove that Y takes its values in E and that it satisfies the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem.

(1) *Finiteness of the number of upcrossings.* Let $x \in (0, +\infty)$ and f be a nonnegative function in $C_c^{(s)}$, extended to ∂ with $f(\partial) = 0$. Our aim is to prove that, for any $a < b \in \mathbb{R}$, the number of upcrossings through (a, b) of $(f(Z_t) - f(x))_{t\in\mathbb{Q}_+}$ is finite $\mathbb{P}_x^Z$-almost surely on any finite time horizon.

Fix $a < b \in \mathbb{R}$ and $\delta \in (0, \frac{b-a}{1+4c}) \cap \mathbb{Q}$, where $c := \sup(\mathcal{L}f)_-$ is finite according to Lemma 1 point (ii). For all $n \geq 1$, let $M_0^{(n)} = 0$ and

$$M_{k+1}^{(n)} - M_k^{(n)} = f(Z_{\delta(k+1)/n}) - f(Z_{\delta k/n}) - \int_0^{\delta/n} Q_u \mathcal{L} f(Z_{\delta k/n})\,\mathrm{d}u.$$

The process $M^{(n)}$ is a discrete time martingale. Hence, setting $N_0^{(n)} = 0$ and

$$\begin{aligned} N_{k+1}^{(n)} - N_k^{(n)} &= f(Z_{\delta(k+1)/n}) - f(Z_{\delta k/n}) + \frac{c\delta}{n} \\ &= M_{k+1}^{(n)} - M_k^{(n)} + \int_0^{\delta/n} Q_u \mathcal{L} f(Z_{\delta k/n})\,\mathrm{d}u + \frac{c\delta}{n} \end{aligned}$$

defines a sub-martingale. In particular, using Lemma 2.5 page 57 in [26], we have (here $U^{(n)}(a, b)$ denotes the number of upcrossings through the interval (a, b) during the n first steps of the sub-martingale $N^{(n)}$):

$$\mathbb{E}_x^Z\big(U^{(n)}(a,b)\big) \leq \frac{\mathbb{E}_x^Z((N_n^{(n)} - a)_+)}{b-a} \leq \frac{\|f\|_\infty + c\delta + |a|}{b-a},$$

since $N_n^{(n)} = f(Z_\delta) - f(x) + c\delta$. In addition, the number of up-crossing through (a, b) of $(f(Z_{\delta k/n}) - f(x))_{k\in\{0,\dots,n\}}$, denoted by $V^{(n)}(a, b, \delta)$ from now on, is bounded from above by the number of up-crossing through $(a + c\delta, b - c\delta)$ of $(N_k^{(n)})_{k\in\{0,\dots,n\}}$. Hence

$$\mathbb{E}_x^Z\big(V^{(n)}(a,b,\delta)\big) \leq \frac{\|f\|_\infty + 2c\delta + |a|}{b-a-2c\delta}.$$

Since, for all $n \geq 1$, $(f(Z_{k/n!}) - f(x))_{k\in\{0,n!\}}$ is a sub-process of $(f(Z_{k/(n+1)!}) - f(x))_{k\in\{0,(n+1)!\}}$, we have $V^{(n!)}(a, b, \delta) \leq V^{((n+1)!)}(a, b, \delta)$ almost surely and hence

$$\mathbb{E}_x^Z\Big(\sup_{n\geq 1} V^{(n)}(a,b,\delta)\Big) \leq \frac{\|f\|_\infty + 2c\delta + |a|}{b-a-2c\delta}.$$

But $\sup_{n\geq 1} V^{(n)}(a, b, \delta)$ is exactly the number of upcrossings through (a, b) of $(f(Z_t) - f(x))_{t\in\mathbb{Q}_+\cap[0,\delta]}$ and hence, denoting by $V(a, b, \delta)$ this number, we have

$$\mathbb{E}_x^Z\big(V(a,b,\delta)\big) \leq \frac{\|f\|_\infty + 2c\delta + |a|}{b-a-2c\delta}.$$

Hence

$$\mathbb{E}_x^Z\big(V\big(a-f(x),b-f(x),\delta\big)\big)\le\frac{\|f\|_\infty+2c\delta+|a-f(x)|}{b-a-2c\delta}\le\frac{2\|f\|_\infty+2c\delta+|a|}{b-a-2c\delta},$$

and, since the upcrossings through $(a-f(x),b-f(x))$ by $(f(Z_t)-f(x))_{t\in\mathbb{Q}_+\cap[0,\delta]}$ is exactly the number $V'(a,b,\delta)$ of upcrossings of (a,b) by $(f(Z_t))_{t\in\mathbb{Q}_+\cap[0,\delta]}$, we deduce that

$$\mathbb{E}_x^Z\big(V'(a,b,\delta)\big)\le\frac{2\|f\|_\infty+2c\delta+|a|}{b-a-2c\delta}.$$

We conclude that the number of upcrossings $V'(a,b,\delta)$ is finite $\mathbb{P}_x^Z$-almost surely. Since this is true for all initial distribution, using the Markov property at times $\delta, 2\delta, \dots$, we obtain that, for all $T\in\mathbb{Q}_+$, the number of upcrossings $V'(a,b,T)$ is finite almost surely. Since this is true for all $a<b\in\mathbb{R}$, this in turn implies that $V(a,b,T)$ is finite $\mathbb{P}_x^Z$-almost surely.

(2) *Construction of a càdlàg representation of* $(Q_t)_{t\in[0,+\infty)}$ *in* $E\cup\{\Delta\}$. Now, using Problem 9(a), page 90 in [26], we deduce that, for all nonnegative functions $f\in C_c^{(s)}(0,+\infty)$ extended to ∂ with $f(\partial)=0$, $\mathbb{P}_x^Z$-almost surely, for all $t\in[0,+\infty)$,

$$\lim_{u\in\mathbb{Q}_+,u>t,u\to t} f(Z_u)\quad\text{and}\quad\lim_{u\in\mathbb{Q}_+,u<t,u\to t} f(Z_u)\tag{29}$$

both exist. Moreover ∂ is an absorbing point for Z, so that $(\mathbf{1}_\partial(Z_t))_{t\in\mathbb{Q}_+}$ is increasing, taking its values in $\{0,1\}$, and hence the above limits also exist for $f=\mathbf{1}_\partial$.

As a consequence, there exists a countable family $\mathcal{H}$ of continuous functions f that separates points in E and such that the above limits exist (recall that $\mathbf{1}_\partial$ is continuous since ∂ is an isolated point). We deduce that, $\mathbb{P}_x^Z$-almost surely, for all $t\in[0,+\infty)$,

$$\lim_{u\in\mathbb{Q}_+,u>t,u\to t} Z_u\quad\text{and}\quad\lim_{u\in\mathbb{Q}_+,u<t,u\to t} Z_u$$

also exist in $(0,+\infty)\cup\{\partial,\Delta\}$, where Δ is a compactification point for $(0,+\infty)$ (and hence for $(0,+\infty)\cup\{\partial\}$). Indeed, let Z_{t_+} and Z'_{t_+} be two accumulation points in $(0,+\infty)\cup\{\partial,\Delta\}$ of $(Z_u)_{u\in\mathbb{Q}_+,u\ge t}$ at $t\in[0,+\infty)$. On the one hand, if $Z_{t_+}\in(0,+\infty)\cup\{\partial\}$ and $Z'_{t_+}\in(0,+\infty)\cup\{\partial\}$ are different, then there exists a function $f\in\mathcal{H}$ such that $f(Z_{t_+})\ne f(Z'_{t_+})$. Since f is continuous, then this contradicts (29). On the other hand, if $Z_{t_+}\in(0,+\infty)\cup\{\partial\}$ and $Z'_{t_+}=\Delta$, then one chooses any function $f\in\mathcal{H}$ such that $f(Z_{t_+})>0$ with compact support, and observe that f extended by 0 at Δ is continuous, so that $f(Z_{t_+})\ne 0=f(Z'_{t_+})$ also contradicts (29). This implies that, almost surely, for all $t\in(0,+\infty)$, the accumulation point in $(0,+\infty)\cup\{\partial,\Delta\}$ of $(Z_{t+u})_{t+u\in\mathbb{Q}_+}$ at $t\in(0,+\infty)$ is unique, which implies the existence of the first limit. The existence of the second limit is proved similarly.

We deduce that Z satisfies almost surely the assumptions of Lemma 2.8, page 58 in [26] and hence we can define the càdlàg random process $(Y_t)_{t\in\mathbb{R}_+}$ with values in $E\cup\{\Delta\}$ as

$$Y_t:=\lim_{u\in\mathbb{Q},u>t,u\to t} Z_u,\quad\mathbb{P}_x^Z\text{-almost surely.}$$

Since $(Z_t)_{t\in\mathbb{Q}_+}$ is (right)-continuous according to Lemma 5, we deduce that $Y_t=Z_t$ for all $t\in\mathbb{Q}_+$ (in particular, $Y_t\in E$ $\mathbb{P}_x^Z$-almost surely, for all $t\in\mathbb{Q}_+$).

Let us now show that, for all $t\ge 0$, $\delta_x Q_t$ is the law of Y_t under $\mathbb{P}_x^Z$. We have, for all $f\in\mathcal{D}(\mathcal{L})$ extended to $E\cup\{\Delta\}$ by $f(\Delta)=0$,

$$\begin{aligned}\mathbb{E}_x^Z\big(f(Y_t)\big)&=\mathbb{E}_x^Z\Big(\lim_{u>t,u\in\mathbb{Q},u\to t} f(Z_u)\Big)\\&=\lim_{u>t,u\in\mathbb{Q},u\to t}\mathbb{E}_x^Z\big(f(Z_u)\big)=\lim_{u>t,u\in\mathbb{Q},u\to t} Q_u f(x)=Q_t f(x),\end{aligned}$$

since $Q_u f(x)$ is continuous in u for all $f \in \mathcal{D}(\mathcal{L})$ by (12). Since $C_c^{(s)} \subset \mathcal{D}(\mathcal{L})$ and $\mathbf{1}_\partial \in \mathcal{D}(\mathcal{L})$, we deduce that $\mathbb{P}_x^Z(Y_t \in A) = \delta_x Q_t \mathbf{1}_A$ for all measurable $A \subset (0,+\infty) \cup \{\partial\}$. Since $\delta_x Q_t \mathbf{1}_E = 1$, we conclude that $\mathbb{P}_x^Z(Y_t \in E) = 1$ and that $\delta_x Q_t$ is the law of Y_t under $\mathbb{P}_x^Z$, for all $t \in [0,+\infty)$.

Let us now prove that Y is a Markov process with respect to its natural filtration $(\mathcal{F}_t^0)_{t\geq 0}$. Fix $u_0 \leq t_0 \in [0,+\infty)$ and consider the Markov process $(Z'_t)_{t\in\mathbb{Q}_+\cup\{u_0,t_0\}}$ with semigroup $(Q_t)_{t\in\mathbb{Q}_+\cup\{u_0,t_0\}}$. Then $(Z'_t)_{t\in\mathbb{Q}_+}$ under $\mathbb{P}_x^{Z'}$ has the same law as $(Z_t)_{t\in\mathbb{Q}_+}$ under $\mathbb{P}_x^Z$. Since Z' and Y are right-continuous at times u_0, t_0 almost-surely (according to Lemma 5 for Z'), we deduce that $(Z'_{u_0}, Z'_{t_0}, (Z'_t)_{t\in\mathbb{Q}_+})$ under $\mathbb{P}_x^{Z'}$ and $(Y_{u_0}, Y_{t_0}, (Z_t)_{t\in\mathbb{Q}_+})$ under $\mathbb{P}_x^Z$ have the same law, for all $x \in E$. Hence, for all bounded measurable functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_x^Z\big(f(Y_{u_0})g(Y_{t_0})\big) &= \mathbb{E}_x^{Z'}\big(f(Z'_{u_0})g(Z'_{t_0})\big) \\
&= \mathbb{E}_x^{Z'}\big(f(Z'_{u_0})Q_{t_0-u_0}g(Z'_{u_0})\big) \\
&= \mathbb{E}_x^Z\big(f(Y_{u_0})Q_{t_0-u_0}g(Y_{u_0})\big).
\end{aligned}$$

The same line of arguments applies for any finite family of times $u_1 \leq \cdots \leq u_k \leq u_0 \leq t_0$, which implies that, for all $0 \leq u \leq t$,

$$\mathbb{E}_x^Z\big(f(Y_t)|\sigma(Y_v, v \leq u)\big) = Q_{t-u} f(Y_u), \quad \mathbb{P}_x^Z\text{-almost surely.}$$

We conclude that Y is indeed a Markov process, with values in $E \cup \{\Delta\}$.

(3) *The càdlàg representation is a solution to the martingale problem in E.* We observe that, for all $t \geq u \geq 0$ and all $f \in \mathcal{D}(\mathcal{L})$, and setting $\mathcal{L}f(\Delta) = 0$,

$$\begin{aligned}
&\mathbb{E}_x^Z\left(f(Y_t) - \int_0^t \mathcal{L}f(Y_v)\,dv|\mathcal{F}_u^0\right) \\
&\quad = Q_{t-u} f(Y_u) - \int_0^u \mathcal{L}f(Y_v)\,\mathrm{d}v - \mathbb{E}_x^Z\left(\int_u^t \mathcal{L}f(Y_v)\,\mathrm{d}v|\mathcal{F}_u^0\right),
\end{aligned}$$

where $\mathcal{F}$ is the natural filtration of Y. But

$$Q_{t-u} f(Y_u) = f(Y_u) + \int_0^{t-u} Q_v \mathcal{L} f(Y_u)\,\mathrm{d}v$$

and

$$\mathbb{E}_x^Z\left(\int_u^t \mathcal{L}f(Y_v)\,\mathrm{d}v|\mathcal{F}_u^0\right) = \int_u^t Q_{v-u}\mathcal{L}f(Y_u)\,\mathrm{d}v$$

(using the fact that $\int_u^t Q_{v-u}|\mathcal{L}f|(Y_u)\,\mathrm{d}v$ is finite, which allows the use of Fubini's theorem). Hence $f(Y_t) - \int_0^t \mathcal{L}f(Y_v)\,\mathrm{d}v$ defines a martingale. We deduce that Y is a càdlàg solution to the martingale problem associated to $\mathcal{L}$ on $E \cup \{\Delta\}$.

But, according to Lemma 5, Z is bounded away from 0 and $+\infty$ almost surely, so that Y (whose values are in the adherence of the values taken by Z almost surely) is also bounded away from 0 and $+\infty$ almost surely. This implies that Y never reaches Δ and hence that Y takes its values in E, $\mathbb{P}_x^Z$-almost surely for all $x \in E$. This entails that Y is a càdlàg solution to the martingale problem in E.

We conclude the proof of the first part of Lemma 7 by observing that Proposition 2 states that the càdlàg solution to the martingale problem $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is unique.

In order to obtain the last claim of Lemma 7, observe that $\mathbf{1}_\partial \in \mathcal{D}(\mathcal{L})$ and that $Q_t\mathbf{1}_E = \mathbf{1}_E$, so that

$$\begin{aligned}\delta_x Q_t \mathbf{1}_{(0,+\infty)} &= \delta_x Q_t \mathbf{1}_E - \delta_x Q_t \mathbf{1}_\partial \\ &= \mathbf{1}_E(x) - \mathbf{1}_\partial(x) - \int_0^t Q_u \mathcal{L}\mathbf{1}_\partial(x)\,\mathrm{d}u \\ &= \mathbf{1}_{(0,+\infty)}(x) - \int_0^t Q_u q(x)\,\mathrm{d}u.\end{aligned}$$

This concludes the proof of Lemma 7. □

2.3. *Conclusion of the proof of Theorem* 1. For the existence, we set $T_t f(x) = e^{bt}h(x)Q_t(f/h)(x)$ for all $f \in \mathcal{D}(\mathcal{A})$ with the convention $f/h(\partial) := 0$, where Q is the semigroup of Proposition 3. For all $f \in \mathcal{D}(\mathcal{A})$, the function $g = f/h$ is in $\mathcal{D}(\mathcal{L})$ or $g = \mathbf{1}_{(0,+\infty)}$, and hence, for all $x \in (0,+\infty)$, if $g \in \mathcal{D}(\mathcal{L})$, then

$$\begin{aligned}\partial_t T_t f(x) &= \partial_t\big[e^{bt}h(x)Q_t g(x)\big] \\ &= be^{bt}h(x)Q_t g(x) + e^{bt}h(x)Q_t\mathcal{L}g(x) \\ &= be^{bt}h(x)Q_t g(x) + e^{bt}h(x)Q_t\left(\frac{\mathcal{A}(hg)}{h} - bg\right)(x) \\ &= e^{bt}h(x)Q_t\left(\frac{\mathcal{A}f}{h}\right)(x) = T_t\mathcal{A}f(x),\end{aligned}$$

understanding differentiation here in the sense of density with respect to Lebesgue measure; if $g = \mathbf{1}_{(0,+\infty)}$, then the same computation holds true according to the last property of Lemma 7. The fact that $T_t B \subset B$ is a straightforward consequence of the fact that $Q_t \mathbf{1}_{(0,+\infty)} \le \mathbf{1}_{(0,+\infty)}$.

Let us now check the uniqueness. Assume that T is a semigroup which solves the above equation for $f \in \mathcal{D}(\mathcal{A})$. Then $h \in \mathcal{D}(\mathcal{A})$ and hence the semigroup defined by $\delta_x R_t := \frac{e^{-bt}\delta_x T_t(\cdot h)}{h(x)}$ ($x \in (0,+\infty)$) satisfies, for all $x \in (0,+\infty)$,

$$\begin{aligned}R_t\mathbf{1}_{(0,+\infty)}(x) &= \frac{e^{-bt}T_t h(x)}{h(x)} \\ &= 1 - b\int_0^t \frac{e^{-bu}T_u h(x)}{h(x)}\,\mathrm{d}u + \int_0^t e^{-bu}\frac{T_u\mathcal{A}h(x)}{h(x)}\,\mathrm{d}u \\ &= 1 + \int_0^t R_u\mathcal{L}\mathbf{1}_{(0,+\infty)}(x)\,\mathrm{d}u \le 1.\end{aligned}$$

Hence $(R_t)_{t\ge 0}$ is a sub-Markov semigroup on the set of bounded measurable functions on $(0,+\infty)$. As usual, we extend R as a Markov semigroup on the set of bounded measurable functions on $E = (0,+\infty)\cup\{\partial\}$, by setting $R_t\mathbf{1}_\partial(x) = 1 - R_t\mathbf{1}_{(0,+\infty)}(x)$ for all $x \in (0,+\infty)$ and $R_t f(\partial) = f(\partial)$ for all bounded measurable functions f on E. For all $f \in C_c^{(s)}$, $fh \in \mathcal{D}(\mathcal{A})$ and hence, for all $x \in (0,+\infty)$,

$$\begin{aligned}R_t f(x) &= \frac{e^{-bt}T_t(fh)}{h(x)} \\ &= f(x) - b\int_0^t \frac{e^{-bu}T_u(fh)}{h(x)}\,\mathrm{d}u + \int_0^t e^{-bu}\frac{T_u\mathcal{A}(fh)(x)}{h(x)}\,\mathrm{d}u \\ &= f(x) + \int_0^t R_u\mathcal{L}f(x)\,\mathrm{d}u,\end{aligned}$$

while $R_t f(\partial) = f(\partial) = f(\partial) + \int_0^t R_u \mathcal{L} f(\partial)\,\mathrm{d}u$. For all $x \in (0,+\infty)$, we have

$$R_t \mathbf{1}_{(0,+\infty)}(x) = 1 - \int_0^t R_u q(x)\,\mathrm{d}u$$

and hence

$$R_t \mathbf{1}_\partial(x) = \int_0^t R_u q(x)\,\mathrm{d}u = \int_0^t R_u \mathcal{L}\mathbf{1}_\partial(x)\,\mathrm{d}u,$$

while $R_t \mathbf{1}_\partial(\partial) = \mathbf{1}_\partial(\partial) = \mathbf{1}_\partial(\partial) + \int_0^t R_u \mathcal{L}\mathbf{1}_\partial(\partial)\,\mathrm{d}u$. Using Lemma 7, we deduce that $R_t = Q_t$ and hence that $T_t f(x) = e^{bt} h(x) Q_t(f/h)(x)$. This concludes the proof of Theorem 1.

2.4. *Proof of Corollary* 1. Fix $x \in (0,+\infty)$. Assume first that $f \geq 0$ and set $\varphi = f/h$ and let $(\varphi_m)_{m\geq 0}$ be a nondecreasing sequence of functions in $C_c^{(s)}$ such that $\varphi_m(x) = \varphi(x)$ for all $x \in (1/m, m)$. We also set $\varphi_m(\partial) = \varphi(\partial) = 0$. Then, for all $m > k \geq 1$, since $\varphi_m \in \mathcal{D}(\mathcal{L})$ and τ_k (defined in the first step of the proof of Proposition 2) is a stopping time, for all $t \geq 0$, and all $x \in (1/k, k)$, we have

$$\mathbb{E}_x\big(\varphi_m(X_{t\wedge\tau_k})\big) = \varphi_m(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \mathcal{L}\varphi_m(X_u)\,\mathrm{d}u\right) = \varphi(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \mathcal{L}\varphi_m(X_u)\,\mathrm{d}u\right).$$

But, almost surely, for all $u < \tau_k$, we have $X_u \in (1/k, k) \subset (1/m, m)$ and hence

$$\begin{aligned}
\mathcal{L}\varphi_m(X_u) = {} & \frac{\partial \varphi_m}{\partial s}(X_u) \\
& + \int_{(0,x)} \varphi_m(y) k_h(X_u, \mathrm{d}y) - \varphi_m(X_u) k_h\big(X_u, (0, X_u)\big) - q(X_u)\varphi_m(X_u) \\
= {} & \frac{\partial \varphi}{\partial s}(X_u) + \int_{(0,x)} \varphi_m(y) k_h(X_u, \mathrm{d}y) - \varphi(X_u) k_h\big(X_u, (0, X_u)\big) - q(X_u)\varphi(X_u) \\
\nearrow {} & \frac{\partial \varphi}{\partial s}(X_u) + \int_{(0,x)} \varphi(y) k_h(X_u, \mathrm{d}y) - \varphi(X_u) k_h\big(X_u, (0, X_u)\big) - q(X_u)\varphi(X_u) \\
= {} & \mathcal{L}\varphi(X_u) \quad \text{when } m \to +\infty.
\end{aligned}$$

The monotone convergence theorem (taking into account $\mathbb{E}_x(\int_0^{t\wedge\tau_k} |\mathcal{L}\varphi_m(X_u)|\,\mathrm{d}u) < +\infty$ for all $m \geq 1$), we deduce that

$$\mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \mathcal{L}\varphi_m(X_u)\,\mathrm{d}u\right) \xrightarrow[m\to+\infty]{} \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \mathcal{L}\varphi(X_u)\,\mathrm{d}u\right).$$

Since $\varphi = f/h$ is bounded, by the dominated convergence theorem, we also deduce that

$$\mathbb{E}_x\big(\varphi_m(X_{t\wedge\tau_k})\big) \xrightarrow[m\to+\infty]{} \mathbb{E}_x\big(\varphi(X_{t\wedge\tau_k})\big)$$

and hence

$$\mathbb{E}_x\big(\varphi(X_{t\wedge\tau_k})\big) = \varphi(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \mathcal{L}\varphi(X_u)\,\mathrm{d}u\right).$$

Assume first that $\mathcal{A}f/h$ is lower bounded by $-a$, where $a > 0$. Then

$$\mathbb{E}_x\big(\varphi(X_{t\wedge\tau_k})\big) + a\mathbb{E}_x(t\wedge\tau_k) = \varphi(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} \big(\mathcal{L}\varphi(X_u) + a\big)\,\mathrm{d}u\right),$$

where $\mathcal{L}\varphi(X_u) + a = \mathcal{A}f(X_u)/h(X_u) + a \geq 0$, so that, by dominated convergence on the left-hand side, and by monotone convergence in the right-hand side, we obtain by letting $k \to +\infty$

$$\mathbb{E}_x\big(\varphi(X_t)\big) + a\mathbb{E}_x(t) = \varphi(x) + \mathbb{E}_x\left(\int_0^t \big(\mathcal{L}\varphi(X_u) + a\big)\,\mathrm{d}u\right)$$

and hence that

$$\text{(30)}\quad \mathbb{E}_x\left(\int_0^t |\mathcal{L}\varphi(X_u)|\,\mathrm{d}u\right) < +\infty \quad \text{and} \quad \mathbb{E}_x(\varphi(X_t)) = \varphi(x) + \mathbb{E}_x\left(\int_0^t \mathcal{L}\varphi(X_u)\,\mathrm{d}u\right).$$

Assume now instead that $\mathcal{A}f/h$ is upper bounded by $a > 0$. Then

$$\mathbb{E}_x(\varphi(X_{t\wedge\tau_k})) - a\mathbb{E}_x(t\wedge\tau_k) = \varphi(x) - \mathbb{E}_x\left(\int_0^{t\wedge\tau_k} (-\mathcal{L}\varphi(X_u) + a)\,\mathrm{d}u\right),$$

where $-\mathcal{L}\varphi(X_u) + a = -\mathcal{A}f(X_u)/h(X_u) + a \geq 0$. As above, this entails that (30) holds true.

In both cases, we deduce from Fubini's theorem that

$$\int_0^t Q_u|\mathcal{L}\varphi|(x)\,\mathrm{d}u < +\infty \quad \text{and} \quad Q_t\varphi(x) = \varphi(x) + \int_0^t Q_u\mathcal{L}\varphi(x)\,\mathrm{d}u.$$

Replacing Q, $\mathcal{L}$ and φ by their respective expressions of T, $\mathcal{A}$ and f, this concludes the proof of Corollary 1.

2.5. *Proof of Corollary* 2. We observe that Assumption 1 is clearly satisfied with $h = h_1 + h_2$, and hence, according to Theorem 1, there exists T a solution to (4). In addition, $\mathcal{A}h_1/h$ is upper bounded by $\mathcal{A}h_1/h_1$ and hence is upper bounded. By Corollary 1, we deduce that

$$\int_0^t T_u|\mathcal{A}h_1|\,\mathrm{d}u < +\infty \quad \text{and} \quad T_t h_1(x) = h_1(x) + \int_0^t T_u\mathcal{A}h_1\,\mathrm{d}u.$$

Since in addition (4) holds true for all $f \in C_c^{(s)}$, we deduce from the uniqueness part of Theorem 1 that $T = T^1$. Similarly, $T = T^2$ which concludes the proof.

3. Long time asymptotics of the solution to the growth-fragmentation equation. In this section, we focus on the existence of leading eigenelements and a spectral gap for the semigroup T solution to (4) acting on the Banach space B. Our approach will be to leverage the representation of T as the h-transform of the semigroup Q of an absorbed Markov process evolving on $E = (0, +\infty) \cup \{\partial\}$, as given in Section 2. More precisely, we will make use of the results developed in [20] for the study of quasi-stationary distributions.

At this stage, we require Assumptions 2 and 3, which appeared in the Introduction. These can be interpreted, respectively, as an irreducibility and local Doeblin condition for the càdlàg Markov process with semigroup Q defined in Proposition 2.

Under Assumptions 1 and 2, the semigroup T from Theorem 1, the semigroup Q from Proposition 3 and the Markov process X from Proposition 4 below are well defined, and we have the following irreducibility result, which is proved in Section 3.2.

Denote by $\mathbb{P}_x$ the law of X with initial distribution δ_x for $x \in (0, +\infty)$, and $\mathbb{P}_\mu = \int \mathbb{P}_x\mu(\mathrm{d}x)$ for a distribution μ on $(0, +\infty)$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the completion of the natural filtration with respect to sets which are null for every $\mathbb{P}_\mu$ (see [22], §25). Moreover, define $H_y = \inf\{t \geq 0, X_t = y\}$, the hitting time of y by X.

PROPOSITION 4. *Assume that Assumption* 1 *holds true. Let X be the unique càdlàg solution of the martingale problem $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. Then X is a strong Markov process with respect to $\mathcal{F}$. If in addition Assumption* 2 *holds, then X is irreducible in $(0, +\infty)$, in the sense that, for all $l < r \in (0, +\infty)$, there exists $t_0 > 0$ such that*

$$\inf_{x,y\in[l,r]} \mathbb{P}_x(H_y \leq t_0) > 0.$$

Some natural examples of growth-fragmentation models may not lead to a process irreducible in $(0, +\infty)$. For instance, if a cell can grow from small sizes to some critical $m > 0$, and subsequent divisions always lead to cells of size at least m, then one may expect to fruitfully study a version of the process X above, restricted either to $(m, +\infty)$ or to $[m, +\infty)$, and find a similar irreducibility result. Practically, this can be ensured by replacing the role of 0 with m throughout our assumptions, but we will not discuss this in detail.

We turn next to Assumption 3. This is sufficient to obtain a local Doeblin condition for X. Assumption 3(a) is adapted from a general, multi-dimensional result developed in [38], Proposition 1. Its application leverages a simple change of variable argument, as detailed in Section 3.3. Assumption 3(b) places regularity conditions on a lower bound of the division kernel, inspired by the concept of T-chain, as introduced in [32], Chapter 6. Its application use irreducibility arguments, as detailed in Section 3.4.

Other approaches to the local Doeblin condition typically revolve around coupling; for instance, see [18] for an approach to the TCP process and [15] for results applied to the mitosis kernel $k(x, \mathrm{d}y) = 2K(x)\delta_{x/2}(\mathrm{d}y)$. The equal mitosis kernel is also considered in Section 6.3.3 of [44], using a similar approach to us; see Remark 4(ii) in Section 3.1 below.

PROPOSITION 5. *Assume that Assumptions 1, 2 and 3 hold true. Then there exists a probability measure υ on $(0, +\infty)$ such that, for any compactly contained interval $L \subset (0, +\infty)$, there exists $t_L > 0$ such that, for all $t \geq t_L$ and all $x \in L$,*

$$\mathbb{P}_x(X_t \in \cdot) \geq c_{L,t}\upsilon(\cdot), \tag{31}$$

where $c_{L,t} > 0$ only depends on L and t and is nonincreasing in t.

If Assumptions 1, 2 and the Doeblin condition (31) hold true, we can introduce the growth coefficient of T, defined by

$$\lambda_0 := \inf\Big\{\lambda \in \mathbb{R}, \liminf_{t\to+\infty} e^{\lambda t} T_t \mathbf{1}_L(x) = +\infty\Big\},$$

with arbitrary $x \in (0, +\infty)$ and nonempty, open, compactly contained interval $L \subset (0, +\infty)$. One easily checks, using the relationship between T and the semigroup of X, that $\lambda_0 = \lambda_0^X - b$, where

$$\lambda_0^X := \inf\Big\{\lambda \in \mathbb{R}, \liminf_{t\to+\infty} e^{\lambda t} \mathbb{P}_x(X_t \in L) = +\infty\Big\}. \tag{32}$$

The fact that λ_0^X (and hence λ_0) does not depend on x nor L is a well-known consequence of the irreducibility property and the Doeblin condition (31).

Our aim is to apply Theorem 3.5 in [20] to X. This requires a Foster–Lyapunov type condition, which will be obtained using the following assumption, where we recall that $C^{(s)}_{\mathrm{loc}}$ denotes the set of functions with a locally bounded derivative with respect to s. (In fact, one may consider situations where ψ is only s-absolutely continuous, as defined in the Appendix).

ASSUMPTION 4. There exist a positive function $\psi \in C^{(s)}_{\mathrm{loc}}$, a constant $\lambda_1 > \lambda_0$ and a compact interval $L \subset (0, +\infty)$ such that $\inf_{x\in(0,+\infty)} \psi/h > 0$ and

$$\mathcal{A}\psi(x) \leq -\lambda_1\psi(x) + C\mathbf{1}_L(x), \quad \forall x \in (0, +\infty),$$

for some constant $C > 0$.

We emphasis that, in most cases, taking $h = \psi$ is the most natural choice, in which case the requirement $\inf_{x\in(0,+\infty)} \psi/h > 0$ of the last assumption is trivial.

We can now state the main result of this section. It is proved in Section 3.5.

THEOREM 3. *Assume that Assumptions 1, 2, 3 and 4 hold true. Then there exist a unique positive measure m on $(0,+\infty)$ and a unique function $\varphi:(0,+\infty)\to(0,+\infty)$ such that $m(\psi)=1$ and $\|\varphi/\psi\|_\infty<+\infty$ and such that, for all $t\geq 0$, $mT_t=e^{\lambda_0 t}m$ and $T_t\varphi=e^{\lambda_0 t}\varphi$. Moreover, for all $f:(0,+\infty)\to\mathbb{R}$ such that $|f|\leq\psi$, we have*

$$\left|e^{\lambda_0 t}T_t f(x)-\varphi(x)m(f)\right|\leq Ce^{-\gamma t}\psi(x),$$

for some constants $C,\gamma>0$.

We call λ_0 the *growth coefficient* of T. Theorem 3 entails the existence of a spectral gap for the semigroup of $(T_t)_{t\geq 0}$ acting on the Banach space $L^\infty(\psi):=\{f:(0,+\infty)\to\mathbb{R},\|f/\psi\|_\infty<+\infty\}$, endowed with the norm $f\mapsto\|f/\psi\|_\infty$. Conversely, if the convergence of Theorem 3 holds true, then $(T_t)_{t\geq 0}$ also satisfies Lyapunov type conditions and Doeblin type conditions (we refer the reader to [6] and [19] for such converse properties) and it is thus expected that Theorem 3 covers most situations where a spectral gap exists in some $L^\infty(\psi)$. However it is clear that our result does not apply in situations with no spectral gap. While a similar approach may be used in this situation, the main limitation is that the theory of quasi-stationary distributions for sub-Markov semigroup without spectral gap is limited and is still as of this day an active area of research.

In practice, checking Assumption 4 requires to find an upper bound on λ_0 and to find a Lyapunov function ψ. We first relate λ_0 to an apparently lower quantity. This result is proved in Section 3.6.

PROPOSITION 6. *If Assumptions 1, 2 and 3 hold true, then*

$$\lambda_0=\inf\left\{\lambda\in\mathbb{R},\int_0^\infty e^{\lambda t}T_t\mathbf{1}_L(x)\,\mathrm{d}t=+\infty\right\}$$

for any $x\in(0,+\infty)$ and any nonempty open compactly embedded subset $L\subset(0,+\infty)$.

Making use of a second Lyapunov-type function ξ, the following result provides a criterion to find upper bounds for λ_0, proved in Section 3.7 (the proof adapts easily to situations where ξ is only s-absolutely continuous). Theorem 3 together with part (b) of this result provides Theorem 2 in the Introduction.

PROPOSITION 7. *Assume that Assumptions 1, 2 and 3 hold true, and that:*

(i) *There exist a positive function $\psi\in C^{(s)}_{loc}$, a constant $\lambda_1\in\mathbb{R}$ and a compact interval $L\subset(0,+\infty)$ such that $\inf_{x\in(0,+\infty)}\psi/h>0$ and*

$$\mathcal{A}\psi(x)\leq-\lambda_1\psi(x)+C\mathbf{1}_L(x),\quad\forall x\in(0,+\infty),\tag{33}$$

for some constant $C>0$.

(ii) *There exists a positive function $\xi\in C^{(s)}_{loc}$ such that $\|\xi/h\|_\infty<+\infty$, $\frac{\xi(x)}{\psi(x)}\xrightarrow[x\to 0,+\infty]{}0$, and such that there exists $\lambda_2\in\mathbb{R}$ such that*

$$\mathcal{A}\xi(x)\geq-\lambda_2\xi(x),\quad\forall x\in(0,+\infty).\tag{34}$$

The following hold:

(a) *if $\lambda_2<\lambda_1$, then $\lambda_0\leq\lambda_2$;*

(b) *if $\lambda_2\leq\lambda_1$ and $\sup_{x\in(0,M)}\int_{(0,x)}\frac{\xi(y)}{\xi(x)}k(x,\mathrm{d}y)<+\infty$ for all $M>0$, then $\lambda_0\leq\lambda_2$, with strict inequality if $x\in(0,+\infty)\mapsto\frac{\mathcal{A}\xi(x)}{\xi(x)}$ is not constant.*

While finding Lyapunov functions can be tricky, we show in the next section that exponentials of s or of $\int_1^\cdot K(y)s(\mathrm{d}y)$ cover several situations and allow to recover and improve on several results in the literature.

REMARK 3. We emphasize that λ_0 may be characterized by other means than its definition. For instance, in [10], Proposition 3.3, it is shown that $\lambda_0 = -\inf\{q \in \mathbb{R}, L_{x_0,x_0}(q) < 1\}$, where L_{x_0,x_0} is defined in terms of a multiplicative functional of an auxiliary Markov process evaluated at the return time to x_0. In particular, [10], Proposition 3.4, provides a upper bound for λ_0. We also refer the reader to [17], Section 2.2, for a situation where the mass conservation does not hold.

3.1. *Applications.* In this section, we apply the results of Sections 2 and 3 to different situations, focusing on Assumptions 1 and 4, since Assumptions 2 and 3 are already explicit (see also Remark 4 below). In Section 3.1.1, we provide a sufficient criterion for Assumptions 1 and 4 in the situation where $s(0+) > -\infty$. In Section 3.1.2, we consider the situation where $\int_{(0,1)} K(y)s(\mathrm{d}y) < +\infty$ and where mass conservation holds true. The last two subsections are dedicated to the study of near-critical cases, the critical case being when K is constant and $s(x) = \ln x$, in which case it is well known that the conclusions of Theorem 3 do not hold true. In Section 3.1.3, we study the case $s(x) \approx \ln x$ and K is not constant. In Section 3.1.4, we study the case $s(x) \neq \ln x$ and K approximately constant.

REMARK 4. Assumption 3 holds in the following situations.

(i) Assume that there exists a positive constant $a > 0$, a nonempty open interval $I \subset (0, +\infty)$ and a probability measure κ on $(0, +\infty)$, such that

$$k(x, \mathrm{d}y) \geq a\kappa(\mathrm{d}y) \quad x \in I.$$

Then Assumption 3(a) and 3(b) both hold true. The fact that 3(b) holds true is immediate. For 3(a), let μ be the Lebesgue measure on $[0, 1]$ and $T(\theta, x) = F_\kappa^{-1}(\theta)$, where F_κ^{-1} is the generalized inverse of the cumulative distribution function of κ. Then $\kappa(\mathrm{d}y) = \int_{[0,1]} \delta_{T(\theta,x)}(\mathrm{d}y)\mu(\mathrm{d}\theta)$, so that (5) holds true. Since $T(\theta, x)$ does not depend on x, the rest of the assumption holds true.

(ii) Assume that k is locally lower bounded by the equal mitosis kernel, that is, there exists a nonempty open interval $I \subset (0, +\infty)$ and $a > 0$ such that

$$k(x, \mathrm{d}y) \geq a\delta_{x/2}(\mathrm{d}y), \quad x \in I,$$

and moreover assume that $s(x) = \int_1^x 1/c(y)\,\mathrm{d}y$ for some positive function $c\colon (0, +\infty) \to (0 + \infty)$, continuous on I, such that $c(x) \neq 2c(x/2)$ for all $x \in I$. Then, Assumption 3(a) holds by taking $T(x) = x/2$. Clearly, Assumption 3(b) does not hold in this situation.

(iii) More generally, consider the situation where $k(x, \cdot) = K(x)p \circ m_x^{-1}$ with $m_x(u) = xu$ and p is a finite measure on $(0, 1)$, with $s(x) = \int_1^x 1/c(y)\,\mathrm{d}y$ for some positive function $c\colon (0, +\infty) \to (0 + \infty)$ continuous on some sub-interval I. Assume in addition K is lower bounded away from 0 on I by a constant $a > 0$ and that there exists measurable $A \subset (0, 1)$ such that $p(A) > 0$ and for all $\theta \in A$,

$$\theta c(x) \neq c(\theta x), \quad x \in I. \tag{35}$$

Then, Assumption 3 holds by setting $T(\theta, x) = \theta x$ and $\mu(\mathrm{d}\theta) = p(\mathrm{d}\theta)/p((0, 1))$.

(iv) Consider the situation where $k(x, \cdot) = K(x)p \circ m_x^{-1}$ with $m_x(u) = xu$ and p is a finite nonzero measure on $(0, 1)$, with $s(x) = \int_1^x 1/c(y)\,\mathrm{d}y$ for some positive function $c\colon (0, +\infty) \to (0 + \infty)$ continuous on $(0, +\infty)$. Assume in addition that K is locally lower bounded away from 0. If $s(0+) < +\infty$, then Assumption 3(a) holds. Indeed, let $\theta_0 \in [0, 1]$

such that $p(U) > 0$ for all neighborhood U of θ_0. Then, since $1/c$ is summable in a neighborhood of 0, there exists $x_0 \in (0, +\infty)$ such that $\theta_0 c(x_0) \neq c(\theta_0 x_0)$, which then holds true for all $(\theta, x) \in A \times I$, where A and I are neighborhoods θ_0 and x respectively. This shows that (35) holds true for all $\theta \in A$ and $x \in I$, with $p(A) > 0$.

(v) Finally, assume that $k(x, \cdot) = K(x) p \circ m_x^{-1}$ with $m_x(u) = xu$ and p is a finite measure on $(0, 1)$ which is not singular with respect to the Lebesgue measure. Assume in addition that K is lower bounded away from 0 on an open subset of $(0, +\infty)$. Then Assumption 3(b) holds true. Indeed, let $\underline{K}$ be a nonzero continuous function such that $0 \leq \underline{K} \leq K$. Then $k(x, \cdot) \geq \underline{K}(x) p_\lambda \circ m_x^{-1}$, where p_λ is the absolutely continuous part of p, and it only remains to prove that $p_\lambda \circ m_x^{-1}$ defines a lower semi-continuous kernel. Let g be the density of p_λ with respect to the Lebesgue measure and let g_n be a sequence of continuous functions which converge to g in $L^1(\mathrm{d}x)$. Then, for all measurable $A \subset (0, +\infty)$ and all $x \in (0, +\infty)$,

$$\begin{aligned} p_\lambda \circ m_x^{-1}(A) &= \int_{(0,1)} \mathbf{1}_{ux \in A}\, g(u)\, \mathrm{d}u \\ &\geq \int_{(0,1)} \mathbf{1}_{ux \in A}\, g_n(u)\, \mathrm{d}u - \|g - g_n\|_{L_1} \\ &= \int_{(0,x)} \mathbf{1}_{y \in A}\, g_n(y/x) \frac{\mathrm{d}y}{x} - \|g - g_n\|_{L_1}. \end{aligned}$$

The first term in the last line is continuous in x, while the second term goes to 0 when $n \to +\infty$, so that, for all $x \in (0, +\infty)$,

$$\liminf_{y \to x} p_\lambda \circ m_y^{-1}(A) \geq p_\lambda \circ m_x^{-1}(A).$$

This shows that Assumption 3(b) holds true. Note that in this particular case, $p_\lambda \circ m_y^{-1}(A)$ is actually continuous with respect to x.

3.1.1. *Entrance boundary.* In this section, we provide a simple criterion for processes with an entrance boundary at 0 (i.e., $s(0+) > -\infty$) and with a locally bounded fragmentation rate, inspired by the main result of [17]. As in this reference, and contrarily to the following sections, the result depends on λ_0.

PROPOSITION 8. *Assume that* $s(0+) > -\infty$, *that* $\sup_{x \in (0,M)} k(x, (0, x)) < +\infty$ *for all* $M > 0$, *that* K *is nonnegative and that*

$$\limsup_{x \to +\infty} k\big(x, (0, x)\big) - K(x) < +\infty. \tag{36}$$

Then Assumption 1 holds true. If in addition Assumptions 2 and 3 hold true, and if

$$\limsup_{x \to +\infty} k\big(x, (0, x)\big) - K(x) < -\lambda_0, \tag{37}$$

then Assumption 4 holds true.

Before proceeding with the proof of Proposition 8, we remark on the strong similarities with Theorem 1.1 of [17]. There, the author reaches the conclusion of Theorem 3, making some additional regularity and further assumptions on s, k and K; these ensure in particular that T is a strongly continuous Feller semigroup on the space of bounded functions vanishing at infinity, which is not in general true for us.

PROOF OF PROPOSITION 8. Let $a < -\limsup_{x \to 0} k(x, (0, x)) - \lambda_0$ such that $a \leq 0$. Let $x_0 \geq 1$ be such that $\exp(-as(0+)) + s(x_0) = 1$ and set, for all $x \in (0, +\infty)$,

$$h(x) = \exp\big(a\big(s(x) - s(0+)\big)\big)\mathbf{1}_{x<1} + 1 \wedge \big(\exp\big(-as(0+)\big) + s(x)\big)\mathbf{1}_{x \geq 1}.$$

Then, for all $x \in (0,1)$,

$$\frac{\mathcal{A}h(x)}{h(x)} = a + \int_{(0,x)} \exp\big(a\big(s(y)-s(x)\big)\big)k(x,\mathrm{d}y) - K(x)$$
$$\le a + \exp\big(a\big(s(0+)-s(x)\big)\big)k\big(x,(0,x)\big),$$

which is uniformly bounded from above on $x \in (0,1)$ by assumption. For $x \in [1, x_0)$, we have

$$\frac{\mathcal{A}h(x)}{h(x)} = \frac{1}{h(x)} + \int_{(0,1)} \frac{\exp(a(s(y)-s(0+)))}{\exp(-as(0+))+s(x)}k(x,\mathrm{d}y)$$
$$+ \int_{[1,x)} \frac{\exp(-as(0+))+s(y)}{\exp(-as(0+))+s(x)}k(x,\mathrm{d}y) - K(x)$$
$$\le \exp\big(as(0+)\big) + \exp\big(as(0+)\big)k\big(x,(0,1)\big) + k\big(x,[1,x)\big),$$

which is uniformly bounded from above on $x \in [1, x_0)$ by assumption. For $x \ge x_0$, we have

$$\frac{\mathcal{A}h(x)}{h(x)} = \int_{(0,1)} \exp\big(a\big(s(y)-s(0+)\big)\big)k(x,\mathrm{d}y)$$
$$+ \int_{[1,x_0)} \big(\exp\big(-as(0+)\big)+s(y)\big)k(x,\mathrm{d}y) + k\big(x,[x_0,x)\big) - K(x)$$
$$\le k\big(x,(0,x)\big) - K(x),$$

which is locally bounded from above and is bounded when $x \to +\infty$ by (36). This entails that $\frac{\mathcal{A}h(x)}{h(x)}$ is bounded from above on $(0,+\infty)$. It is clearly locally bounded, and, in addition, for all $M > 0$,

$$\sup_{x\in(0,M)} k_h\big(x,(0,x)\big) \le \sup_{x\in(0,M)} \int_{(0,x)} \exp\big(as(0+)\big)k(x,\mathrm{d}y)$$

which is finite by assumption. We conclude that Assumption 1 holds true.

We now work under the additional assumptions and set $\psi = h$. We have $s(x) \to s(0+)$ when $x \to 0$, and hence

$$\limsup_{x\to 0} \frac{\mathcal{A}\psi(x)}{\psi(x)} \le \limsup_{x\to 0}\big[a + \exp\big(a\big(s(0+)-s(x)\big)\big)k\big(x,(0,x)\big)\big]$$
$$= a + \limsup_{x\to 0} k\big(x,(0,x)\big) < -\lambda_0.$$

Using (37), we also obtain

$$\limsup_{x\to +\infty} \frac{\mathcal{A}\psi(x)}{\psi(x)} \le \limsup_{x\to +\infty}\big[k\big(x,(0,x)\big) - K(x)\big] < -\lambda_0.$$

This concludes the proof of Proposition 8. □

3.1.2. *Pseudo-entrance boundary and mass conservation.* In this section, we consider the situation where $\int_{(0,1)} K(x)s(\mathrm{d}x) < +\infty$. Informally, this means that a PDMP with drift determined by s and jump rate K has a positive, lower bounded probability to reach 1 before its first jump when starting from any $x \in (0,1)$.

For simplicity, we consider the situation $k(x,\cdot) = K(x)p \circ m_x^{-1}$ where $m_x(u) = xu$, p is a measure on $(0,1)$ such that $\int_{(0,1)} up(\mathrm{d}u) = 1$. We also assume that K is right-continuous.

PROPOSITION 9. *Assume that* $\sup_{x\in(0,M)} K(x) < +\infty$ *for each* $M > 0$. *Assume in addition that Assumptions* 2 *and* 3 *hold true*, *that* p *is a finite measure*, *that*

$$\int_{(0,1)} K(x)s(\mathrm{d}x) < +\infty$$

and that there exists $\alpha > 1$ *such that*, *for all* $u \in (0, 1)$,

$$\liminf_{x\to+\infty} \int_{ux}^{x} K(y)s(\mathrm{d}y) > \frac{-\alpha \ln u}{1 - \int_{(0,1)} v^{\alpha} p(\mathrm{d}v)}. \tag{38}$$

Then, *Assumptions* 1 *and* 4 *hold*, $\lambda_0 < 0$ *and the conclusions of Theorem* 3 *hold true*.

Suppose moreover that, for all $x \in (0, +\infty)$, $s(x) = \int_1^x \frac{1}{c(y)}\,\mathrm{d}y$ where $c : (0, +\infty) \to (0, +\infty)$ is a right-continuous and locally bounded function. In the case of uniform mass repartition, where $p(\mathrm{d}u) = 2\,\mathrm{d}u$, the right-hand term in (38) reaches its minimal value $(-\ln u)(3 + 2\sqrt{2})$ at $\alpha = 1 + \sqrt{2}$. In particular, (38) holds true if

$$\liminf_{y\to+\infty} \frac{yK(y)}{c(y)} > 3 + 2\sqrt{2}.$$

Before turning to the proof of this proposition, it is interesting to compare it with the findings of [15]. In this recent paper, the authors use advanced methods from functional analysis to derive the existence of an eigenfunction h. This gives them access to a (conservative) Markov process using an h-transform (see also [34, 36], where similar approaches were used to study nonconservative semigroups). This allows them to study the growth fragmentation equation under mild conditions. The main drawback of this approach is that it requires the preliminary proof of the existence and fine properties of a positive right eigenfunction h, which typically requires additional assumptions on regularity and asymptotic behaviour of the coefficients. On the contrary, our approach, based on the study of nonconservative Markov processes, only requires the existence of a Lyapunov function h, and the existence of an eigenfunction is then a consequence of our theorem, instead of a preliminary step in the proof. This lets us consider more general situations.

More precisely, in the case where p is the uniform measure over (0, 1) (where Assumption 3 is clearly satisfied by Remark 4(i)), Theorem 1.3 in [15] states that the conclusions of our Theorem 3 hold true, assuming in addition (compared to Proposition 9) that c is locally Lipschitz, that $\limsup_{x\to+\infty} \frac{c(x)}{x} < +\infty$, that $c(x) = o(x^{-p})$ when $x \to 0$ for some $p \geq 0$, that K is continuous on $[0, +\infty)$, that $xK(x)/c(x) \to 0$ when $x \to 0$ and that $xK(x)/c(x) \to +\infty$ when $x \to +\infty$. Similarly, the mitosis kernel case considered in [15] is a special case of Proposition 9 (using this time Remark 4(ii) instead of Remark 4(i)).

Our result also covers and extends the setting considered in [8], where the authors assume either (1) that $c(x) = x$ and p is absolutely continuous with respect to the Lebesgue measure (this is a particular case of Remark 4(iv), our result shows in particular that it is sufficient for p to have a nonzero absolute continuous part with respect to the Lebesgue measure) or (2) that $c \equiv 1$ and p has compact support in (0, 1) (this is a particular case of Remark 4(iii), and we show that the condition on p can be dispensed of entirely).

We can also compare Proposition 9 with Theorem 4.3 in the recent paper [6], where the authors consider the special case where $c \equiv 1$ (which means that $s(x) = x - 1$) and K is a continuously differentiable increasing function, and under the additional assumption that p is lower bounded by a uniform measure over a subinterval of [0, 1] or by a Dirac measure (these situations clearly satisfy Assumption 3(a) via Remarks 4(iv) and 4(ii) respectively). In this situation, both assumptions of Proposition 9 are clearly satisfied, with $\liminf_{x\to+\infty} \int_{ux}^{x} K(y)s(\mathrm{d}y) = +\infty$ for all $u \in (0, 1)$ and Theorem 4.3 in [6] is thus a special case of Proposition 9.

REMARK 5. In the proof, we make use of the functions ψ and h defined by

$$\psi(x) = h(x) = \exp\Big(\int_{(x,1)} a_0 K(y)s(\mathrm{d}y)\Big)\mathbf{1}_{x<1} + \exp\Big(\int_{(1,x)} a_\infty K(y)s(\mathrm{d}y)\Big)\mathbf{1}_{x\geq 1},$$

where $a_0, a_\infty \in \mathbb{R}$, so that, for all $x < 1$,

$$\frac{\mathcal{A}\psi(x)}{\psi(x)} = K(x)\Big(\int_{(0,1)} \exp\Big(a_0 \int_{(ux,x)} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) - 1 - a_0\Big)$$

and similarly for $x \geq 1$ (see (42) for the exact expression). Our assumptions are then used to derive asymptotics on $\frac{\mathcal{A}\psi(x)}{\psi(x)}$ when $x \to 0$ and $x \to +\infty$. In this situation, the main point of the mass conservation assumption is to ensure that $x \in (0, +\infty) \mapsto x$ is a natural candidate for ξ in Proposition 7, and is thus used to derive a lower bound for λ_0. The strategy developed in the proof, and in particular the use of ψ with this form, is relevant even outside of the structure for k assumed in Proposition 9.

For example, let us assume that Assumptions 2 and 3 hold, and let us make the following assumptions: K is locally bounded, $\sup_{x>0} \frac{k(x,(0,x)}{K(x)} < +\infty$, $\int \frac{y}{x} k(x, \mathrm{d}y) = K(x)$ (conservation of mass), there exists $\alpha > 1$ such that $m_\alpha := \limsup_{x\to\infty} \int (y/x)^\alpha k(x, \mathrm{d}y)/K(x) < 1$, c is right continuous, $\int_{(0,1)} \frac{K(x)}{c(x)} \mathrm{d}x < +\infty$ (pseudo-entrance boundary), and there exists $a_\infty \in (0, m_\alpha)$ such that, for all $x \geq y \geq 1$ with y large enough,

$$\exp\Big(-a_\infty \int_y^x K(z)s(\mathrm{d}z)\Big) \leq \Big(\frac{y}{x}\Big)^\alpha. \tag{39}$$

We emphasize that this implies that $\frac{1}{\ln x}\int_1^x K(z)s(\mathrm{d}z)$ goes to $+\infty$ when $x \to +\infty$. Then, using the same ψ as in the proof, we obtain for all $x \geq 1$,

$$\begin{aligned}\frac{\mathcal{A}\psi(x)}{\psi(x)} = K(x)\Big(&\int_{(0,1)} \exp\Big(a_0 \int_y^1 K(z)s(\mathrm{d}z) - a_\infty \int_1^x K(z)s(\mathrm{d}z)\Big)\frac{1}{K(x)}k(x, \mathrm{d}y)\\ &+ \int_{[1,x)} \exp\Big(-a_\infty \int_y^x K(z)s(\mathrm{d}z)\Big)\frac{1}{K(x)}k(x, \mathrm{d}y) - 1 + a_\infty\Big).\end{aligned}$$

The first term in the parenthesis goes to 0 when $x \to +\infty$ since $\int_0^1 K(z)s(\mathrm{d}z) < +\infty$, $\int_1^x K(z)s(\mathrm{d}z) \to +\infty$ when $x \to +\infty$ and the total mass of $\frac{1}{K(x)}k(x,\cdot)$ is uniformly bounded. For the other terms, we obtain using (39) that, for all $x \geq y \geq 1$ with y large enough,

$$\limsup_{x\to+\infty} \int_{[1,x)} \exp\Big(-a_\infty \int_y^x K(z)s(\mathrm{d}z)\Big)\frac{1}{K(x)}k(x, \mathrm{d}y) - 1 + a_\infty \leq m_\alpha - 1 + a_\infty < 0.$$

This entails that

$$\lim_{x\to+\infty} \frac{\mathcal{A}\psi(x)}{\psi(x)} < 0.$$

Choosing $a_0 > \sup_{x>0} \frac{k(x,(0,x)}{K(x)}$ and proceeding as in the proof, we deduce that $\lambda_0 < 0$ and the conclusions of Theorem 3 hold true.

In particular, this recovers and improve the convergence of the model in [23], where the authors prove the (a priori nongeometric) convergence for a more specific model under stronger assumptions inherited from [25] (in order to obtain the leading eigenelements) and additional regularity assumptions (see (2.5)–(2.7) in [23] and (2.1)–(2.9) in [25]). We emphasize that, although we proved that there is actually a spectral gap under the assumptions of [23] and [25], the authors of the former use the latter to obtain the existence, uniqueness and additional properties on the eigenelements of the adjoint operator, so their methods may apply to situations where there is no spectral gap and thus where our result does not hold true. Similarly, we also cover the setting of [14], where the authors prove a convergence with geometric rate for two specific models under additional regularity assumptions (see Hypotheses 1.1 to 1.7 therein), some of which are also derived from their use of [25].

To conclude this remark, we consider the sharp and quite explicit result of [30], where the authors consider the special case where $s(x) = \ln x$, $K(x) = xR(x)$ and $k(x, \mathrm{d}u) = xR(x)\frac{2u}{x^2}\,\mathrm{d}u$. There is no mass conservation in this case, but instead conservation of the number of fragments ($\mathcal{A}1 \equiv 0$). The authors show that the condition

$$\int_0^\infty y \exp\Big(-\int_1^x R(y)\,\mathrm{d}y\Big)\,\mathrm{d}x < +\infty$$

holds true if and only if $\|\delta_x T_t - m\|_{\mathrm{TV}}$ goes to 0 when $t \to +\infty$ (without a geometric rate). In order to ensure that Theorem 3 applies to this case (and thus to ensure geometric convergence), and using the same Lyapunov function as above (observing that $\lambda_0 = 0$), we require that there exists $a_\infty \in (0, 1)$ such that

$$\int_{(0,1)} R(y)\,\mathrm{d}y < +\infty \quad \text{and} \quad \limsup_{x\to+\infty} \int_{(1,x)} \exp\Big(-a_\infty \int_y^x R(z)\,\mathrm{d}z\Big)\frac{2y\,\mathrm{d}y}{x^2} - 1 + a_\infty < 0.$$

The last property holds true for instance if $\int_y^x R(z)\,\mathrm{d}z \geq (2+\varepsilon)\ln\frac{y}{x}$ for some $\varepsilon > 0$ and all $x > y$ with y large enough. In this case, it is clear that the condition of [30] holds true. It is also the case that their condition (and of course ours) does not hold true if $\int_y^x R(z)\,\mathrm{d}z \leq 2\ln\frac{y}{x}$ for all $x > y$ with y large enough.

PROOF OF PROPOSITION 9. For all $a \in \mathbb{R}$, we set $p_a := \int_{(0,1)} u^a\, p(\mathrm{d}u)$.

(1) *Identification of $h = \psi$.* For all $u \in (0, 1)$, we define

$$\varepsilon_u := \liminf_{x\to+\infty} \frac{\int_{ux}^x K(y)s(\mathrm{d}y)}{-\ln u} - \frac{\alpha}{1-p_\alpha}$$

and set

$$\ell := \frac{\alpha}{1-p_\alpha} + \frac{\varepsilon_{1/2}}{2}.$$

Note that $\varepsilon_u > 0$ by assumption and hence $\alpha/\ell < 1 - p_\alpha$ and

$$\lim_{a\to 1-p_\alpha} \int_{(0,1)} u^{a(\varepsilon_u+\alpha/(1-p_\alpha))}\, p(\mathrm{d}u) = \int_{(0,1)} u^{(1-p_\alpha)\varepsilon_u+\alpha}\, p(\mathrm{d}u) < p_\alpha.$$

In particular, there exists $a_\infty \in (\frac{\alpha}{\ell}, 1-p_\alpha)$ such that

$$\int_{(0,1)} u^{a_\infty(\varepsilon_u+\alpha/(1-p_\alpha))}\, p(\mathrm{d}u) < p_\alpha. \tag{40}$$

We also fix $a_0 > p_0 - 1$ and define the function

$$\psi(x) = \begin{cases} \exp\Big(-a_0 \int_1^x K(y)\mathrm{s}(\mathrm{d}y)\Big) & \text{if } x \leq 1, \\ \exp\Big(a_\infty \int_1^x K(y)s(\mathrm{d}y)\Big) & \text{if } x \geq 1. \end{cases}$$

We have, for all $x < 1$,

$$\frac{\mathcal{A}\psi(x)}{\psi(x)} = K(x)\Big(\int_{(0,1)} \exp\Big(a_0 \int_{ux}^x K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) - 1 - a_0\Big).$$

Since $\exp(a_0 \int_{ux}^x K(y)s(\mathrm{d}y)) \leq \exp(a_0 \int_0^1 K(y)s(\mathrm{d}y))$, with

$$\int_{(0,1)} \exp\Big(a_0 \int_0^1 K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) = \exp\Big(a_0 \int_0^1 K(y)s(\mathrm{d}y)\Big)p_0 < +\infty$$

and since $\int_{ux}^{x} K(y)s(\mathrm{d}y) \to 0$ as $x \to 0$, we deduce from the dominated convergence theorem that

$$\lim_{x\to 0}\int_{(0,1)} \exp\Big(a_0 \int_{ux}^{x} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) - 1 - a_0 = p_0 - 1 - a_0 < 0.$$

Hence there exists $x_0 > 0$ such that

$$\frac{\mathcal{A}\psi(x)}{\psi(x)} \le 0, \quad \text{for all } x \in (0, x_0). \tag{41}$$

For all $x \ge 1$, we have

$$\begin{aligned}\frac{\mathcal{A}\psi(x)}{\psi(x)} = K(x)\Big(&\int_{(0,1/x)} \exp\Big(a_0 \int_{ux}^{1} K(y)s(\mathrm{d}y) - a_\infty \int_{1}^{x} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u)\\ &+ \int_{[1/x,1)} \exp\Big(-a_\infty \int_{ux}^{x} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) - 1 + a_\infty\Big).\end{aligned} \tag{42}$$

On the one hand, we have (noting that $a_\infty > 0$)

$$\begin{aligned}&\int_{(0,1/x)} \exp\Big(a_0 \int_{ux}^{1} K(y)s(\mathrm{d}y) - a_\infty \int_{1}^{x} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u)\\ &\quad\le \exp\Big(a_0 \int_{0}^{1} K(y)s(\mathrm{d}y)\Big)p\big((0,1/x)\big) \xrightarrow[x\to+\infty]{} 0.\end{aligned}$$

On the other hand, for all $u \in (0, 1)$,

$$\limsup_{x\to+\infty} \mathbf{1}_{u\in[1/x,1)} \exp\Big(-a_\infty \int_{ux}^{x} K(y)s(\mathrm{d}y)\Big) = u^{a_\infty(\varepsilon_u+\alpha/(1-p_\alpha))}$$

and hence, by Fatou's lemma and using (40),

$$\limsup_{x\to+\infty}\int_{[1/x,1)} \exp\Big(-a_\infty \int_{ux}^{x} K(y)s(\mathrm{d}y)\Big)p(\mathrm{d}u) - 1 + a_\infty \le p_\alpha - 1 + a_\infty < 0.$$

We deduce that there exists $x_\infty \ge 1$ such that, for all $x \ge x_\infty$,

$$\frac{\mathcal{A}\psi(x)}{\psi(x)} \le 0. \tag{43}$$

Taking $h = \psi$, we observe that $x \in (0, +\infty) \mapsto \frac{\mathcal{A}h(x)}{h(x)}$ is locally bounded, and we deduce from (41) and (43) that it is bounded from above. Moreover, the above calculations show that, for all $M > 0$,

$$\sup_{x\in(0,M)} \int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) < +\infty.$$

We conclude that Assumption 1 holds true.

(2) *Identification of ξ and conclusion.* We choose $\xi(x) := x$ for all $x \in (0, +\infty)$. We first prove that $\xi(x) = x = o(\psi(x))$ close to 0 and $+\infty$. Since ψ is bounded away from 0 in a vicinity of 0, this is immediate for x close to 0. Now, according to our assumptions and the definition of ℓ, there exists $x_1 \ge 1$ (which is fixed from now on) such that, for all $x \ge x_1$,

$$\int_{x/2}^{x} K(y)s(\mathrm{d}y) \ge \ell \ln 2.$$

For any $x > x_1$, let $n \geq 0$ such that $2^{-n}x \geq x_1 \geq 2^{-(n+1)}x$ (in particular $n \ln 2 \geq \ln x - \ln x_1 - \ln 2$). Then

$$\int_1^x K(y)s(\mathrm{d}y) \geq \int_1^{2^{-n}x} K(y)s(\mathrm{d}y) + \int_{2^{-n}x}^{2^{-(n-1)}x} K(y)s(\mathrm{d}y) + \cdots + \int_{2^{-1}x}^{x} K(y)s(\mathrm{d}y)$$
$$\geq n\ell \ln 2 \geq \ell \ln x - \ell \ln(2x_1).$$

Since $a_\infty > \alpha/\ell$, we deduce that, for all $x > x_1$,

$$a_\infty \int_1^x K(y)s(\mathrm{d}y) \geq \alpha \ln x - a_\infty \ell \ln(2x_1).$$

This shows that $\liminf_{x\to+\infty} \psi(x)/x^\alpha > 0$ and hence, since $\alpha > 1$ by assumption, that $x = o(\psi(x))$ when $x \to +\infty$.

We also observe that, for all $M > 0$, $\sup_{x\in(0,M)} \int_{(0,x)} \frac{\xi(y)}{\xi(x)} k(x, \mathrm{d}y) = \sup_{x\in(0,M)} K(x) < +\infty$, by assumption. Finally, for all $x \in (0, +\infty)$,

$$\frac{\mathcal{A}\xi(x)}{\xi(x)} = \frac{1}{x}\frac{\partial \xi}{\partial s}(x) = \frac{c(x)}{x}.$$

Since $c(x)/x$ is not zero, we deduce that it is either lower bounded by a positive constant or that it is not constant. Using Proposition 7 together with (41) and (43), we deduce that $\lambda_0 < 0$. This also entails that Assumption 4 holds true, which concludes the proof. □

3.1.3. *Critical case, s comparable to* $\ln x$. It is well known that, when K is constant and $s(x) = \ln x$, the results of Theorem 3 do not hold true in general (see, for instance, [25], end of §2). In this section, we consider first the situation where $s(x) = \ln x$ and K is not constant, and then the situation where $s(x)/\ln x$ has positive limit inferior when $x \to 0$ and $x \to +\infty$ and finite limit superior when $x \to +\infty$.

As in the previous section, we consider for simplicity the situation where $k(x, \cdot) = K(x) p \circ m_x^{-1}$, with p a positive measure on $(0, 1)$ such that $\int_{(0,1)} up(\mathrm{d}u) = 1$; we do not assume that p is a finite measure. We assume that K is right-continuous and nonnegative.

PROPOSITION 10. *Assume that Assumptions 2 and 3 hold true. Assume in addition that $s(x) = \ln x$ for all $x \in (0, +\infty)$ and that there exist $\alpha < 1 < \beta$ such that $\int_{(0,1)} u^\alpha p(\mathrm{d}u) < \infty$ and*

$$(44)\qquad \limsup_{x\to 0} K(x) < \frac{1-\alpha}{\int_{(0,1)} u^\alpha p(\mathrm{d}u) - 1} \quad \textit{and} \quad \liminf_{x\to\infty} K(x) > \frac{\beta - 1}{1 - \int_{(0,1)} u^\beta p(\mathrm{d}u)}.$$

Then Assumptions 1 and 4 hold true.

We note that in the case of uniform mass repartition, that is, $p(\mathrm{d}u) = 2\,\mathrm{d}u$, condition (44) reduces to

$$\limsup_{x\to 0} K(x) < 2 < \liminf_{x\to\infty} K(x).$$

This may be compared with the conditions in Section 6 of [12], whose effectiveness in this setting relies on [9], Theorem 1.2. Similar conditions can be found in [9], Section 3.6. We leave as an open problem to check whether this condition is sharp; one natural approach to this question would be to follow the strategy developed in [16].

Proposition 10 is actually a particular case of the following result, which applies when the drift $c(x)$ is only approximately linear in x. We assume here that, for all $x \in (0, +\infty)$, $s(x) = \int_1^x \frac{1}{c(y)}\,\mathrm{d}y$, where $c : (0, +\infty) \to (0, +\infty)$ is a right-continuous and locally bounded function.

PROPOSITION 11. *Assume that Assumptiosn 2 and 3 hold true. Assume in addition that there exist $\alpha, \beta \geq 0$ such that*

$$\alpha < \inf_{x>0} \frac{c(x)}{x} \quad \textit{and} \quad \int_{(0,1)} u^{\alpha \inf_{x<1} \frac{x}{c(x)}} p(\mathrm{d}u) < +\infty \tag{45}$$

and

$$\beta > \limsup_{x\to+\infty} \frac{c(x)}{x} \quad \textit{and} \quad \int_{(0,1)} u^{\beta \inf_{x\geq 1} \frac{x}{c(x)}} p(\mathrm{d}u) < +\infty. \tag{46}$$

If

$$\limsup_{x\to 0} K(x) < \frac{\inf_x \frac{c(x)}{x} - \alpha}{\int_{(0,1)} u^{\alpha \liminf_{x\to 0} \frac{x}{c(x)}} p(\mathrm{d}u) - 1} \tag{47}$$

and

$$\liminf_{x\to+\infty} K(x) > \frac{\beta - \inf_x \frac{c(x)}{x}}{1 - \int_{(0,1)} u^{\beta \liminf_{x\to+\infty} \frac{x}{c(x)}} p(\mathrm{d}u)}, \tag{48}$$

then Assumptions 1 and 4 hold true.

PROOF. For all $a \in \mathbb{R}$, we set $p_a := \int_{(0,1)} u^a p(\mathrm{d}u)$.

Note that $\limsup_{x\to 0} K(x) < +\infty$ and hence, since K is locally bounded, K is bounded on $(0, M)$, for all $M > 0$. We define, for all $x \in (0, +\infty)$,

$$\psi(x) = h(x) = \exp(\alpha s(x))\mathbf{1}_{x<1} + \exp(\beta s(x))\mathbf{1}_{x\geq 1} \quad \text{and} \quad \xi(x) = x.$$

In particular, for all $x \in (0, +\infty)$,

$$\frac{\mathcal{A}\xi(x)}{\xi(x)} = \frac{c(x)}{x}.$$

We first prove that $\psi/\xi \to +\infty$ when $x \to 0$ and $+\infty$. According to (45), there exists $x_0 \in (0, 1)$ and $\varepsilon > 0$ such that, for all $y \in (0, x_0)$, $\alpha/c(y) \leq (1-\varepsilon)/y$, so that, for all $x \in (0, x_0)$,

$$\begin{aligned}\alpha s(x) - \ln x &= \int_{(x,1)} \left(\frac{-\alpha}{c(y)} + \frac{1}{y}\right) \mathrm{d}y \\ &\geq \varepsilon \int_{(x,x_0)} \frac{1}{y} \,\mathrm{d}y + \int_{(x_0,1)} \left(\frac{-\alpha}{c(y)} + \frac{1}{y}\right) \mathrm{d}y \xrightarrow[x\to 0]{} +\infty.\end{aligned}$$

This shows that $\psi/\xi \to +\infty$ when $x \to 0$. Similarly, (46) implies that there exists $x_\infty \geq 1$ and $\varepsilon > 0$ such that, for all $y > x_\infty$, $\beta/c(y) \geq (1+\varepsilon)/y$, so that, for all $x > x_\infty$,

$$\begin{aligned}\beta s(x) - \ln x &= \int_{(1,x)} \left(\frac{\beta}{c(y)} - \frac{1}{y}\right) \mathrm{d}y \\ &\geq \int_{(1,x_\infty)} \left(\frac{\beta}{c(y)} - \frac{1}{y}\right) + \varepsilon \int_{(x_\infty,x)} \frac{1}{y} \,\mathrm{d}y \xrightarrow[x\to+\infty]{} +\infty.\end{aligned} \tag{49}$$

This shows that $\psi/\xi \to +\infty$ when $x \to +\infty$.

We observe that, for all $x \in (0,1)$,

$$\frac{\mathcal{A}h(x)}{h(x)} = \alpha + \int_{(0,x)} \frac{h(y)}{h(x)} k(x,\mathrm{d}y) - K(x)$$
$$= \alpha + K(x)\Big(\int_{(0,1)} \exp\big(\alpha\big(s(ux)-s(x)\big)\big)p(\mathrm{d}u) - 1\Big).$$

We have, for all $u \in (0,1)$ and $x \in (0,1)$,

$$s(ux) - s(x) \le -\Big(\inf_{y\in(0,1)} \frac{y}{c(y)}\Big)\int_{ux}^{x} \frac{1}{y}\,\mathrm{d}y = \Big(\inf_{y\in(0,1)} \frac{y}{c(y)}\Big)\ln u,$$

so that $\exp(\alpha(s(ux)-s(x))) \le u^{\alpha \inf_{y\in(0,1)} \frac{y}{c(y)}}$, which does not depend on x and is integrable with respect to $p(\mathrm{d}u)$ by Assumption (45). We conclude that

$$\sup_{x\in(0,1)} \int_{(0,x)} \frac{h(y)}{h(x)} k(x,\mathrm{d}y) < +\infty. \tag{50}$$

In addition, for all $u \in (0,1)$,

$$\limsup_{x\to 0}\big(s(ux)-s(x)\big) \le \limsup_{x\to 0}\Big(\inf_{y\in(0,x)} \frac{y}{c(y)}\Big)\ln u = \liminf_{x\to 0} \frac{x}{c(x)} \ln u.$$

Using Fatou's lemma, we deduce that

$$\limsup_{x\to 0} \int_{(0,x)} \frac{h(y)}{h(x)} k(x,\mathrm{d}y) - K(x) \le \limsup_{x\to 0} K(x)\Big(\int_{(0,1)} u^{\alpha \liminf_{y\to 0} \frac{y}{c(y)}} p(\mathrm{d}u) - 1\Big).$$

We conclude, using in addition (47) and the fact that $\alpha \liminf_{x\to 0} \frac{x}{c(x)} < 1$, that

$$\begin{aligned}\limsup_{x\to 0} \frac{\mathcal{A}h(x)}{h(x)} &\le \alpha + \limsup_{x\to 0} K(x)\Big(\int_{(0,1)} u^{\alpha \liminf_{y\to 0} \frac{y}{c(y)}} p(\mathrm{d}u) - 1\Big)\\ &< \inf_x \frac{\mathcal{A}\xi(x)}{\xi(x)}.\end{aligned} \tag{51}$$

For all $x \ge 1$, we have

$$\begin{aligned}\frac{\mathcal{A}h(x)}{h(x)} &= \beta + \int_{(0,x)} \frac{h(y)}{h(x)} k(x,\mathrm{d}y) - K(x)\\ &= \beta + K(x)\Big(\int_{(0,1/x)} \exp\big(\alpha s(ux) - \beta s(x)\big)p(\mathrm{d}u)\\ &\quad + \int_{(1/x,1)} \exp\big(\beta\big(s(ux)-s(x)\big)\big)p(\mathrm{d}u) - 1\Big).\end{aligned} \tag{52}$$

According to (49), there exists $x'_\infty \ge 1$ such that, for all $x \in (x'_\infty, +\infty)$, $\beta s(x) \ge \ln x$, so that, for all $x \in (x'_\infty, +\infty)$ and $u \in (0, 1/x)$,

$$\alpha s(ux) - \beta s(x) \le \alpha\Big(\inf_{y\in(0,1)} \frac{y}{c(y)}\Big)(\ln u + \ln x) - \ln x \le \alpha\Big(\inf_{y\in(0,1)} \frac{y}{c(y)}\Big)\ln u,$$

since $\alpha \inf_{y\in(0,1)} \frac{y}{c(y)} < 1$ by (45). Since, by (45), $u^{\alpha \inf_{y\in(0,1)} \frac{y}{c(y)}}$ is integrable with respect to $p(\mathrm{d}u)$, we deduce by dominated convergence that

$$\int_{(0,1/x)} \exp\big(\alpha s(ux) - \beta s(x)\big)p(\mathrm{d}u) \xrightarrow[x\to+\infty]{} 0.$$

For all $x > 1$ and $u \in (1/x, 1)$, we have

$$s(ux) - s(x) \leq \left(\inf_{y \geq 1} \frac{y}{c(y)}\right) \ln u,$$

so that $\exp(\beta(s(ux) - s(x))) \leq u^{\beta \inf_{y\geq 1} \frac{y}{c(y)}}$, which does not depend on x and is integrable with respect to $p(\mathrm{d}u)$ by (46). We conclude that

$$\sup_{x \in [1,M)} \int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) < +\infty, \quad \forall M > 1. \tag{53}$$

Similarly as above, we have in addition, for all $u \in (0, 1)$,

$$\limsup_{x \to +\infty} \bigl(s(ux) - s(x)\bigr) = \liminf_{x \to +\infty} \frac{x}{c(x)} \ln u.$$

Using again Fatou's lemma, we obtain

$$\limsup_{x \to +\infty} \int_{(1/x,1)} \exp\bigl(\beta\bigl(s(ux) - s(x)\bigr)\bigr) p(\mathrm{d}u) \leq \int_{(0,1)} u^{\beta \liminf_{x\to+\infty} \frac{x}{c(x)}} p(\mathrm{d}u).$$

Using (52), we deduce that

$$\begin{aligned} \limsup_{x \to +\infty} \frac{\mathcal{A}h(x)}{h(x)} &\leq \beta + \limsup_{x \to +\infty} K(x) \left(\int_{(0,1)} u^{\beta \liminf_{x\to+\infty} \frac{x}{c(x)}} p(\mathrm{d}u) - 1 \right) \\ &< \inf_x \frac{\mathcal{A}\xi(x)}{\xi(x)}, \end{aligned} \tag{54}$$

where we used (48) and the fact that $\beta \liminf_{x \to +\infty} \frac{x}{c(x)} > 1$ for the last inequality.

By (50) and (53), we deduce that the first part of Assumption 1 holds true. Since $\mathcal{A}h/h$ is locally bounded, we deduce from (51) and (54) that it is bounded from above. We conclude that Assumption 1 holds true.

Finally, using Proposition 7, we deduce from (51) and (54) that Assumption 4 holds true. □

To once again give an explicit example, we offer:

COROLLARY 3. *Assume that Assumptions 2 and 3 hold true. Let $p(\mathrm{d}u) = 2\,\mathrm{d}u$ and*

$$c(x) = \begin{cases} c_0 x & x \leq x_c, \\ c_\infty x & x > x_c, \end{cases}$$

for some $x_c > 0$ and $0 < c_\infty < c_0 < \infty$. Assume that

$$\limsup_{x \to 0} K(x) < 3c_0 - c_\infty - 2\sqrt{2c_0(c_0 - c_\infty)} \tag{55}$$

and $\liminf_{x\to\infty} K(x) > 2c_\infty$. Then, the conditions of Proposition 11 are satisfied.

PROOF. With this particular choice of c, the conditions of Proposition 11 are that there exist $\alpha \in [0, c_\infty)$ and $\beta > c_\infty$, such that

$$\limsup_{x \to 0} K(x) < \frac{c_\infty - \alpha}{\int_{(0,1)} u^{\alpha/c_0} 2\,\mathrm{d}u - 1} = \frac{(\alpha + c_0)(c_\infty - \alpha)}{c_0 - \alpha} \tag{56}$$

and

$$\liminf_{x \to +\infty} K(x) > \frac{\beta - c_\infty}{1 - \int_{(0,1)} u^{\beta/c_\infty} 2\,\mathrm{d}u} = \beta + c_\infty. \tag{57}$$

The maximum of $\alpha \mapsto \frac{(\alpha+c_0)(c_\infty-\alpha)}{c_0-\alpha}$ on the domain $\alpha \in [0, c_\infty)$ is given by the right-hand side of (55), which shows that the condition given suffices for (56) to hold.

Since $\liminf_{x\to\infty} K(x) > 2c_\infty$, one can find $\beta > c_\infty$ such that (57) holds, which concludes the proof. □

3.1.4. *Critical case, K comparable to a constant.* We consider now the situation where K is the constant function 1, and then the situation when K is bounded away from 0 and bounded from above by 1.

As in the previous section, we consider for simplicity the situation where $k(x,\cdot) = K(x) p \circ m_x^{-1}$, with p a positive measure on $(0, 1)$ such that $\int_{(0,1)} up(\mathrm{d}u) = 1$. We also assume that, for all $x \in (0, +\infty)$, $s(x) = \int_1^x \frac{1}{c(y)} \mathrm{d}y$ where $c : (0, +\infty) \to (0, +\infty)$ is a right-continuous and locally bounded function.

PROPOSITION 12. *Assume that Assumptions 2 and 3 hold true, and that $K \equiv 1$. Assume in addition that there exists $\delta > 0$ such that $\int_{(0,1)} u^{-\delta} p(\mathrm{d}u) < +\infty$. If $s(0+) = -\infty$ and*

$$\limsup_{x\to+\infty} \frac{c(x)}{x} < -\int_{(0,1)} \ln u p(\mathrm{d}u) < \liminf_{x\to 0} \frac{c(x)}{x}, \tag{58}$$

then Assumptions 1 and 4 hold true.

In [16], the author considers the case where $p(\mathrm{d}u)$ is absolutely continuous with respect to the Lebesgue measure and where there exist positive constants a_- and a_+ such that

$$c(x) = \begin{cases} a_- x & \text{if } x < 1, \\ a_+ x & \text{if } x \geq 1. \end{cases}$$

In this case, our assumption reads

$$a_+ < -\int_{(0,1)} \ln u p(\mathrm{d}u) < a_-,$$

which is sharp, according to [16], in the sense that, if one of the inequalities fails, then $e^{\lambda_0 t} T_t f$ does not converge (for some bounded, compactly supported function f). Additional properties, and in particular fine estimates on the limiting profile of $e^{\lambda_0 t} T_t$, can be found in the above reference.

The previous result is a particular case of the following proposition, where we do not assume any more that K is constant. Here K is a locally bounded right-continuous function.

PROPOSITION 13. *Assume that Assumptions 2 and 3 holds true. Assume in addition that there exists $\delta > 0$ such that $\int_{(0,1)} u^{-\delta} p(\mathrm{d}u) < +\infty$, and $0 < \inf K \leq 1$. If $s(0+) = -\infty$,*

$$\inf K = \limsup_{x\to 0} K(x) = \limsup_{x\to+\infty} K(x) \tag{59}$$

and

$$\limsup_{x\to+\infty} \frac{c(x)}{x} < -\int_{(0,1)} \ln u p(\mathrm{d}u) < \liminf_{x\to 0} \frac{c(x)}{x}, \tag{60}$$

then Assumptions 1 and 4 hold true.

We start with a simple technical lemma, whose proof is standard and thus omitted.

LEMMA 8. *If there exists* $\delta > 0$ *such that* $\int_{(0,1)} u^{-\delta} p(\mathrm{d}u) < +\infty$ *and constants* a_0, a_1 *such that*

$$a_0 < -\int_{(0,1)} \ln u p(\mathrm{d}u) < a_1,$$

then there exists $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon + \int_{(0,1)} (u^{\varepsilon/a_0} - 1) p(\mathrm{d}u) < 0 \quad \textit{and} \quad \int_{(0,1)} u^{-\varepsilon/a_1} p(\mathrm{d}u) < \varepsilon + p\big((0,1)\big).$$

PROOF OF PROPOSITION 13. Let $\xi(x) = 1$ for all $x \in (0, +\infty)$ and set

$$\psi(x) = h(x) = \exp\big(-\alpha s(x)\big)\mathbf{1}_{x<1} + \exp\big(\beta s(x)\big)\mathbf{1}_{x\geq 1},$$

where $\alpha > 0$ and $\beta > 0$ are (small enough) constant which will be chosen later. We already observe that, by assumption, $\psi(x)/\xi(x) \to +\infty$ when $x \to 0$ and when $x \to +\infty$. In addition,

$$\frac{\mathcal{A}\xi(x)}{\xi(x)} = K(x)\bigg(\int_{(0,1)} p(\mathrm{d}u) - 1\bigg) \geq \inf K \int_{(0,1)} (1-u) p(\mathrm{d}u). \tag{61}$$

For all $x < 1$, we have

$$\int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) = K(x) \int_{(0,1)} \exp\big(-\alpha\big(s(ux) - s(x)\big)\big) p(\,\mathrm{d}u),$$

where

$$\exp\big(-\alpha\big(s(ux) - s(x)\big)\big) \leq u^{-\alpha \sup_{y\in(0,x)} \frac{y}{c(y)}}.$$

On the one hand, choosing $\alpha < \delta / \sup_{y\in(0,1)} \frac{y}{c(y)}$, we deduce that

$$\int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) \leq \sup_{y\in(0,1)} K(y) \int_{(0,1)} u^{-\delta} p(\mathrm{d}u), \tag{62}$$

and, on the other hand, letting $x \to 0$ and using Fatou's lemma, we deduce that

$$\begin{aligned}
\limsup_{x\to 0} \frac{\mathcal{A}h(x)}{h(x)} &= -\alpha + \limsup_{x\to 0} \int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) - K(x) \\
&\leq -\alpha + \limsup_{x\to 0} K(x) \bigg(\int_{(0,1)} u^{-\alpha \limsup_{x\to 0} \frac{y}{c(y)}} p(\mathrm{d}u) - 1\bigg) \\
&= -\alpha + \inf K \bigg(\int_{(0,1)} u^{-\alpha \limsup_{x\to 0} \frac{y}{c(y)}} p(\mathrm{d}u) - 1\bigg).
\end{aligned}$$

According to Lemma 8 and the second inequality in (60), there exists $\alpha_0 > 0$ such that, for all $\alpha < \alpha_0$,

$$\int_{(0,1)} u^{-\alpha \limsup_{x\to 0} \frac{y}{c(y)}} p(\,\mathrm{d}u) < \alpha + p\big((0,1)\big).$$

This implies, choosing $\alpha < \alpha_0 \wedge (\delta / \sup_{y\in(0,1)} \frac{y}{c(y)})$ (which we will assume from now on) and using in addition (61), that

$$\limsup_{x\to 0} \frac{\mathcal{A}h(x)}{h(x)} < -\alpha(1 - \inf K) + \inf K\big(p(0,1) - 1\big) < \inf_x \frac{\mathcal{A}\xi(x)}{\xi(x)}. \tag{63}$$

For all $x \geq 1$, we have

$$\int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) = K(x) \int_{(0,1/x)} \exp\bigl(-\alpha s(ux) - \beta s(x)\bigr) p(\mathrm{d}u) \\ + K(x) \int_{(1/x,1)} \exp\bigl(\beta\bigl(s(ux) - s(x)\bigr)\bigr) p(\mathrm{d}u) - K(x),$$

where

$$\exp\bigl(-\alpha s(ux) - \beta s(x)\bigr) \leq (ux)^{-\alpha \sup_{y\in(0,ux)} \frac{y}{c(y)}} \leq u^{-\delta}$$

and

$$\exp\bigl(\beta\bigl(s(ux) - s(x)\bigr)\bigr) \leq u^{\beta \inf_{y\in(ux,x)} \frac{y}{c(y)}}.$$

On the one hand, we deduce that

$$\int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) \leq \sup_{y\in(0,M)} K(y) \int_{(0,1)} u^{-\delta} p(\mathrm{d}u) \tag{64}$$

and, on the other hand, choosing $\beta < 1/\inf_{y\geq 1} \frac{y}{c(y)}$, letting $x \to +\infty$ and using Fatou's lemma, we deduce that

$$\begin{aligned} \limsup_{x\to+\infty} \frac{\mathcal{A}h(x)}{h(x)} &= \beta + \limsup_{x\to+\infty} \int_{(0,x)} \frac{h(y)}{h(x)} k(x, \mathrm{d}y) - K(x) \\ &\leq \beta + \limsup_{x\to+\infty} K(x) \Bigl(\int_{(0,1)} u^{\beta \liminf_{y\to+\infty} \frac{y}{c(y)}} p(\mathrm{d}u) - 1 \Bigr) \\ &= \beta + \inf K \Bigl(\int_{(0,1)} u^{\beta \liminf_{y\to+\infty} \frac{y}{c(y)}} p(\mathrm{d}u) - 1 \Bigr). \end{aligned}$$

According to Lemma 8 and the first inequality in (60), there exists $\beta_0 > 0$ such that, for all $\beta < \beta_0$,

$$\int_{(0,1)} u^{\beta \liminf_{y\to+\infty} \frac{y}{c(y)}} p(\mathrm{d}u) < -\beta + p\bigl((0,1)\bigr).$$

Choosing $\beta < \beta_0 \wedge (1/\inf_{y\geq 1} \frac{y}{c(y)})$, we deduce that

$$\limsup_{x\to+\infty} \frac{\mathcal{A}h(x)}{h(x)} \leq \beta(1 - \inf K) + \inf K\bigl(p(0,1) - 1\bigr)$$

and hence, choosing β small enough and using (61),

$$\limsup_{x\to+\infty} \frac{\mathcal{A}h(x)}{h(x)} < \inf_x \frac{\mathcal{A}\xi(x)}{\xi(x)}. \tag{65}$$

By (62) and (64), and observing that our assumptions imply that K is uniformly bounded, we deduce that the first part of Assumption 1 holds true. In addition, $\mathcal{A}h/h$ is locally bounded and, by (63) and (65), it is thus bounded from above. We conclude that Assumption 1 is verified.

Finally, (63) and (65) in combination with Proposition 7 entail that Assumption 4 holds true. This concludes the proof of Proposition 13. □

3.2. *Proof of Proposition* 4. Since the process X is a PDMP, it is a strong Markov process with respect to its completed natural filtration according to Theorem 25.5 in [22] (its proof remains correct under our assumptions).

Let us now prove the irreducibility of X. Fix $x_0 \in (0,+\infty)$ and set

$$A := \{x \in (0,+\infty), \mathbb{P}_x(H_{x_0} < +\infty) > 0\}.$$

We first note that A is nonempty since $x_0 \in A$. Our strategy is to prove that A is open and closed in $(0,+\infty)$, so that $A = (0,+\infty)$ since $(0,+\infty)$ is connected.

(1) $A \cap (0, x_0)$ *is open.* For all $x < x_0 \in (0,+\infty)$, $m_x := \sup_{z\in[x,x_0]} k_h(z,(0,z))$ is finite according to Assumption 1. Setting $t_x = s(x_0) - s(x)$, we deduce from the construction of the process (see Step 1 in the proof of Proposition 2) that

$$\begin{aligned}\mathbb{P}_x(H_{x_0} \le t_x) &\ge \mathbb{P}_x\big(\text{the process } X \text{ does not jump during the time interval } [0,t_x]\big)\\ &\ge e^{-m_x t_x} > 0.\end{aligned}$$

In particular, $(0,x_0) \subset A$ so that $A \cap (0,x_0)$ is open.

(2) A *contains a neighbourhood of* x_0. According to the previous step, for all $\varepsilon \in (0,x_0)$, $(x_0-\varepsilon, x_0] \subset A$. It remains to prove that there exists $\varepsilon > 0$ such that $(x_0, x_0+\varepsilon) \subset A$. According to Assumption 2, the Lebesgue measure of $s(\{y \in (x_0,+\infty), k(y,(0,x_0)) > 0\})$ is positive. Since

$$\{y \in (x_0,+\infty), k\big(y,(0,x_0)\big) > 0\} = \bigcup_{n\ge1,m\ge1} \{y \in (x_0,n), k\big(y,(0,x_0)\big) > 1/m\},$$

we deduce that there exists a bounded $I_0 \subset (x_0,+\infty)$ such that

$$\lambda_1\big(s(I_0)\big) > 0 \quad \text{and} \quad \inf_{y\in I_0} k\big(y,(0,x_0)\big) > 0.$$

Choosing $\varepsilon > 0$ small enough, we deduce that, for all $x \in (x_0, x_0+\varepsilon)$, $\lambda_1(s(I_0 \cap (x,+\infty))) > 0$.

We also have, denoting by σ the first jump time of X and using the strong Markov property at time σ,

$$\mathbb{P}_x(H_{x_0} < +\infty) \ge \mathbb{E}_x\big(\mathbf{1}_{\sigma<+\infty}\mathbb{P}_{X_\sigma}(H_{x_0} < +\infty)\big). \tag{66}$$

Since $\mathbb{P}_y(H_{x_0} < +\infty) > 0$ for all $y \in (0,x_0)$, it is sufficient to prove that $\mathbb{P}(\sigma < +\infty \text{ and } X_\sigma \in (0,x_0)) > 0$ to conclude that $\mathbb{P}_x(H_{x_0} < +\infty) > 0$. By construction of the process X, we have

$$\begin{aligned}&\mathbb{P}_x\big(\sigma < +\infty \text{ and } X_\sigma \in (0,x_0)\big)\\ &\quad\ge \mathbb{P}_x\big(\sigma < +\infty \text{ and } X_{\sigma-} \in I_0 \text{ and } X_\sigma \in (0,x_0)\big)\\ &\quad\ge \mathbb{P}_x\big(s^{-1}\big(s(x)+\sigma\big) \in I_0\big)\frac{\inf_{y\in I_0} k_h(y,(0,x_0))}{\sup_{y\in I_0} k_h(y,(0,y)) + q(y)}\end{aligned}\tag{67}$$

since $s^{-1}(s(x)+t)$ is the position of the process X_{t-} under $\mathbb{P}_x$, conditionally to $t \le \sigma$. We also have

$$\begin{aligned}\mathbb{P}_x\big(s^{-1}\big(s(x)+\sigma\big) \in I_0\big) &= \mathbb{P}_x\big(\sigma \in s(I_0) - s(x)\big)\\ &= \int_{s(I_0)-s(x)} \frac{1}{[k_h+q](s^{-1}(s(x)+t))}\\ &\quad\times \exp\left(-\int_0^t [k_h+q]\big(s^{-1}\big(s(x)+u\big)\big)\,\mathrm{d}u\right)\mathrm{d}t > 0.\end{aligned}\tag{68}$$

Using (66), (67) and (68), we deduce that, for all $x \in (x_0, x_0 + \varepsilon)$,

$$\mathbb{P}_x(H_{x_0} < +\infty) > 0.$$

This concludes the second step of the proof.

(3) *$A \cap (x_0, +\infty)$ is open.* Fix $x \in A \cap (x_0, +\infty)$. Then, for all $\varepsilon \in (0, x)$, for all $y \in (x - \varepsilon, x)$, we have using the strong Markov property at time H_x,

$$\mathbb{P}_y(H_{x_0} < +\infty) \geq \mathbb{P}_y(H_x < +\infty)\mathbb{P}_x(H_{x_0} < +\infty) > 0,$$

since $y < x$ and $x \in A$. In particular, $(x - \varepsilon, x) \subset A$. Moreover, since X is right-continuous,

$$\lim_{y \to x, y > x} \mathbb{P}_x(H_y < H_{x_0}) = 1.$$

Hence there exists $\varepsilon > 0$ such that, for all $y \in (x, x + \varepsilon)$,

$$\mathbb{P}_x(H_y < H_{x_0}) \geq 1 - \mathbb{P}_x(H_{x_0} < +\infty)/2.$$

This implies that

$$\mathbb{P}_x(H_y < H_{x_0} \text{ and } H_{x_0} < +\infty) > 0.$$

Since, by the strong Markov property applied at time H_y, we have $\mathbb{P}_x(H_y < H_{x_0} \text{ and } H_{x_0} < +\infty) = \mathbb{P}_x(H_y < H_{x_0})\mathbb{P}_y(H_{x_0} < +\infty)$, we deduce that, for all $y \in (x, x + \varepsilon)$,

$$\mathbb{P}_y(H_{x_0} < +\infty) > 0.$$

This concludes the third step of the proof.

(4) *A is closed in $(0, +\infty)$.* We prove that A is sequentially closed in $(0, +\infty)$. Let $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{Z}_+}$ be a sequence converging to a point $x \in (0, +\infty)$.

If there exists $n \in \mathbb{Z}_+$ such that $x \leq x_n$, then $\mathbb{P}_x(H_{x_n} < +\infty) > 0$ and hence, using the Markov property at time H_{x_n}, we deduce that $\mathbb{P}_x(H_{x_0} < +\infty) > 0$ and hence that $x \in A$.

Assume now that $x_n < x$ for all $n \in \mathbb{Z}_+$. Without loss of generality, we assume that $(x_n)_{n\in\mathbb{Z}_+}$ is nondecreasing. According to Assumption 2, the Lebesgue measure of $s(\{y \in (x, +\infty), k_h(y, (0, x)) > 0\})$ is positive. Since

$$\{y \in (x, +\infty), k_h(y, (0, x)) > 0\} = \bigcup_{n\geq 1, m\geq 1, p\geq 1} \{y \in (x, p), k_h(y, (0, x_n)) > 1/m\},$$

we deduce that there exists a bounded $I_1 \subset (x, +\infty)$ and $n \in \mathbb{Z}_+$ such that

$$\lambda_1(s(I_1)) \quad \text{and} \quad \inf_{y\in I_1} k_h(y, (0, x_n)) > 0.$$

Using the same procedure as in Step 2 above, we deduce that $\mathbb{P}_x(H_{x_n} < +\infty) > 0$. Using the strong Markov property at time H_{x_n} and the fact that $x_n \in A$, we deduce that $\mathbb{P}_x(H_{x_0} < +\infty) > 0$, so that $x \in A$.

(5) *Conclusion.* Steps 1, 2, 3 and 4 above imply that A is both open and closed in the connected set $(0, +\infty)$, so that $A = (0, +\infty)$ and, for all $x, y \in (0, +\infty)$

$$\mathbb{P}_x(H_y < +\infty) > 0.$$

Now let $l < r \in (0, +\infty)$ and set $t_{l,r} = s(r) - s(l)$. Then, for all $x \leq y \in [l, r]$,

$$\mathbb{P}_x(H_y < t_{l,r}) \geq \mathbb{P}_l(\sigma \geq t_{l,r}) > 0.$$

Moreover, since $\mathbb{P}_r(H_l < +\infty) > 0$, we deduce that there exists $t'_{l,r} > 0$ such that $\mathbb{P}_r(H_l < t'_{l,r}) > 0$. Using the strong Markov property, we deduce that, for all $x > y \in [l, r]$,

$$\begin{aligned}\mathbb{P}_x(H_y < t_{l,r} + t'_{l,r} + t_{l,r}) &\geq \mathbb{P}_x(H_r < t_{l,r})\mathbb{P}_r(H_l < t'_{l,r})\mathbb{P}_l(H_y < t_{l,r}) \\ &\geq \mathbb{P}_l(\sigma \geq t_{l,r})\mathbb{P}_r(H_l < t'_{l,r})\mathbb{P}_l(\sigma \geq t_{l,r}) > 0.\end{aligned}$$

Setting $t_0 = t_{l,r} + t'_{l,r} + t_{l,r}$, this concludes the proof of Proposition 4.

3.3. *Proof of Proposition 5 under Assumption 3(a).* This proof is a direct adaptation of the proof of Proposition 1 in [38], where the problem is already solved when s is of the form $\int_1^x 1/c(y)\,\mathrm{d}y$, with c continuous and positive, and the measure μ is a Dirac measure.

Let $I=(\mathfrak{a},\mathfrak{b})$ and fix $t_1>0$ small enough so that $\phi(\mathfrak{a},t)\in I$ for all $t\in(0,t_1)$. Let us denote by $r(x)=k_h(x,(0,x))+q(x)=b+K(x)-\frac{1}{h(x)}\frac{\partial h}{\partial s}(x)$ the jump rate of X at position $x\in(0,+\infty)$; recall the definition of b in (10). Restricting to the event where the process jumps only one time in the time interval $(0,t_1)$, we deduce that, for all $t\in(0,t_1)$ and all positive measurable function f vanishing on the cemetery point ∂,

$$\begin{aligned} Q_t f(\mathfrak{a}) &\geq \int_0^t\int_{[0,1]} f\bigl(\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)ae^{-\int_0^u r(\phi(\mathfrak{a},u))\,\mathrm{d}v}e^{-\int_0^u r(\phi(T(\phi(\mathfrak{a},u)),t-u))\,\mathrm{d}v} \\ &\qquad\times\frac{h(T(\theta,\phi(\mathfrak{a},u)))}{h(\phi(\mathfrak{a},u))}\mu(\mathrm{d}\theta)\,\mathrm{d}u \\ &= \int_{[0,1]}\int_0^t f\bigl(\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)ae^{-\int_0^u r(\phi(\mathfrak{a},u))\,\mathrm{d}v}e^{-\int_0^u r(\phi(T(\phi(\mathfrak{a},u)),t-u))\,\mathrm{d}v} \\ &\qquad\times\frac{h(T(\theta,\phi(\mathfrak{a},u)))}{h(\phi(\mathfrak{a},u))}\,\mathrm{d}u\mu(\mathrm{d}\theta). \end{aligned}$$

By assumption, h is upper bounded on I and r is uniformly bounded away from ∞ on compact subsets of $(0,+\infty)$, so that there exists a constant $a_1>0$ such that

$$\begin{aligned} Q_t f(\mathfrak{a}) &\geq a_1\int_{[0,1]}\int_0^t f\bigl(\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)h\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr)\bigr)\,\mathrm{d}u\mu(\mathrm{d}\theta) \\ &\geq a_1\int_{[0,1]}\int_0^t f\circ s^{-1}\bigl(s\circ\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)\,\mathrm{d}u m_\theta\mu(\mathrm{d}\theta), \end{aligned}$$

where $m_\theta=\inf_{u\in[0,t_1]}h(T(\theta,\phi(\mathfrak{a},u)))>0$, where we used the fact that, for any fixed θ, $u\mapsto h(T(\theta,\phi(\mathfrak{a},u)))$ is positive and continuous. We observe that, for all $\theta\in[0,1]$, $s\circ\phi(T(\theta,\phi(\mathfrak{a},u)),t-u)=s(T(\theta,\phi(\mathfrak{a},u)))+t-u$, so that (recall (15)) for all $\theta\in[0,1]$,

$$\frac{\mathrm{d}s\circ\phi(T(\theta,\phi(\mathfrak{a},u)),t-u)}{\mathrm{d}u}=\frac{\mathrm{d}s\circ T(\theta,\phi(\mathfrak{a},u))}{\mathrm{d}u}-1=\frac{\partial s\circ T(\theta,\cdot)}{\partial s}\bigl(\phi(\mathfrak{a},u)\bigr)-1\neq 0,$$

with the left-hand side continuous in u and in particular bounded away from 0 and ∞ for $u\in(0,t_1)$. We are now in position to use the change of variable $y=s(\phi(T(\theta,\phi(\mathfrak{a},u)),t-u))$, and deduce that, for all $\theta\in[0,1]$, there exists a positive constant $a_2(\theta)>0$ such that, for all $t\in(0,t_1)$,

$$\int_0^t f\circ s^{-1}\bigl(s\circ\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)\,\mathrm{d}u\geq a_2(\theta)\left|\int_{s(\phi(T(\theta,\mathfrak{a}),t))}^{s(T(\theta,\phi(\mathfrak{a},t)))} f\circ s^{-1}(y)\,\mathrm{d}y\right|.$$

We have $\phi(T(\theta,\mathfrak{a}),t_1)\neq T(\theta,\phi(\mathfrak{a},t_1))$ and hence, by continuity of both terms at t_1, there exist two values $s_1(\theta)<s_2(\theta)$ and a fixed time $t_1'(\theta)\in(0,t_1)$ such that, for all $t\in(t_1'(\theta),t_1)$,

$$\int_0^t f\circ s^{-1}\bigl(s\circ\phi\bigl(T\bigl(\theta,\phi(\mathfrak{a},u)\bigr),t-u\bigr)\bigr)\,\mathrm{d}u\geq a_2(\theta)m_\theta\int_{s_1(\theta)}^{s_2(\theta)} f\circ s^{-1}(y)\,\mathrm{d}y$$

and hence

$$Q_t f(\mathfrak{a})\geq\int_{[0,1]} a_2(\theta)m_\theta\mathbf{1}_{t\in(t_1'(\theta),t_1)}\int_{s_1(\theta)}^{s_2(\theta)} f\circ s^{-1}(y)\,\mathrm{d}y\mu(\mathrm{d}\theta).$$

Let $t_1'\in(0,t_1)$ such that

$$\int_{[0,1]} a_2(\theta)m_\theta\mathbf{1}_{t_1'\in(t_1'(\theta),t_1)}\int_{s_1(\theta)}^{s_2(\theta)} f\circ s^{-1}(y)\,\mathrm{d}y\mu(\mathrm{d}\theta)>0$$

for all positive continuous f, and define the positive measure υ' on $(0,+\infty)$ by

$$\upsilon'(f)=\int_{[0,1]} a_2(\theta) m_\theta \mathbf{1}_{t_1'\in(t_1'(\theta),t_1)} \int_{s_1(\theta)}^{s_2(\theta)} f\circ s^{-1}(y)\,\mathrm{d}y\mu(\mathrm{d}\theta)$$

so that, for all $t\in(t_1',t_1)$, setting $a_2=\upsilon'((0,+\infty))$ and $\upsilon=\frac{1}{a_2}\upsilon'$,

$$\mathbb{E}_{\mathfrak{a}}\big(f(X_t)\big)=Q_t f(\mathfrak{a})\geq a_2\upsilon(f). \tag{69}$$

Now, since υ is a nonzero measure on $(0,\infty)$, we have, by the irreducibility property proved in Proposition 4,

$$\mathbb{P}_\upsilon(H_{\mathfrak{a}}<+\infty)=\int_{(0,+\infty)}\upsilon(\mathrm{d}y)\mathbb{P}_y(H_{\mathfrak{a}}<\infty)>0,$$

where $H_{\mathfrak{a}}=\inf\{t\geq 0: X_t=\mathfrak{a}\}$. In particular, there exists $t_2>0$ such that

$$a_3:=\mathbb{P}_\upsilon\big(H_{\mathfrak{a}}\in\big[t_2,t_2+\big(t_1-t_1'\big)/2\big]\big)>0.$$

Hence, using the strong Markov property at time $H_{\mathfrak{a}}$, we deduce that for all $t\in[t_2+t_1',t_2+(t_1'+t_1)/2]$,

$$\mathbb{E}_\upsilon\big(f(X_t)\big)\geq\mathbb{E}_\upsilon\big[\mathbf{1}_{H_{\mathfrak{a}}\in[t_2,t_2+(t_1-t_1')/2]}\mathbb{E}_{\mathfrak{a}}\big[f(X_{t-u})\big]|_{H_{\mathfrak{a}}}\big]\geq a_3a_2\upsilon(f). \tag{70}$$

Iterating the above inequality (i.e., applying the Markov property successively at times tk/n, $k=1,\dots,n-1$) we deduce that

$$\mathbb{E}_\upsilon\big(f(X_t)\big)\geq(a_3a_2)^n\upsilon(f),\quad t\in\big[n\big(t_2+t_1'\big),n\big(t_2+\big(t_1'+t_1\big)/2\big)\big]. \tag{71}$$

We set $n_1=\lfloor\frac{2t_2+2t_1'}{t_1-t_1'}\rfloor+1$ (so that $(n+1)(t_2+t_1')\leq n(t_2+(t_1'+t_1)/2)$ for all $n\geq n_1$), and define $t_3=n_1(t_2+t_1')$. For any $t\geq t_3$, the integer $n=\lfloor\frac{t}{t_2+t_1'}\rfloor$ satisfies $t\in[n(t_2+t_1'),(n+1)(t_2+t_1')]$ and $n\geq n_1$, so that $t\in[n(t_2+3t_1/2),n(t_2+(t_1'+t_1)/2)]$. Hence, setting

$$\beta_t=(a_3a_2)^{\lfloor\frac{t}{t_2+t_1'}\rfloor}>0,\quad t\geq t_3,$$

we deduce from (71) that

$$\mathbb{E}_\upsilon\big(f(X_t)\big)\geq\beta_t\upsilon(f),\quad t\geq t_3.$$

Using again the irreducibility property stated in Proposition 4, we know that, for any compactly contained interval $L\subset(0,+\infty)$ containing $\mathfrak{a}$, there exists a constant $t_4(L)>0$ such that

$$a_4(L):=\inf_{x\in L}\mathbb{P}_x\big(H_{\mathfrak{a}}\leq t_4(L)\big)>0.$$

Hence Markov's property applied at time $H_{\mathfrak{a}}$ and the above inequalities gives, for $t\geq t_1+t_3+t_4(L)$ and $x\in L$,

$$\begin{aligned}\mathbb{E}_x\big(f(X_t)\big)&\geq\mathbb{E}_x\big[\mathbf{1}_{H_{\mathfrak{a}}\leq t_4(L)}\mathbb{E}_{\mathfrak{a}}\big[f(X_{t-u})\big]|_{u=H_{\mathfrak{a}}}\big]\\&\geq\mathbb{E}_x\big[\mathbf{1}_{H_{\mathfrak{a}}\leq t_4(L)}a_2\mathbb{E}_\upsilon\big[f(X_{t-t_1-u})\big]|_{u=H_{\mathfrak{a}}}\big]\\&\geq\mathbb{E}_x[\mathbf{1}_{H_{\mathfrak{a}}\leq t_4(L)}a_2\beta_{t-t_1-H_{\mathfrak{a}}}]\upsilon(f)\\&\geq c_{L,t}\upsilon(f),\end{aligned}$$

where $c_{L,t}:=a_4(L)a_2\beta_{t-t_1-t_4(L)}$. This concludes the proof of Proposition 5 under Assumption 3(a).

3.4. *Proof of Proposition 5 under Assumption 3(b).* (1) *We find a lower bound for Q by a simpler semigroup S.*

Let $I = (\mathfrak{a}, \mathfrak{b})$ such that $\beta(x, (0, x))$ is positive for all $x \in I$ and fix $t_1 > 0$ small enough so that $\phi(\mathfrak{a}, t) \in I$ for all $t \in (0, t_1)$. Without loss of generality, we assume that there exists a compact set $A \subset (0, \mathfrak{a})$ such that $\beta(x, A^c) = 0$ for all $x \in (0, +\infty)$, and that $\beta(x, \cdot) = 0$ for all $x \geq 2\mathfrak{b}$. Let $(S_t)_{t\geq 0}$ be the semigroup of a PDMP on $[\min A, +\infty)$, with jump kernel $\beta(x, \mathrm{d}y)$ and flow directed by ϕ. Then, using the fact that h, q and $k_h(x, (0, x))$ are locally bounded, we deduce that, for all $t \geq 0$, there exists a constant $a_1(t) > 0$ such that, for all nonnegative function f with support in $(\min A, 2\mathfrak{b})$,

$$Q_t f(x) \geq a_1(t) S_t f(x) \quad \forall x \in (\mathfrak{a}, \mathfrak{b}).$$

(2) *We prove a Feller-type property for S.*

Let $t_0 > 0$ be such that $\phi(\min A, t_0) > 2\mathfrak{b}$, so that the process with semi-group S has jumped at least once before time t_0 or remains outside $[\min A, 2\mathfrak{b}]$. Then for all $t \geq t_0$, all measurable $B \subset [\min A, 2\mathfrak{b}]$ and all $x \in [\min A, +\infty)$,

$$S_t \mathbf{1}_B(x) = \int_0^t \int_{(0,\phi(x,u))} S_{t-u} \mathbf{1}_B\big(\phi(y, t-u)\big) e^{-\int_0^u r_\beta(\phi(x,u))\,\mathrm{d}v} \beta\big(\phi(x, u), \mathrm{d}y\big)\,\mathrm{d}u$$

with $r_\beta(z) := \beta(z, (0, +\infty))$. Using the property (6) for β, one deduces that it also holds for S_t: for all $t \geq t_1$, for all $x_0 \in [\min A, 2\mathfrak{b}]$

$$S_t \mathbf{1}_B(x_0) > 0 \quad \text{implies that} \quad \liminf_{x \to x_0, x < x_0} S_1 \mathbf{1}_B(x) > 0. \tag{72}$$

(3) *We find a recurrent set.* To be more specific, in this part, we obtain a set C of arbitrarily small radius with the property that $S_t \mathbf{1}_C(x) > 0$ for all t sufficiently large and $x \in C$. We begin with a useful inequality, before making a case distinction.

Restricting to the event where the process jumps exactly once, and does so in the time interval $[0, t_1]$, we deduce that, for all positive measurable function f and all $x \in I$, for all $t \geq t_1$, there exists a constant $a_2(t) > 0$ such that

$$S_t f(x) \geq a_2(t) \delta_x R^t f, \quad \text{where } \delta_x R^t f := \int_0^{t_1} \int_{(0,+\infty)} f\big(\phi(y, t-u)\big) \beta\big(\phi(x, u), \mathrm{d}y\big)\,\mathrm{d}u.$$

We set $\mathfrak{c} = \frac{\mathfrak{a}+\mathfrak{b}}{2}$. The term $\delta_\mathfrak{c} R^{t_1}$ defines a measure such that $\delta_\mathfrak{c} R^{t_1}((0, \mathfrak{a})) > 0$, and hence there exists $t_2 > t_1$ such that $\delta_\mathfrak{c} R^{t_2}((\mathfrak{c} - \varepsilon, \mathfrak{c}]) > 0$ for all $\varepsilon > 0$. Let $\mathfrak{c}' \in (\mathfrak{a}, \mathfrak{c})$ be such that $\phi(\mathfrak{c}', t_2 - t_1) > \mathfrak{c}$. Then there exists a constant $a_3 > 0$ such that, for all $x \in (\mathfrak{c}', \mathfrak{c})$ and all positive function $f : (0, +\infty) \to [0, +\infty)$,

$$\delta_x R^{t_2} f \geq a_3 \delta_\mathfrak{c} R^{t_2 - t(x)} f, \tag{73}$$

where $t(x) \in (0, t_2 - t_1)$ is such that $\phi(x, t(x)) = \mathfrak{c}$. In particular, for all $x \in (\mathfrak{c}', \mathfrak{c})$, noting that $\phi(\mathfrak{c}', t(x)) < \phi(x, t(x)) = \mathfrak{c}$, we have

$$\begin{aligned} \delta_x S_{t_2}\big((\mathfrak{c}', \mathfrak{c}]\big) &\geq a_2(t_2) \delta_x R^{t_2}\big((\mathfrak{c}', \mathfrak{c}]\big) \\ &\geq a_2(t_2) a_3 \delta_\mathfrak{c} R^{t_2 - t(x)}\big((\mathfrak{c}', \mathfrak{c}]\big) \\ &= a_2(t_2) a_3 \delta_\mathfrak{c} R^{t_2}\big(\big(\phi(\mathfrak{c}', t(x)), \phi(\mathfrak{c}, t(x))\big]\big) \\ &\geq a_2(t_2) a_3 \delta_\mathfrak{c} R^{t_2}\big(\big(\phi(\mathfrak{c}', t(x)), \mathfrak{c}\big]\big) > 0, \end{aligned} \tag{74}$$

where (here and later) we define the measure $\delta_x S_t(A) = S_t \mathbf{1}_A(x)$.

Case (a): $\delta_\mathfrak{c} S_{t_2}(\{\mathfrak{c}\}) > 0$. When this is true, we can prove Proposition 5 in a straightforward way. If this holds then, using (72), we deduce that there exists $\mathfrak{c}'' \in (\mathfrak{c}', \mathfrak{c})$ such that $\inf_{x \in (\mathfrak{c}'', \mathfrak{c}]} \delta_x S_{t_2}(\{\mathfrak{c}\}) > 0$. Fixing $t_3 \in (t_1, t_2)$ such that

$$\inf_{t \in [t_3, t_2]} \delta_\mathfrak{c} S_t\big((\mathfrak{c}'', \mathfrak{c}]\big) > 0,$$

we deduce that there exists a constant $a_4 > 0$ such that, for all $t \in [t_3 + t_2, 2t_2]$,

$$\delta_{\mathfrak{c}} S_t \geq a_4 \delta_{\mathfrak{c}}.$$

We can now follow the same strategy as in the previous section, since the equation above is essentially (70) with $\upsilon = \delta_{\mathfrak{c}}$. This allows us deduce that Q satisfies (31), which completes the proof of Proposition 5 in this case.

Case (b): $\delta_{\mathfrak{c}} S_{t_2}(\{c\}) = 0$. Take $\mathfrak{c}'' \in (\mathfrak{c}', \mathfrak{c})$ such that $\phi(\mathfrak{c}', t(\mathfrak{c}'')) = \mathfrak{c}''$. Then there exists $\varepsilon > 0$ such that $\delta_{\mathfrak{c}} R^{t_2}((\mathfrak{c}'', \mathfrak{c} - \varepsilon)) > 0$. According to (74), we have

$$\begin{aligned}\delta_x R^{t_2}\big((\mathfrak{c}', \mathfrak{c} - \varepsilon)\big) &\geq a_3 \delta_{\mathfrak{c}} R^{t_2 - t(x)}\big((\mathfrak{c}', \mathfrak{c} - \varepsilon)\big)\\ &= a_3 \delta_{\mathfrak{c}} R^{t_2}\big((\phi(\mathfrak{c}', t(x)), \phi(\mathfrak{c} - \varepsilon, t(x))]\big).\end{aligned}$$

If $x \in (\mathfrak{c}', \mathfrak{c}'']$, then $\phi(\mathfrak{c}', t(x)) < \mathfrak{c}$ and $\phi(\mathfrak{c} - \varepsilon, t(x)) > \phi(\mathfrak{c}'', t(x)) \geq \mathfrak{c}$ so that

$$\delta_x R^{t_2}\big((\mathfrak{c}', \mathfrak{c} - \varepsilon)\big) \geq a_3 \delta_{\mathfrak{c}} R^{t_2}\big((\phi(\mathfrak{c}', t(x)), \mathfrak{c})\big) > 0.$$

If $x \in (\mathfrak{c}'', \mathfrak{c})$, then $\phi(\mathfrak{c}', t(x)) \leq \phi(\mathfrak{c}', t(\mathfrak{c}'')) = \mathfrak{c}''$ and $\phi(\mathfrak{c} - \varepsilon, t(x)) > \mathfrak{c} - \varepsilon$, hence

$$\delta_x R^{t_2}\big((\mathfrak{c}', \mathfrak{c} - \varepsilon)\big) \geq a_3 \delta_{\mathfrak{c}} R^{t_2}\big((\mathfrak{c}'', \mathfrak{c} - \varepsilon)\big) > 0.$$

Choosing $t_4 > t_2$ such that $\phi(\mathfrak{c} - \varepsilon, t_4 - t_2) = \mathfrak{c}$, we deduce that

$$\delta_x S_t\big((\mathfrak{c}', \mathfrak{c}]\big) \geq a_2(t) \delta_x R^t\big((\mathfrak{c}', \mathfrak{c}]\big) \geq a_2(t) \delta_x R^{t_2}\big((\mathfrak{c}', \mathfrak{c} - \varepsilon]\big) > 0, \quad \forall x \in (\mathfrak{c}', \mathfrak{c}], \forall t \in [t_2, t_4].$$

Hence there exists $t_5 > 0$ such that, for all $t \geq t_5$,

$$S_t\big((\mathfrak{c}', \mathfrak{c}]\big)(x) > 0, \quad \forall x \in (\mathfrak{c}', \mathfrak{c}].$$

Since $\mathfrak{c}'$ can be replaced in this argument by any point arbitrarily close to $\mathfrak{c}$ and since S is irreducible on $(\mathfrak{c}', \mathfrak{c})$, we deduce that, for all $\varepsilon > 0$, there exists $t(\varepsilon)$ such that, for all $t \geq t(\varepsilon)$,

$$S_t\big((\mathfrak{c} - \varepsilon, \mathfrak{c}]\big)(x) > 0, \quad \forall x \in (\mathfrak{c}', \mathfrak{c}]. \tag{75}$$

(4) *Conclusion.* In this part, we give an adaptation of [32], Proposition 6.2.1, in order to conclude.

Define the measures $\delta_x \bar{S}_n := \mathbf{1}_{(\mathfrak{c}', \mathfrak{c}]}(x) \delta_x S_{nt_1}(\cdot \cap (\mathfrak{c}', \mathfrak{c}])$ for $x \in (0, +\infty)$ and $n \in \mathbb{N}$. For all measurable $B \subset (\mathfrak{c}', \mathfrak{c}]$ such that $\delta_{\mathfrak{c}} \bar{S}_1(B) > 0$, we deduce from (72) that there exists $\varepsilon > 0$ such that $\inf_{x \in (\mathfrak{c} - \varepsilon, \mathfrak{c}]} \delta_x \bar{S}_1(B) > 0$ and hence from (75) that, for any $n_0 \geq 1$ such that $n_0 t_1 \geq t(\varepsilon)$,

$$\delta_x \bar{S}_{n_0 + 1}(B) > 0.$$

This shows that $(\bar{S}_n)_{n \in \mathbb{N}}$ is ψ-irreducible with $\psi = \delta_{\mathfrak{c}} \bar{S}_1$. In particular, by [32], Theorem 5.2.2, and its proof, $(\bar{S}_n)_{n \in \mathbb{N}}$ admits a "small set" $C \subset (\mathfrak{c}', \mathfrak{c}]$ such that $\delta_{\mathfrak{c}} \bar{S}_1(C) > 0$; this means that there exists $m \geq 1$, a constant $\eta > 0$ and a probability measure υ on $(\mathfrak{c}', \mathfrak{c}]$ such that $\delta_x \bar{S}_m \geq \eta \upsilon$ for all $x \in C$. Since $\delta_{\mathfrak{c}} \bar{S}_1(C) > 0$, we deduce from (72) that there exists a neighborhood U of $\mathfrak{c}$ in $(\mathfrak{c}, ', \mathfrak{c}]$ such that $\inf_{x \in U} \delta_x \bar{S}_1(C) > 0$. In particular,

$$\begin{aligned}\delta_x Q_{(m+1)t_1} &\geq a_1\big((m+1)t\big) \delta_x S_{(m+1)t_1}\\ &\geq a_1\big((m+1)t\big) \delta_x \bar{S}_{m+1} \geq \eta \inf_{x \in U} \delta_x \bar{S}_1(C) \upsilon, \quad \forall x \in U.\end{aligned}$$

Since there exists $t_6 > t_7$ such that $\inf_{t \in [t_6, t_7]} \delta_{\mathfrak{c}} Q_t U > 0$, we deduce as above that (31) holds true. This concludes the proof of Proposition 5.

3.5. *Proof of Theorem* 3. Our aim is to prove that Assumptions 1, 2, 3 and 4 together imply that Assumption F of [20] is satisfied for the Markov semigroup $(Q_t)_{t\in[0,+\infty)}$. Let us recall this assumption.

ASSUMPTION (F). There exist positive real constants γ_1, γ_2, c_1, c_2 and c_3, $t_1, t_2 \in [0,+\infty)$, a measurable function $\psi_1 : (0,+\infty) \to [1,+\infty)$, and a probability measure ν on a measurable subset $L \subset (0,+\infty)$ such that:

(F0) *(A strong Markov property).* Defining

$$H_L := \inf\{t \geq 0, X_t \in L\}, \tag{76}$$

assume that for all $x \in (0,+\infty)$, $X_{H_L} \in L$, $\mathbb{P}_x$-almost surely on the event $\{H_L < \infty\}$ and for all $t > 0$ and all measurable $f : (0,+\infty) \cup \{\partial\} \to \mathbb{R}_+$,

$$\mathbb{E}_x\big[f(X_t)\mathbf{1}_{H_L\leq t<\zeta}\big] = \mathbb{E}_x\big[\mathbf{1}_{H_L\leq t\wedge\zeta}\mathbb{E}_{X_{H_L}}\big[f(X_{t-u})\mathbf{1}_{t-u<\zeta}\big]|_{u=H_L}\big].$$

(F1) *(Local Dobrushin coefficient).* $\forall x \in L$,

$$\mathbb{P}_x(X_{t_1} \in \cdot) \geq c_1\nu(\cdot \cap L).$$

(F2) *(Global Lyapunov criterion).* We have $\gamma_1 < \gamma_2$ and

$$\begin{aligned} \mathbb{E}_x\big(\psi_1(X_{t_2})\mathbf{1}_{t_2<H_L\wedge\zeta}\big) &\leq \gamma_1^{t_2}\psi_1(x), \quad \forall x \in (0,+\infty) \\ \mathbb{E}_x\big(\psi_1(X_t)\mathbf{1}_{t<\zeta}\big) &\leq c_2, \quad \forall x \in L, \forall t \in [0,t_2], \\ \gamma_2^{-t}\mathbb{P}_x(X_t \in L) &\xrightarrow[t\to+\infty]{} +\infty, \quad \forall x \in L. \end{aligned}$$

(F3) *(Local Harnack inequality).* We have

$$\sup_{t\geq 0}\frac{\sup_{y\in L}\mathbb{P}_y(t<\zeta)}{\inf_{y\in L}\mathbb{P}_y(t<\zeta)} \leq c_3.$$

We prove in the following subsections that F0, F1, F2 and F3 are satisfied, in this order, with the aim to apply the following result, which is Theorem 3.5 in [20] combined with the continuous time adaptation of Theorem 1.7 in [20].

THEOREM 4 ([20]). *Under Assumption* (F), $(X_t)_{t\in[0,+\infty)}$ *admits a quasi-stationary distribution* ν_{QS} *on* $(0,+\infty)$, *which is the unique one satisfying* $\nu_{QS}(\psi_1) < \infty$ *and* $\mathbb{P}_{\nu_{QS}}(X_t \in L) > 0$ *for some* $t \in [0,+\infty)$. *In addition, there exists a constant* $\lambda_0^X \geq 0$ *such that* $\lambda_0^X \leq \log(1/\gamma_2) < \log(1/\gamma_1)$ *and* $\mathbb{P}_{\nu_{QS}}(t < \zeta) = e^{-\lambda_0^X t}$ *for all* $t \geq 0$, *and there exists a function* $\eta : (0,+\infty) \to [0,+\infty)$ *lower bounded away from* 0 *on* L *and such that*

$$\big|\eta(x) - e^{\lambda_0^X t}\mathbb{P}_x(t<\zeta)\big| \leq Ce^{-\gamma t}\psi_1(x), \quad \forall x \in (0,+\infty) \tag{77}$$

and such that $\mathbb{E}_x(\eta(X_t)\mathbf{1}_{t<\zeta}) = e^{-\lambda_0^X t}\eta(x)$ *for all* $x \in (0,+\infty)$ *and* $t \geq 0$. *Finally, setting* $E' = \{x \in (0,+\infty), \eta(x) > 0\}$, *we have, for all* $f : E' \to \mathbb{R}$ *such that* $\|f\eta/\psi_1\|_\infty < +\infty$,

$$\left|\frac{e^{\lambda_0^X t}}{\eta(x)}\mathbb{E}_x\big(\eta(X_t)f(X_t)\mathbf{1}_{t<\zeta}\big) - \nu_{QS}(\eta f)\right| \leq Ce^{-\gamma t}\frac{\psi_1(x)}{\eta(x)}\|f\eta/\psi_1\|_\infty, \quad \forall x \in E', \tag{78}$$

for some constants $\gamma > 0$ *and* $C > 0$.

Note that, in the above result, it is clear that λ_0^X is the same as the one defined in (32). We conclude by proving that the property obtained from this result entails Theorem 3.

In what follows, we only consider functions f vanishing on the cemetery point, so that $Q_t f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{t<\zeta}) = \mathbb{E}_x(f(X_t))$ for all $x \in E$, where ζ is the first hitting time of ∂. Moreover, b and h are the objects defined in Section 2.1.

3.5.1. *Proof of* F0 *and* F1. The completed natural filtration of X is right continuous (see Theorem 25.3 in [22]). Hence the Début Theorem (see, for instance, Lemma 75.1 in [42]) implies that H_L is a stopping time with respect to this filtration. By Proposition 4, we deduce that F0 holds true for any compact interval $L \subset (0, +\infty)$ (this set shall be chosen in Section 3.5.2).

According to Proposition 5, the condition F1 holds true for L, assuming in addition (and without loss of generality) that L large enough so that $\upsilon(L) \geq 1/2$.

3.5.2. *Proof of F*2. Take $\psi_1 = \psi/h$ (we assume without loss of generality that $\psi \geq h$), extended to ∂ by the value 0. We deduce from Assumption 4 that there exists $\lambda_1^X > \lambda_0^X$ and a compact interval $L \subset (0, +\infty)$ such that

$$\mathcal{L}\psi_1(x) \leq -\lambda_1^X \psi_1(x) + C\mathbf{1}_L(x), \quad \forall x \in (0, +\infty).$$

Let $(f_k)_{k\geq 2}$ be a nondecreasing sequence of nonnegative functions in $C_c^{(s)}$ such that, for all $k \geq 2$, $f_k(x) = \psi_1(x)$ for all $x \in (1/k, k)$. We deduce that, for all $x \in (1/k, k)$,

$$\begin{aligned}\mathcal{L} f_k(x) &= \frac{\partial f}{\partial s}(x) + k_h(x, f_k) - f_k(x)k_h\big(x, (0, x)\big) - q(x) f_k(x) \\ &= \frac{\partial \psi_1}{\partial s}(x) + k_h(x, f_k) - \psi_1(x)k_h\big(x, (0, x)\big) - q(x)\psi_1(x) \\ &\leq \mathcal{L}\psi_1(x) \leq -\lambda_1^X \psi_1(x) + C\mathbf{1}_L(x) = -\lambda_1^X f_k(x) + C\mathbf{1}_L(x).\end{aligned}$$

Since f_k, extended by the value 0 on ∂, belongs to the domain of the extended infinitesimal generator of X, we deduce that

$$M_t^k := e^{\lambda_1^X t} f_k(X_t) - f_k(x) - \int_0^t e^{\lambda_1^X u}\big(\lambda_1^X + \mathcal{L} f_k(X_u)\big)\,\mathrm{d}u$$

is a local martingale. Since $e^{\lambda_1^X u}(\lambda_1^X + \mathcal{L} f_k(X_u))$ is uniformly bounded on $[0, t]$ (where we used Lemma 1(iii)), we deduce that it is a martingale. In particular, for any $2 \leq k' \leq k$, denoting by $\tau_{k'} = \inf\{t \geq 0, X_t \text{ or } X_{t-} \notin (1/k', k')\}$, we have, using the optional stopping theorem,

$$\mathbb{E}\big(e^{\lambda_1^X t\wedge\zeta\wedge\tau_{k'}\wedge H_L} f_k(X_{t\wedge\zeta\wedge\tau_{k'}\wedge H_L})\big) \leq f_k(x), \quad \forall x \in \big(1/k', k'\big).$$

Letting $k \to +\infty$, we deduce that

$$\mathbb{E}\big(e^{\lambda_1^X t\wedge\zeta\wedge\tau_{k'}\wedge H_L} \psi_1(X_{t\wedge\zeta\wedge\tau_{k'}\wedge H_L})\big) \leq \psi_1(x), \quad \forall x \in \big(1/k', k'\big).$$

Using Fatou's lemma and the nonexplosion of the process X, we conclude by letting $k' \to +\infty$ that

$$\mathbb{E}\big(e^{\lambda_1^X t\wedge\zeta\wedge H_L} \psi_1(X_{t\wedge\zeta\wedge H_L})\big) \leq \psi_1(x), \quad \forall x \in (0, +\infty).$$

This entails that

$$\mathbb{E}\big(e^{\lambda_1^X t} \psi_1(X_t)\mathbf{1}_{t<\zeta\wedge H_L}\big) \leq \psi_1(x), \quad \forall x \in (0, +\infty),$$

which implies the first line of F2 for any $t_2 > 0$ and $\gamma_1 = e^{-\lambda_1^X}$.

The same procedure, but replacing λ_1^X by $-C$, stopping the process at time $t \wedge \zeta \wedge \tau_{k'}$ instead of $t \wedge \zeta \wedge \tau_{k'} \wedge H_L$ and using the fact that $\mathcal{L} f_k \leq C$ for all $x \in (1/k, k)$, one deduces that, for all $t \geq 0$.

$$\mathbb{E}\big(\psi_1(X_t)\mathbf{1}_{t<\zeta}\big) \leq e^{Ct}\psi_1(x), \quad \forall x \in (0, +\infty).$$

This implies the second line of F2.

Finally, choosing any $\gamma_2 \in (e^{-\lambda_1^X}, e^{-\lambda_0^X})$, the last line of F2 is a direct consequence of the definition of λ_0^X.

3.5.3. *Proof of F3.* The irreducibility property of Proposition 4 implies that there exists $t_L > 0$ such that $\inf_{x,y\in L} \mathbb{P}_x(H_y < t_L) > 0$. Moreover, for any fixed $x_0 \in L$, $\mathbb{P}_{x_0}(t_L < \zeta) > 0$, hence

$$c_3 := \inf_{x,y\in L} \mathbb{P}_x(H_y < t_L)\mathbb{P}_{x_0}(t_L < \zeta) > 0.$$

For all $t \geq t_L$ and all $x, y \in L$, we obtain, using the fact that $\mathbb{P}_x(t < \zeta)$ is decreasing with respect to t and the strong Markov property at time H_y,

$$\begin{aligned}\mathbb{P}_x(t < \zeta) &\geq \mathbb{E}_x\big(\mathbf{1}_{H_y\leq t}\mathbb{P}_y(t-u < \zeta)|_{u=H_y}\big)\\ &\geq \mathbb{P}_x(H_y \leq t_L)\mathbb{P}_y(t < \zeta) \geq c_3\mathbb{P}_y(t < \zeta).\end{aligned}$$

For $t < t_L$, we observe that, for all $x, y \in L$, using the strong Markov property at time H_{x_0},

$$\mathbb{P}_x(t < \zeta) \geq \mathbb{P}_x(H_{x_0} < +\infty)\mathbb{P}_{x_0}(t_L < H_{x_0}) \geq c_3 \geq c_3\mathbb{P}_y(t < \zeta).$$

Hence,

$$\sup_{t\geq 0} \frac{\sup_{y\in L} \mathbb{P}_y(t < \zeta)}{\inf_{x\in L} \mathbb{P}_x(t < \zeta)} \leq \frac{1}{c_3} < \infty. \tag{79}$$

This concludes the proof of F3.

3.5.4. *Conclusion of the proof of Theorem* 3. We proved in the above subsections that the semigroup Q satisfies the conditions of Theorem 4. The Doeblin property obtained in Proposition 5 entails that η is positive on $(0, +\infty)$ (and in particular $E' = (0, +\infty)$). Hence, for all $f \in L^\infty(\psi_1)$ and all $t \geq 0$, applying (78) to f/η, we deduce that

$$\left|\frac{e^{\lambda_0^X t}}{\eta(x)}\mu Q_t f - \nu_{\mathrm{QS}}(f)\right| \leq Ce^{-\gamma t}\frac{\psi_1(x)}{\eta(x)}\|f/\psi_1\|_\infty, \quad \forall x \in E'.$$

Since $\delta_x Q_t f = e^{-bt}\frac{1}{h(x)}\delta_x T_t(fh)$, we obtain, taking $f = g/h$ with $g \in L^\infty(\psi) = L^\infty(\psi_1 h)$,

$$\begin{aligned}\big|e^{(\lambda_0^X-b)t}\delta_x T_t g - \nu_{\mathrm{QS}}(g/h)\eta(x)h(x)\big| &\leq Ce^{-\gamma t}\psi_1(x)h(x)\big\|g/(h\psi_1)\big\|_\infty\\ &= Ce^{-\gamma t}\psi(x)\|g/\psi\|_\infty.\end{aligned}$$

Finally, using that $\lambda_0 = \lambda_0^X - b$ and setting $m(g) := \nu_{\mathrm{QS}}(g/h)$ and $\varphi(x) = \eta(x)h(x)$ we deduce that, for all $g \in L^\infty(\psi)$,

$$\big|e^{\lambda_0 t}\delta_x T_t g - m(g)\varphi(x)\big| \leq Ce^{-\gamma t}\psi(x).$$

Integrating with respect to μ such that $\mu(\psi) < +\infty$ concludes the proof.

3.6. *Proof of Proposition* 6. Let $\lambda_0' = \inf\{\lambda \in \mathbb{R}, \int_0^\infty e^{\lambda t} T_t \mathbf{1}_L(x)\,\mathrm{d}t = +\infty\}$, where $x \in (0, +\infty)$ is fixed and $L \subset (0, +\infty)$ is a nonempty, compactly embedded open interval. We clearly have $\lambda_0 \geq \lambda_0'$. Let us prove the converse inequality.

Fix $\lambda > \lambda_0'$, so that $\int_0^\infty e^{\lambda t} T_t \mathbf{1}_L(x)\,\mathrm{d}t = +\infty$ for some $x \in (0, +\infty)$ and some compactly embedded nonempty interval $L \subset (0, +\infty)$. In particular, setting $\lambda^X = \lambda + b$, we have $\int_0^\infty e^{\lambda^X t}\mathbb{P}_x(X_t \in L)\,\mathrm{d}t = +\infty$. For any $y \in (0, +\infty)$, there exists, according to Proposition 4, $u_0 > 0$ such that $\mathbb{P}_y(H_x \leq u_0) > 0$, and hence, using the strong Markov property at time H_x,

$$\begin{aligned}\int_{u_0}^\infty e^{\lambda^X t}\mathbb{P}_y(X_t \in L)\,\mathrm{d}t &\geq \int_{u_0}^\infty e^{\lambda^X t}\mathbb{E}_y\big(\mathbf{1}_{H_x\leq u_0}\mathbb{P}_x(X_{t-u} \in L)|_{u=H_x}\big)\,\mathrm{d}t\\ &= \mathbb{E}_y\left(\mathbf{1}_{H_x\leq u_0}\int_{u_0}^\infty e^{\lambda^X t}\mathbb{P}_x(X_{t-u} \in L)\Big|_{u=H_x}\,\mathrm{d}t\right)\\ &\geq \mathbb{E}_y\left(\mathbf{1}_{H_x\leq u_0}\int_{u_0}^\infty e^{\lambda^X v}\mathbb{P}_x(X_v \in L)\,\mathrm{d}t\right) = +\infty.\end{aligned}$$

In particular, $\int_0^\infty e^{\lambda^X t}\mathbb{P}_y(X_t \in L)\,\mathrm{d}t = +\infty$ for all $y \in (0, +\infty)$. This implies that the probability measure υ from Proposition 5 satisfies

$$\int_0^\infty e^{\lambda^X t}\mathbb{P}_\upsilon(X_t \in L)\,\mathrm{d}t = +\infty. \tag{80}$$

Consider t_L, υ and $c_{L,t}$ from Proposition 5. Then, for all $T \geq t_L + 1$ and all $x \in L$, we have, applying the Markov property at time $t_L + u$ for all $u \in [0, 1]$,

$$\mathbb{P}_x(X_T \in L) \geq c_{L,t_L+u}\mathbb{P}_\upsilon(X_{T-t_L-u} \in L) \geq c_{L,t_L+1}\mathbb{P}_\upsilon(X_{T-t_L-u} \in L)$$

and hence

$$\begin{aligned} e^{\lambda^X T}\mathbb{P}_x(X_T \in L) &\geq c_{L,t_L+1}e^{\lambda^X T}\int_0^1 \mathbb{P}_\upsilon(X_{T-t_L-u} \in L)\,\mathrm{d}u \\ &\geq c_{L,t_L+1}\int_0^1 e^{\lambda^X(T-t_L-u)}\mathbb{P}_\upsilon(X_{T-t_L-u} \in L)\,\mathrm{d}u \\ &= c_{L,t_L+1}\int_{T-t_L-1}^{T-t_L} e^{\lambda^X t}\mathbb{P}_\upsilon(X_t \in L)\,\mathrm{d}t. \end{aligned}$$

Now, according to (80), for any fixed $\varepsilon > 0$, there exists $T_\varepsilon \in \{0, 1, \ldots\}$ such that $T_\varepsilon \geq t_L + 1$ and

$$\int_{T_\varepsilon-t_L-1}^{T_\varepsilon-t_L} e^{(\lambda^X+\varepsilon)t}\mathbb{P}_\upsilon(X_t \in L)\,\mathrm{d}t \geq \frac{1}{c_{L,t_L+1}}$$

and hence such that

$$e^{(\lambda^X+\varepsilon)T_\varepsilon}\mathbb{P}_x(X_{T_\varepsilon} \in L) \geq 1. \tag{81}$$

We define the function $w_\varepsilon : (0, +\infty) \to [0, +\infty)$ by

$$w_\varepsilon(x) = \sum_{i=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)i}\mathbb{P}_x(X_i \in L) = \sum_{i=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)i}Q_i\mathbf{1}_L(x),$$

where we recall that Q is the semigroup associated to the Markov process X. We thus have

$$\begin{aligned} Q_1 w_\varepsilon(x) &= \sum_{i=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)i}Q_{i+1}\mathbf{1}_L(x) \\ &= e^{-(\lambda^X+\varepsilon)}\sum_{i=1}^{T_\varepsilon} e^{(\lambda^X+\varepsilon)i}Q_i\mathbf{1}_L(x) \\ &= e^{-(\lambda^X+\varepsilon)}\big(w_\varepsilon(x) + e^{(\lambda^X+\varepsilon)T_\varepsilon}Q_{T_\varepsilon}\mathbf{1}_L(x) - \mathbf{1}_L(x)\big). \end{aligned}$$

But, by (81), $e^{(\lambda^X+\varepsilon)T_\varepsilon}Q_{T_\varepsilon}\mathbf{1}_L(x) \geq 1$ for all $x \in L$, and hence we obtain, for all $x \in (0, +\infty)$,

$$Q_1 w_\varepsilon(x) \geq e^{-(\lambda^X+\varepsilon)}w_\varepsilon(x)$$

and hence, by iteration,

$$Q_n w_\varepsilon(x) \geq e^{-(\lambda^X+\varepsilon)n}w_\varepsilon(x), \quad \forall n \in \{0, 1, \ldots\}.$$

Since $w_\varepsilon(x) > 0$ for all $x \in L$, we deduce that

$$e^{-(\lambda^X+\varepsilon)n}Q_n w_\varepsilon(x) \xrightarrow[n\to+\infty]{} +\infty, \quad \forall x \in L. \tag{82}$$

Proposition 4 and 5 entail that there exists $t_0 > 0$ such that

$$c_0 := \mathbb{P}_\upsilon(X_{t_0} \in L) > 0.$$

We can assume without loss of generality that $T_\varepsilon > t_L + t_0$. Hence, for all $y \in L$, we have according to the Markov property and by Proposition 5, for all $u \geq t$ such that $u - t \geq t_0 + t_L$,

$$\mathbb{P}_y(X_{u-t} \in L) \geq \mathbb{E}_y\big(\mathbb{P}_{X_{u-t-t_0}}(X_{t_0} \in L)\big) \geq c_{L,u-t-t_0}\mathbb{P}_\upsilon(X_{t_0} \in L) \geq c_{L,u-t-t_0}c_0.$$

Using again the Markov property, we thus observe that, for all $u > t_0 + t_L$, all $x \in (0, +\infty)$ and all $t \in [0, u - t_0 - t_L]$,

$$\begin{aligned}\mathbb{P}_x(X_u \in L) &\geq \mathbb{E}_x\big(\mathbf{1}_{X_t \in L}\mathbb{P}_{X_t}(X_{u-t} \in L)\big)\\ &\geq \mathbb{P}_x(X_t \in L)c_{L,u-t-t_0}c_0.\end{aligned}$$

In particular, for all $u > t_0 + t_L + T_\varepsilon$ and all $k \in \{0, 1, \dots, T_\varepsilon - 1\}$,

$$\mathbb{P}_x(X_u \in L) \geq \mathbb{P}_x(X_{\lfloor u\rfloor - T_\varepsilon + k} \in L)c_{L,u-\lfloor u\rfloor+T_\varepsilon-k-t_0}c_0 \geq \mathbb{P}_x(X_{\lfloor u\rfloor - T_\varepsilon + k} \in L)c_{L,1+T_\varepsilon-t_0}c_0.$$

Hence, setting $\delta_\varepsilon = \sum_{k=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)k}$, we have

$$\begin{aligned}e^{(\lambda^X+2\varepsilon)u}\mathbb{P}_x(X_u \in L) &= \frac{e^{(\lambda^X+2\varepsilon)u}}{\delta_\varepsilon}\sum_{k=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)k}\mathbb{P}_x(X_u \in L)\\ &\geq \frac{e^{(\lambda^X+2\varepsilon)u}}{\delta_\varepsilon}c_{L,1+T_\varepsilon-t_0}c_0\sum_{k=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)k}\mathbb{P}_x(X_{\lfloor u\rfloor-T_\varepsilon+k} \in L)\\ &= \frac{e^{(\lambda^X+2\varepsilon)u}}{\delta_\varepsilon}c_{L,1+T_\varepsilon-t_0}c_0\sum_{k=0}^{T_\varepsilon-1} e^{(\lambda^X+\varepsilon)k}Q_{\lfloor u\rfloor-T_\varepsilon+k}\mathbf{1}_L(x)\\ &\geq \frac{e^{(\lambda^X+2\varepsilon)(\lfloor u\rfloor-T_\varepsilon)}}{\delta_\varepsilon}c_{L,1+T_\varepsilon-t_0}c_0Q_{\lfloor u\rfloor-T_\varepsilon}w_\varepsilon(x).\end{aligned}$$

By (82), this shows that $e^{(\lambda^X+2\varepsilon)u}\mathbb{P}_x(X_u \in L)$ goes to infinity when $u \to +\infty$. In particular, $\lambda^X + 2\varepsilon \geq \lambda_0^X$. Since this is true for all $\varepsilon > 0$, we deduce that $\lambda^X \geq \lambda_0^X$ and hence $\lambda \geq \lambda_0$. Since this is true for all $\lambda > \lambda_0'$, we deduce that $\lambda_0' \geq \lambda_0$, which concludes the proof of the proposition.

3.7. *Proof of Proposition* 7. (1) *Proof of (a)* We set $\psi_1 = \psi/h$ and $\psi_2 = \xi(x)/h$, both extended by the value 0 at point ∂. We observe that (up to a change in the constant $C > 0$)

$$\mathcal{L}\psi_1 \leq -(\lambda_1 + b)\psi_1 + C\mathbf{1}_L \quad \text{and} \quad \mathcal{L}\psi_2 \geq -(\lambda_2 + b)\psi_2.$$

Since ψ_2 is continuous and positive, it is lower bounded on the compact interval L, and hence we have $\mathcal{L}\psi_1 \leq -(\lambda_1 + b)\psi_1 + C'\psi_2$, for some constant $C' > 0$. Hence setting $F = \psi_1 - \frac{C'}{\lambda_1-\lambda_2}\psi_2$, we obtain

$$\mathcal{L}F \leq -(\lambda_1 + b)\psi_1 + C'\psi_2 + \frac{(\lambda_2 + b)C'}{\lambda_1 - \lambda_2}\psi_2 = -(\lambda_1 + b)F.$$

Fix $x \in (0, +\infty)$. Using the same approach as in Section 3.5.2 (note that F is lower bounded on $(0, +\infty)$ and positive in a neighbourhood of $\{0, +\infty\}$), we deduce that, for all $t \geq 0$,

$$\mathbb{E}_x\big[e^{(\lambda_1+b)t}F(X_t)\big] \leq F(x).$$

In particular, for all $t \geq 0$,

$$\mathbb{E}_x\big(\psi_1(X_t)\big) \leq \frac{C'}{\lambda_1 - \lambda_2}\mathbb{E}_x\big(\psi_2(X_t)\big) + e^{-(\lambda_1+b)t}F(x).$$

For all $M > 0$, there exists a compact interval $L_M \subset (0, +\infty)$ such that $\psi_1 \geq M\psi_2$ on the set $E \setminus L_M$. Hence, for all $t \geq 0$,

$$M\mathbb{E}_x\big(\psi_2(X_t)\mathbf{1}_{X_t \notin L_M}\big) \leq \frac{C'}{\lambda_1 - \lambda_2}\mathbb{E}_x\big(\psi_2(X_t)\big) + e^{-(\lambda_1+b)t}F(x),$$

so that, choosing $M = \frac{C'}{\lambda_1 - \lambda_2} + 1$,

$$\mathbb{E}_x\big(\psi_2(X_t)\mathbf{1}_{X_t \notin L_M}\big) \leq \frac{C'}{\lambda_1 - \lambda_2}\mathbb{E}_x\big(\psi_2(X_t)\mathbf{1}_{X_t \in L_M}\big) + e^{-(\lambda_1+b)t}F(x),$$

which entails

$$\begin{aligned} \mathbb{E}_x\big(\psi_2(X_t)\big) &\leq \left(1 + \frac{C'}{\lambda_1 - b}\right)\mathbb{E}_x\big(\psi_2(X_t)\mathbf{1}_{X_t \in L_M}\big) + e^{-(\lambda_1+b)t}F(x) \\ &\leq \left(1 + \frac{C'}{\lambda_1 - b}\right)\mathbb{P}_x(X_t \in L_M) + e^{-(\lambda_1+b)t}F(x). \end{aligned} \tag{83}$$

In addition, by Corollary 1 (and more precisely its proof), we have, for all $t \geq 0$,

$$\mathbb{E}_x\big(\psi_2(X_t)\big) = \psi_2(x) + \int_0^t \mathbb{E}_x\big(\mathcal{L}\psi_2(X_u)\big)\,\mathrm{d}u \geq \psi_2(x) - (\lambda_2 + b)\int_0^t \mathbb{E}_x\big(\psi_2(X_u)\big)\,\mathrm{d}u$$

and hence, by Grönwall's lemma,

$$\psi_2(x) \leq e^{(\lambda_2+b)t}\mathbb{E}_x\big(\psi_2(X_t)\big).$$

The last inequality and (83) and the fact that $\lambda_2 + b < \lambda_1 + b$ imply that, for any fixed $\lambda' \in (\lambda_2, \lambda_1)$,

$$e^{(\lambda'+b)t}\mathbb{P}_x(X_t \in L_M) \xrightarrow[t \to +\infty]{} +\infty.$$

In particular $\lambda_0^X \leq \lambda' + b$, so that $\lambda_0 \leq \lambda'$ for any $\lambda' \in (\lambda_2, \lambda_1)$, which concludes the proof of Proposition 7 (a).

(2) *Proof of (b)* The intuition for this part is that we wish to consider a semigroup generated by the operator $f \mapsto \frac{\mathcal{A}(f\xi)}{\xi}$, represent this in terms of a Markov process Y together with a potential term $e^{\int_0^t d(Y_s)\,\mathrm{d}s}$, and use [11] to bound its growth coefficient. However, we must be cautious: ξ cannot be used in place of h in Assumption 1, so we cannot use our established existence and uniqueness results, and moreover, the potential term we would get from the calculation above is not bounded. Instead, we first define a process Y with the desired properties, and then consider introducing a truncated potential (d^M below) to allow us to apply [11]. Once this is done, we relate this back to the original semigroup T by applying Theorem 1 to a truncated version of $\mathcal{A}$, and this allows us to bound λ_0.

We consider the right-continuous PDMP Y with drift s and jump kernel $\bar{k}(x, \mathrm{d}y) = \frac{\xi(y)}{\xi(x)}k(x, \mathrm{d}y)$. The function $V = \psi/\xi$ satisfies

$$\begin{aligned} &\frac{\partial V}{\partial s}(x) + \int_{(0,x)}\big(V(y) - V(x)\big)\bar{k}(x, \mathrm{d}y) \\ &\quad = V(x)\left(\frac{1}{\psi(x)}\frac{\partial \psi}{\partial s}(x) - \frac{1}{\xi(x)}\frac{\partial \xi}{\partial s}(x) + \int_{(0,x)}\frac{\psi(y)}{\psi(x)}k(x, \mathrm{d}y) - \bar{k}\big(x, (0,x)\big)\right) \end{aligned}$$

$$= V(x)\left(\frac{\mathcal{A}\psi}{\psi} - \frac{\mathcal{A}\xi}{\xi}\right)$$
$$\le V(x)\big(-\lambda_1 + C\mathbf{1}_L(x) + \lambda_2\big)$$
$$\le -\lambda V(x) + C\max_L V\mathbf{1}_L(x),$$

where $\lambda = \lambda_1 - \lambda_2 \ge 0$. Since in addition $V(x) \to +\infty$ when $x \to 0$ or $x \to +\infty$, and since the jump rate $\bar{k}(x, (0, x))$ is locally bounded, this entails that Y is nonexplosive and recurrent. Its extended infinitesimal generator, denoted by $\mathcal{L}^Y$, satisfies, for all $f \in C_c^{(s)}$,

$$\mathcal{L}^Y f(x) = \frac{\partial f}{\partial s}(x) + \int_{(0,x)} \big(f(y) - f(x)\big)\bar{k}(x, \mathrm{d}y), \quad \forall x \in (0, +\infty).$$

Let $M > \inf_{x>0} \mathcal{A}\xi(x)/\xi(x)$ and define $d^M : x \in (0, +\infty) \mapsto d(x) \wedge M$. Consider the semigroup

$$S_t^M f(x) := \mathbb{E}_x\left(\exp\left(\int_0^t d^M(Y_s)\,\mathrm{d}s\right) f(Y_t)\right), \quad \forall t \ge 0, x \in (0, +\infty).$$

According to Proposition 2.1 in [17] (see also Proposition 3.4 in [11]), if Y is recurrent, then $-\lambda_0(S^M) \ge \inf_{x>0} d(x)$, with strict inequality if d is not constant, where $\lambda_0(S^M)$ is the growth coefficient of S^M (beware of the difference of sign convention in the definition of the growth coefficient in the cited works).

We therefore need to prove that $\lambda_0 \le \lambda_0(S^M)$. For all $f \in C_c^{(s)}$ and for $f \equiv 1$, we have

$$S_t^M f(x) = f(x) + \mathbb{E}_x\left(\int_0^t \exp\left(\int_0^u d^M(Y_s)\,\mathrm{d}s\right)\big(d^M(Y_u) f(Y_u) + \mathcal{L}^Y f(Y_u)\big)\,\mathrm{d}u\right)$$
$$= f(x) + \int_0^t S_u^M\big(d^M f + \mathcal{L}^Y f\big)(x)\,\mathrm{d}u.$$

Let

$$K^M(x) = \bar{k}\big(x, (0, x)\big) - d^M(x)$$

and

$$\mathcal{B}^M f(x) = d^M(x) f(x) + \mathcal{L}^Y f(x) = \frac{\partial f}{\partial s}(x) + \int_{(0,x)} f(y)\bar{k}(x, \mathrm{d}y) - K^M(x) f(x).$$

The operator $\mathcal{B}^M$ is a growth-fragmentation operator just like $\mathcal{A}$, and indeed, it satisfies Assumption 1 with $h' \equiv 1$ instead of h. In particular, according to Theorem 1, S^M is the unique semigroup such that, for all $f \in C_c^{(s)}$ and for $f \equiv 1$, for all $t \ge 0$ and all $x \in (0, +\infty)$,

$$S_t^M f(x) = f(x) + \int_0^t S_u^M\big(\mathcal{B}^M f\big)(x)\,\mathrm{d}u.$$

We now define, for all $f \in \mathcal{D}(\mathcal{A})$ and all $x \in (0, +\infty)$,

$$\mathcal{A}^M f(x) = \mathcal{A}f(x) - \big(d(x) - d^M(x)\big) f(x).$$

Then

$$\frac{\mathcal{A}h(x)}{h(x)} - d(x) \le \frac{\mathcal{A}^M h(x)}{h(x)} \le \frac{\mathcal{A}h(x)}{h(x)}$$

with d locally bounded, since ξ is locally lower bounded away from 0 and since $\frac{\partial \xi}{\partial s}$ and $\int_{(0,x)} \xi(y) k(x, \mathrm{d}y)$ are locally bounded by assumption. Hence one easily checks that $\mathcal{A}^M$

satisfies Assumption 1. Let T^M be the associated semigroup (whose existence and uniqueness is ensured by Theorem 1). Since $\xi \in L^\infty(h)$ and $\mathcal{A}\xi/h \geq -\lambda_2\xi/h$ is lower bounded, we deduce from Corollary 1 that

$$T_t^M\xi(x) = \xi(x) + \int_0^t T_u^M(\mathcal{A}^M\xi)(x)\,\mathrm{d}u.$$

In particular, the semigroup $\widetilde{T}^M$ defined, for all $f \in C_c^{(s)}$ and for $f \equiv 1$, by

$$\widetilde{T}_t^M f(x) = \frac{1}{\xi(x)} T_t^M(\xi f)(x), \quad \forall t \geq 0, \forall x \in (0, +\infty),$$

satisfies, for all such f, x and t,

$$\widetilde{T}_t^M f(x) = f(x) + \int_0^t \frac{1}{\xi(x)} T_u^M\big(\mathcal{A}^M(\xi f)\big)(x)\,\mathrm{d}u = f(x) + \int_0^t \widetilde{T}_u^M\big(\widetilde{\mathcal{A}}^M f\big)(x)\,\mathrm{d}u,$$

where

$$\begin{aligned}
\widetilde{\mathcal{A}}^M f(x) &= \frac{\mathcal{A}^M(f\xi)(x)}{\xi(x)} \\
&= \frac{\partial f}{\partial s}(x) + \int_{(0,x)} f(y)\bar{k}(x,\,\mathrm{d}y) - K(x)f(x) + \frac{1}{\xi(x)}\frac{\partial \xi}{\partial s}(x)f(x) \\
&\quad - \big(d(x) - d^M(x)\big)f(x) \\
&= \frac{\partial f}{\partial s}(x) + \int_{(0,x)} f(y)\bar{k}(x,\,\mathrm{d}y) + \left(\frac{\mathcal{A}\xi(x)}{\xi(x)} - \bar{k}\big(x,(0,x)\big) - d(x) + d^M(x)\right)f(x) \\
&= \mathcal{B}^M f(x).
\end{aligned}$$

This entails that, for all nonnegative measurable function $f : (0, +\infty) \to [0, +\infty)$, all $x \in (0, +\infty)$ and all $t \geq 0$,

$$S_t^M f(x) = \widetilde{T}_t^M f(x) = \frac{1}{\xi(x)} T^M(\xi f)(x).$$

But, according to the representation of T^M as the $1/h$ transform of a sub-Markov process (see Proposition 2 and the conclusion of the proof of Theorem 1 in Section 2.3), we have

$$\frac{1}{\xi(x)} T^M(\xi f)(x) = \frac{h(x)e^{b^M t}}{\xi(x)} \mathbb{E}_x\left(f(X_t^M)\frac{\xi(X_t^M)}{h(X_t^M)}\mathbf{1}_{X_t^M \neq \partial}\right),$$

where X^M is a $(0, +\infty) \cup \{\partial\}$-valued PDMP with drift determined by s, jump kernel $\frac{h(y)}{h(x)}k(x,\,\mathrm{d}y)$ and killing rate (i.e., jump rate toward ∂)

$$q^M(x) = b^M - \frac{\mathcal{A}^M h(x)}{h(x)}, \quad \text{with } b^M = \sup_{x\in(0,+\infty)} \frac{\mathcal{A}^M h(x)}{h(x)} \leq b.$$

Moreover, $b^M - q^M(x) = \frac{\mathcal{A}^M h(x)}{h(x)} \leq \frac{\mathcal{A}h(x)}{h(x)} = b - q(x)$, so

$$\begin{aligned}
S_t^M f(x) &= \frac{1}{\xi(x)} T_t^M(\xi f)(x) \\
&= \frac{h(x)e^{b^M t}}{\xi(x)} \mathbb{E}_x\left(\exp\left(-\int_0^t q^M(Z_u)\,\mathrm{d}u\right) f(Z_t)\frac{\xi(Z_t)}{h(Z_t)}\right)
\end{aligned}$$

$$\leq \frac{h(x)e^{bt}}{\xi(x)} \mathbb{E}_x\left(\exp\left(-\int_0^t q(Z_u)\,\mathrm{d}u\right) f(Z_t)\frac{\xi(Z_t)}{h(Z_t)}\right)$$
$$= \frac{1}{\xi(x)} T_t(\xi f)(x),$$

where Z is a (conservative) PDMP with drift determined by s and jump kernel $\frac{h(y)}{h(x)}k(x, \mathrm{d}y)$.

We hence obtain, immediately from the definition, that $\lambda_0(S^M) \geq \lambda_0$, which is what we needed to prove.

APPENDIX

Let s be continuous (strictly) increasing function from $(0, +\infty)$ to $\mathbb{R}$ such that $s(+\infty) = +\infty$, let Q be a nonnegative kernel from $(0, +\infty) \cup \{\partial\}$ to $(0, +\infty) \cup \{\partial\}$ such that $Q(\partial, (0, +\infty) \cup \{\partial\}) = 0$ and $Q(x, [x, +\infty)) = 0$ for all $x > 0$, where $\partial \notin (0, +\infty)$ is an isolated point. From now on, we set $E = (0, +\infty) \cup \{\partial\}$. We consider the PDMP X with state space E, directed by the flow ϕ defined by (14) (with $\phi(\partial, t) = \partial$ for all $t \geq 0$) between its jumps and with jump kernel Q (note that ∂ is an absorption point for X).

In the following results, $C_b(E)$ denotes the set of bounded real valued continuous functions on E and $C_0(E)$ the set of bounded continuous function vanishing at infinity. We emphasize that its statement and proof can be easily adapted to the case where X takes its values in $[0, +\infty)$ or $\mathbb{R}$.

The first part of the following proposition is proved by Davis in [22], Theorem 27.6, when ϕ is generated by a Lipschitz vector field and $x \mapsto Q(x, (0, +\infty) \cup \{\partial\})$ is continuous and bounded. In our case, we do not assume this regularity, but use instead the fact that our state space is one dimensional.

PROPOSITION 14. *Assume that* $\sup_{x\in(0,M)} Q(x, E) < +\infty$ *for all* $M > 0$. *Then the semigroup T of X maps $C_b(E)$ to itself.*

If in addition $s(0+) = -\infty$, $\sup_{x\in E} Q(x, E) < +\infty$ *and, for all* $M > 0$, *we also have* $\limsup_{x\to+\infty} Q(x, (0, M) \cup \{\partial\}) = \limsup_{x\to 0} Q(x, \{\partial\}) = 0$, *then the semigroup of X is Feller, meaning that it maps $C_0(E)$ to itself and is strongly continuous on $C_0(E)$.*

PROOF. We start by showing the first part, and then the second part of Proposition 14.

(1) *T maps $C_b(E)$ to itself.* Our proof is a simple adaptation of the proof of [22], Theorem 27.6, to our particular one-dimensional setting. Since $s(+\infty) = +\infty$, the explosion time of $\phi(x, \cdot)$ (denoted by $t_*(x)$ in the cited reference) is equal to infinity for all $x \in E$. Moreover, since $\sup_{x\in(0,M)} Q(x, E) < +\infty$, the process X is nonexplosive (as detailed in the first step of the proof of Proposition 2) and well defined for all time $t \geq 0$, for any initial distribution. Finally, $Q(x, E)$ is uniformly bounded over $x \in E$.

The only difference with the proof of [22], Theorem 27.6, is that, in our case, it is not immediate that, for any $\psi \in C_b(\mathbb{R}_+ \times E)$ and $f \in C_b(E)$, the term

$$G\psi(x, t) := f(\phi(x, t))e^{-\Lambda(t,x)} + \int_0^t \int_E \psi(t - u, y) Q(\phi(x, u), \mathrm{d}y)e^{-\Lambda(x,u)}\,\mathrm{d}u,$$

where

$$\Lambda(x, t) := \int_0^t Q(\phi(x, u), E)\,\mathrm{d}u,$$

is continuous in $(t, x) \in [0, +\infty) \times E$ and bounded. The rest of the proof is identical to the one of [22], Theorem 27.6, and we thus only need to prove that $G\psi \in C_b(\mathbb{R}_+ \times E)$ to conclude.

First note that $\|G\psi\|_\infty \le \|f\|_\infty + \|\psi\|_\infty$, so that it is bounded. It only remains to prove that $G\psi$ is continuous. Since $Q(\partial, \mathrm{d}y) = 0$ and since $\phi(\partial, t) = \partial$ for all $t \ge 0$, we have $G\psi(\partial, t) = f(\partial)$ for all $t \ge 0$ and hence $G\psi$ is continuous on $\{\partial\} \times [0, +\infty)$. Now let $(x, t) \in (0, +\infty) \times [0, +\infty)$ and $(\varepsilon, h) \in \mathbb{R} \times \mathbb{R}$ such that $(x+\varepsilon, t+h) \in (0, +\infty) \times [0, +\infty)$. We have, for all $u \ge 0$, denoting $\delta_{x,\varepsilon} := s(x+\varepsilon) - s(x)$

$$\begin{aligned}\phi(x+\varepsilon, u) &= s^{-1}\bigl(s(x+\varepsilon) + u\bigr)\\ &= s^{-1}\bigl(s(x) + \bigl(u + s(x+\varepsilon) - s(x)\bigr)\bigr) = \phi(x, u + \delta_{x,\varepsilon}).\end{aligned} \tag{84}$$

In particular,

$$\begin{aligned}\Lambda(x+\varepsilon, t+h) &= \int_0^{t+h} Q\bigl(\phi(x+\varepsilon, u), E\bigr)\,\mathrm{d}u\\ &= \int_0^{t+h} Q\bigl(\phi(x, u+\delta_{x,\varepsilon}), E\bigr)\,\mathrm{d}u\\ &= \int_{\delta_{x,\varepsilon}}^{t+h+\delta_{x,\varepsilon}} Q\bigl(\phi(x, u), E\bigr)\,\mathrm{d}u,\end{aligned}$$

so that Λ is continuous and more precisely

$$\bigl|\Lambda(x+\varepsilon, t+h) - \Lambda(x, t)\bigr| \le (2\delta_{x,\varepsilon} + h) \sup_{y \in (0, \phi(x, t+h+\delta_{x,\varepsilon}))} Q(y, E). \tag{85}$$

Using again (84), we also obtain

$$\begin{aligned}&\int_0^{t+h} \int_E \psi(t-u, y) Q\bigl(\phi(x+\varepsilon, u), \mathrm{d}y\bigr) e^{-\Lambda(x+\varepsilon, u)}\,\mathrm{d}u\\ &\quad= \int_0^{t+h} \int_E \psi(t-u, y) Q\bigl(\phi(x, u+\delta_{x,\varepsilon}), \mathrm{d}y\bigr) e^{-\Lambda(x+\varepsilon, u)}\,\mathrm{d}u\\ &\quad= \int_{\delta_{x,\varepsilon}}^{t+h+\delta_{x,\varepsilon}} \int_E \psi(t-u-\delta_{x,\varepsilon}, y) Q\bigl(\phi(x, u), \mathrm{d}y\bigr) e^{-\Lambda(x+\varepsilon, u-\delta_{x,\varepsilon})}\,\mathrm{d}u.\end{aligned}$$

By dominated convergence, continuity of ψ and of Λ, we deduce that the last term converges to $\int_0^t \int_E \psi(t-u, y) Q(\phi(x, u), \mathrm{d}y) e^{-\Lambda(x,u)}\,\mathrm{d}u$ when $(\varepsilon, h) \to 0$. In particular, using this and the continuity of ϕ, of f and of Λ, we deduce that $G\psi(x, t)$ is indeed continuous in (x, t), which concludes the proof of the first part of Proposition 14.

(2) *T maps $C_0(E)$ to itself.* We assume that $s(0+) = -\infty$, that $\sup_{x \in E} Q(x, E) < +\infty$ and that, for all $M > 0$, $\limsup_{x \to +\infty} Q(x, (0, M) \cup \{\partial\}) = \limsup_{x \to 0} Q(x, \{\partial\}) = 0$.

Let $f \in C_0(E)$, fix $\varepsilon > 0$, and let n_0 be large enough that $\sup_{x \in (0, 1/n_0) \cup (n_0, +\infty)} f(x) \le \varepsilon$.

Denoting by $T_1 < T_2 < \cdots$ the successive jump times of X, we deduce from the boundedness of $Q(\cdot, E)$, that, for all $t \ge 0$,

$$\sup_{x \in E} \mathbb{P}(T_n \le t | X_0 = x) \xrightarrow[n \to +\infty]{} 0.$$

Fix n_1 such that $\sup_{x \in E} \mathbb{P}(T_{n_1} \le t | X_0 = x) \le \varepsilon$. Since the process X is almost-surely nondecreasing between the jumps, its law at time t on the event $T_n \le t < T_{n+1}$ stochastically dominates the nth iterate of Q, denoted by Q^n (consider that ∂ is below 0). By assumption we have $\limsup_{x \to +\infty} Q(x, (0, n_0) \cup \{\partial\}) = 0$, so that, for all $n \ge 0$, $\limsup_{x \to +\infty} Q^n(x, (0, n_0) \cup \{\partial\}) = 0$, and hence there exists $n_2 \ge 1$ such that, for all $n \in \{0, \dots, n_1\}$,

$$\sup_{x \ge n_2} \mathbb{P}(X_t < n_0 T_n \le t \le T_{n+1} | X_0 = x) \le \varepsilon/(n_1 + 1).$$

In particular,

$$\sup_{x\geq n_2}\mathbb{P}(X_t<n_0|X_0=x)\leq \sup_{x\geq n_2}\sum_{n=0}^{n_1}\mathbb{P}(X_t<n_0T_n\leq t<T_{n+1}|X_0=x) + \sup_{x\geq n_2}\mathbb{P}(T_{n_1}\leq t|X_0=x)\leq 2\varepsilon.$$

As a consequence,

$$\sup_{x\geq n_2}\mathbb{E}\big(f(X_t)|X_0=x\big)\leq 2\varepsilon\|f\|_\infty+\varepsilon.$$

Since the existence of n_2 is true for any fixed $\varepsilon>0$, we deduce that

$$\mathbb{E}\big(f(X_t)|X_0=x\big)\xrightarrow[x\to+\infty]{}0.$$

Now, since $s(x)\xrightarrow[x\to 0]{}-\infty$, we deduce that $\phi(x,t)\to 0$ when $x\to 0$. Since $X_t\leq\phi(x,t)$ or $X_t=\partial$ almost surely when it starts from x at time 0, we deduce that, if f vanishes at 0, then so does $T_tf(x)=\mathbb{E}(f(X_t)\mathbf{1}_{X_t\neq\partial}|X_0=x)$ when $x\to 0$. Moreover, the jumping rate from y to ∂ goes to 0 when $y\to 0$, so that $\mathbb{P}(X_t=\partial|X_0=x)\to 0$ when $x\to 0$. Finally, we deduce that $T_tf(x)\to 0$ when $x\to 0$ or $x\to+\infty$.

We conclude that T_t maps the space of continuous functions vanishing at 0 and infinity to itself.

(3) *T is strongly continuous.* We proceed under the same assumptions as in step (2). Let f be in the space of continuous functions vanishing at 0 and infinity. Fix $\varepsilon>0$. Since $Q(\cdot,E)$ is uniformly bounded, say by a constant C, then the probability that the process has no jumps between times 0 and t is larger than e^{-tC}, for any $t\geq 0$. Hence

(86) $$\big|T_tf(x)-f\big(\phi(x,t)\big)\big|\leq 1-e^{-tC}\|f\|_\infty.$$

Since f vanishes at infinity, there exists n_3 large enough so that $f(x)\leq\varepsilon$ for all $x\geq n_3$ or $x<1/n_3$. Since we have $\phi(x,t)\geq x$ for all starting position $x\geq n_3$ and $t\geq 0$, we deduce that

(87) $$\sup_{x\geq n'}\big|f\big(\phi(x,t)\big)-f(x)\big|\leq 2\sup_{x\geq n'}\big|f(x)\big|\leq 2\varepsilon.$$

Similarly, $\phi(x,t)\leq\phi(1/(n_3+1),t)$ for all starting position $x\leq 1/(n_3+1)$. Since $\phi(1/(n_3+1),t)\to 1/(n_3+1)$ when $t\to 0$, there exists $t_0>0$ such that $\phi(x,t)\leq 1/n_3$ for all $x\leq 1/(n_3+1)$ and $t\in[0,t_0]$. We deduce that

(88) $$\sup_{x\leq 1/(n_3+1)}\big|f\big(\phi(x,t)\big)-f(x)\big|\leq 2\varepsilon.$$

Finally, $\phi(x,t)$ converges to x when $t\to 0$, uniformly on compact sets and f is uniformly continuous on $[1/(n_3+1),n_3]$, so that there exists $t_1>0$ such that

$$\sup_{x\in[1/(n_3+1),n_3]}\big|f\big(\phi(x,t)\big)-f(x)\big|\leq\varepsilon,\quad \forall t\leq t_1.$$

Using the last equation and inequalities (86), (87) and (88), we deduce that there exists $t_2>0$ such that, for all $t\leq t_2$, (note that the case $x=\partial$ is trivial)

$$\sup_{x\in E}\big|T_tf(x)-f(x)\big|\leq 3\varepsilon.$$

Since this is true for any $\varepsilon>0$, we deduce that T_tf converges to f in the uniform topology. This means that T is a strongly continuous semigroup on $C_0(E)$ and concludes the proof of Proposition 14. □

In the following result, we characterize the infinitesimal generator of X when its semigroup is Feller. We recall that, if $s^{-1} : (-\infty, +\infty) \to (0, +\infty)$, then, given a function $f : (0, +\infty) \to \mathbb{R}$ such that $f \circ s^{-1}$ is absolutely continuous, the function $f \circ s^{-1}$ is λ_1-almost everywhere differentiable and that, for any function g equals to this derivative λ_1-almost everywhere, we have

$$f \circ s^{-1}(t) - f \circ s^{-1}(u) = \int_u^t g(v)\,\mathrm{d}v, \quad \forall u \le t \in (-\infty, +\infty).$$

One easily checks that, as a consequence, f is differentiable with respect to s, $\lambda_1 \circ s$-almost everywhere, with derivative $h = g \circ s$, and that

$$f(y) - f(x) = \int_x^y h(z)\,\mathrm{d}s(z), \quad \forall x \le y \in (0, +\infty).$$

In this case, we will say that f is s-absolutely continuous and that h is a s-derivative of f. We consider the domain $\mathcal{D}(\mathcal{U})$, defined as the set functions $f : E \to \mathbb{R}$ such that $f|_{(0,+\infty)}$ is an s-absolutely continuous function admitting a s-derivative h such that $x \mapsto h(x) + \int_{(0,x)} (f(y) - f(x))Q(x, \mathrm{d}y)$ is an element of $C_0(E)$, where we set $\frac{\partial f}{\partial s}(\partial) = 0$. We also define the operator $\mathcal{U} : \mathcal{D}(\mathcal{U}) \to C_0(E)$ by

$$\mathcal{U}f(x) = \frac{\partial f}{\partial s}(x) + \int_{(0,x)} \bigl(f(y) - f(x)\bigr)Q(x, \mathrm{d}y), \quad \forall x \in E,$$

where $\frac{\partial f}{\partial s}$ is the s-derivative of f extended with $\frac{\partial f}{\partial s}(\partial) = 0$ and such that $x \in E \mapsto \frac{\partial f}{\partial s}(x) + \int_{(0,x)} (f(y) - f(x))Q(x, \mathrm{d}y) \in C_0(E)$.

PROPOSITION 15. *Assumes that $s(0+) = -\infty$, that $\sup_{x \in E} Q(x, E) < +\infty$ and that, for all $M > 0$, we have $\limsup_{x \to +\infty} Q(x, (0, M) \cup \{\partial\}) = \limsup_{x \to 0} Q(x, \{\partial\}) = 0$. Then the infinitesimal generator of the semigroup T of X acting on $C_0(E)$ is given by $(\mathcal{U}, \mathcal{D}(\mathcal{U}))$. Moreover, for all bounded $f : E \to \mathbb{R}$ such that $f|_{(0,+\infty)}$ is s-absolutely continuous, denoting by $\partial f/\partial s$ any s-derivative of $f|_{(0,+\infty)}$ extended with $\frac{\partial f}{\partial s}(\partial) = 0$,*

$$M_t^f := f(X_t) - f(x) - \int_0^t Q_u \widetilde{\mathcal{U}} f(X_s)\,\mathrm{d}s,$$

with

$$\widetilde{\mathcal{U}} f(y) := \frac{\partial f}{\partial s}(y) + \int_{(0,x)} \bigl(f(y) - f(x)\bigr)Q(x, \mathrm{d}y),$$

defines a local martingale under $\mathbb{P}_x$, for any $x \in E$.

PROOF. Let us denote by $\mathcal{G}$ the infinitesimal generator of T and by $\mathcal{D}(\mathcal{G})$ its domain. Our aim is to prove that $(\mathcal{G}, \mathcal{D}(\mathcal{G})) = (\mathcal{U}, \mathcal{D}(\mathcal{U}))$.

We make use of the fact that the proof of Theorem 26.14 in [22] adapts directly to our situation where the flow ϕ is generated by s on $(0, +\infty)$ and by 0 on ∂, instead of a Lipschitz flow $\mathcal{X}$ on $\mathbb{R}^d$. The only adaptation lies in the fact that, between two successive jumps, say at times T_{i-1} and T_i, and for any function f such that $f|_{(0,+\infty)}$ is s-absolutely continuous, we have (for the second equality, recall that $\frac{\mathrm{d}f}{\mathrm{d}u}(\phi(y, u)) = \frac{\partial f}{\partial s}(\phi(y, u))$ as soon as the derivative is well defined) if $X_{T_{i-1}} \in (0, +\infty)$

$$\begin{aligned} f(X_{T_i -}) - f(X_{T_{i-1}}) &= \int_0^{T_i - T_{i-1}} \frac{\mathrm{d}f}{\mathrm{d}u}\bigl(\phi(X_{T_{i-1}}, u)\bigr)\,\mathrm{d}u \\ &= \int_0^{T_i - T_{i-1}} \frac{\partial f}{\partial s}\bigl(\phi(X_{T_{i-1}}, u)\bigr)\,\mathrm{d}u \end{aligned}$$

$$= \int_{T_{i-1}}^{T_i} \frac{\partial f}{\partial s}(X_v)\,\mathrm{d}v,$$

instead of $f(X_{T_i-}) - f(X_{T_{i-1}}) = \int_{T_{i-1}}^{T_i} \mathcal{X} f(X_v)\,\mathrm{d}v$ in [22], while $f(X_t) - f(X_{T_{i-1}}) = 0$ for all $t \geq T_{i-1}$ if $X_{T_{i-1}} = \partial$.

In particular, this result implies that any bounded $f : E \to \mathbb{R}$ such that $f|_{(0,+\infty)}$ is s-absolutely continuous is in the domain of the extended infinitesimal generator of X, say $\mathcal{U}'$, and that $\mathcal{U}' f(x) = \frac{\partial f}{\partial s}(x) + \int_{(0,x)} (f(y) - f(x)) Q(x, \mathrm{d}y)$, for any s-derivative $\partial f/\partial s$ of s (note that Conditions 2. and 3. of Theorem 26.14 in [22] are trivially satisfied in our case, respectively because the boundary of the domain is not reached and because the number of jumps is finite in any finite time horizon almost surely). This proves that M^f is a local martingale under $\mathbb{P}_x$, for all $x \in E$.

In particular, given $f \in \mathcal{D}(\mathcal{U})$, the stochastic process defined, for all $t \geq 0$, by

$$M_t^f = f(X_t) - f(x) - \int_0^t \mathcal{U} f(X_u)\,\mathrm{d}u$$

is a local martingale under $\mathbb{P}_x$, for all $x \in E$. Since f and $\mathcal{U} f$ are bounded and M^f is càdlàg, we deduce that it is a martingale and thus, taking the expectation, we obtain

$$\frac{T_t f(x) - f(x)}{t} = \frac{1}{t} \int_0^t T_u \mathcal{U} f(x)\,\mathrm{d}u, \qquad \forall x \in E.$$

Moreover, since T is strongly continuous on $C_0(E)$ by Proposition 14 and since $\mathcal{U} f \in C_0(E)$ by assumption, for all $x \in E$,

$$\left| \frac{1}{t} \int_0^t T_u \mathcal{U} f(x)\,\mathrm{d}u - \mathcal{U} f(x) \right| \leq \frac{1}{t} \int_0^t \| T_u \mathcal{U} f - \mathcal{U} f \|_\infty \,\mathrm{d}u \xrightarrow[t \to 0]{} 0.$$

We conclude that, for any $f \in \mathcal{D}(\mathcal{U})$, we have $f \in \mathcal{D}(\mathcal{G})$ and $\mathcal{G} f = \mathcal{U} f$.

Reciprocally, assume that $f \in \mathcal{D}(\mathcal{G})$. Then f is in the domain of the extended infinitesimal generator of X, so that, according to Theorem 26.14 in [22], $f|_{(0,+\infty)}$ is s-absolutely continuous and

$$M_t = f(X_t) - f(x) - \int_0^t \left[\frac{\partial f}{\partial s}(X_u) + \int_{(0,X_u)} \big(f(y) - f(X_u)\big) Q(X_u, \mathrm{d}y) \right] \mathrm{d}u$$

is a local martingale under $\mathbb{P}_x$ for all $x \in E$, where $\frac{\partial f}{\partial s}$ is an s-derivative of $f|_{(0,+\infty)}$ extended by $\frac{\partial f}{\partial s}(\partial) = 0$. Moreover, denoting by $\mathcal{G}$ the infinitesimal generator of X, we have that

$$M'_t = f(X_t) - f(x) - \int_0^t \mathcal{G} f(X_u)\,\mathrm{d}u$$

is a martingale under $\mathbb{P}_x$. In particular, $M_t - M'_t$ is a continuous local martingale with bounded total variation and hence it is constant $\mathbb{P}_x$-almost surely. We deduce that, $\mathbb{P}_x$-almost surely,

$$\frac{\partial f}{\partial s}(X_u) + \int_{(0,X_u)} \big(f(y) - f(X_u)\big) Q(X_u, \mathrm{d}y) = \mathcal{G} f(X_u), \quad \lambda_1(\mathrm{d}u)\text{-almost everywhere.}$$

Since the jump rate Q is bounded, we know that, for any $t > 0$, with positive probability, $X_u = \phi(x, u)$ for all $u \in [0, t]$. This and the previous equality entails that $\frac{\partial f}{\partial s}(z) + \int_{(0,z)} (f(y) - f(z)) Q(z, \mathrm{d}y)$ equals $\mathcal{G} f(z)$ for $\lambda_1 \circ s$-almost every $z \geq x$. Since this is true for all $x > 0$, we deduce that, up to a modification of $\partial f/\partial s$ on a $\lambda_1 \circ s$-negligible set, $z \mapsto \frac{\partial f}{\partial s}(z) + \int_{(0,z)} (f(y) - f(z)) Q(z, \mathrm{d}y) = \mathcal{G} f(z) \in C_0(E)$, so that $f \in \mathcal{D}(\mathcal{U})$ and $\mathcal{G} f = \mathcal{U} f$. $\square$

We conclude this appendix by two results on the uniqueness of the martingale problem for compactly supported and/or regular functions. Here uniqueness refers to the uniqueness of the finite-dimensional distributions. In particular, it entails that two càdlàg solutions to the martingale problem are indistinguishable.

PROPOSITION 16. *Take the assumptions of the previous proposition. Let D be the space of compactly supported functions $f : E \to \mathbb{R}$ such that $f|_{(0,+\infty)}$ is s-absolutely continuous and such that $\partial f/\partial s$ is bounded, with the extension $\frac{\partial f}{\partial s}(\partial) = 0$. Then the $(\mathcal{U}, D)$ martingale problem is well posed, and its unique solution is the Markov process X.*

PROOF. Note that X is a solution to the $(\mathcal{U}, D)$ martingale problem, so that the problem admits at least one solution.

Assume now that Y is a solution to the $(\mathcal{U}, D)$ martingale problem. Then, for all $h \in D$ and all $x \in E$,

$$h(Y_t) - h(Y_0) - \int_0^t \left[\frac{\partial h}{\partial s}(Y_u) + \int_{(0,Y_u)} h(y)Q(Y_u, \mathrm{d}y) - h(Y_u)Q(Y_u, E)\right] \mathrm{d}u$$

is a $\mathbb{P}_x$-martingale. Let $f \in \mathcal{D}(\mathcal{U})$ such that $f(1) = 0$. Note that $\partial f/\partial s$ is bounded. For all $n \geq 2$, let h_n be the s-absolutely continuous compactly supported function defined by

$$h_n(x) = \begin{cases} f(\partial) & \text{if } x = \partial, \\ \displaystyle\int_1^x \frac{\partial f}{\partial s}(y) s(\mathrm{d}y) & \text{if } x \in \left(\frac{1}{n}, n\right), \\ \big(f(n) - s(x) + s(n)\big)_+ & \text{if } x \geq n \text{ and } f(n) \geq 0, \\ -\big(f(n) + s(x) - s(n)\big)_- & \text{if } x \geq n \text{ and } f(n) \leq 0, \\ \left(f\left(\frac{1}{n}\right) + s(x) - s\left(\frac{1}{n}\right)\right)_+ & \text{if } x \leq \frac{1}{n} \text{ and } f\left(\frac{1}{n}\right) \geq 0, \\ -\left(f\left(\frac{1}{n}\right) + s(n) - s(x)\right)_- & \text{if } x \leq \frac{1}{n} \text{ and } f\left(\frac{1}{n}\right) \leq 0. \end{cases}$$

Then h_n is bounded by $\|f\|_\infty$ and $h_n(x)$ converges toward $f(x)$ for all $x \in E$. Moreover, $\partial h_n/\partial s$ is bounded by $\|\partial f/\partial s\|_\infty \vee 1$ and $\partial h_n/\partial s(x)$ converges toward $\partial f/\partial s(x)$ for all $x \in E$, with the extension $\partial h_n/\partial s(\partial) = 0$. Finally, since $Q(x, \cdot)$ is a bounded measure and h_n is uniformly bounded in n, $Q(\cdot, h_n) - h_n(\cdot)Q(\cdot, E)$ is bounded and, by dominated convergence, $Q(x, h_n) - h_n(x)Q(x, E)$ converges toward $Q(x, f) - f(x)Q(x, E)$ for all $x \in E$. We deduce that $(h_n, \mathcal{U}h_n)$ converges toward $(f, \mathcal{U}f)$ in the bounded point-wise sense, and hence that $f(Y_t) - f(x) - \int_0^t \mathcal{U}f(Y_u)\,\mathrm{d}u$ is a martingale. If $f(1) \neq 0$, then one derives the same result by considering the function $f - f(1)$.

Since this is true for all $f \in \mathcal{D}(\mathcal{U})$, we deduce that Y satisfies the $(\mathcal{U}, \mathcal{D}(\mathcal{U}))$ martingale problem (see, for instance, Proposition 4.3.1 in [26]). But $(\mathcal{U}, \mathcal{D}(\mathcal{U}))$ is the infinitesimal generator of the strongly continuous semigroup T, and hence its martingale problem is well-posed (this is a consequence of Hille–Yosida Theorem 1.2.6 and Theorem 4.4.1 in [26]). As a consequence the finite dimensional laws of X and Y are the same, which concludes the proof of Proposition 16. □

PROPOSITION 17. *Take the assumptions of the previous proposition. Let D' be the space of functions $f \in D$, such that $\partial f/\partial s$ is continuous,*[1] *with the extension $\partial f/\partial s(\partial) = 0$. Then the $(\mathcal{U}, D')$ martingale problem is well posed, and its unique solution is the Markov process X.*

[1] The proof still holds true under stronger regularity conditions on $\partial f/\partial s$.

PROOF. Similar to the proof of the previous proposition, we know that X is a solution to the $(\mathcal{U}, D')$ martingale problem, and our aim is to show the uniqueness of the solution by a density argument.

Assume that Y is a solution to the $(\mathcal{U}, D')$ martingale problem and let S be its semigroup, so that, for all $h \in D'$,

$$S_t h(x) = h(x) + \int_0^t S_u \frac{\partial h}{\partial s}(x)\,\mathrm{d}u + \int_0^t S_u Q(\cdot, h)(x)\,\mathrm{d}u$$
$$- h(x) \int_0^t S_u Q(\cdot, E)(x)\,\mathrm{d}u, \quad \forall x \in E.$$

Let $f \in D$ (where D is defined in Proposition 16) and denote by $[a, b] \subset (0, +\infty)$, $a < b$, a compact interval containing the support of $f|_{(0,+\infty)}$. Fix $t > 0$ and let g_n be a bounded sequence of continuous functions with support in $[a/2, 2b] \cup \{\partial\}$ such that $g_n \to \partial f/\partial s$ in $L^1(\mu_t + \lambda_1)$, where

$$\mu_t(A) := \int_0^t S_u \mathbf{1}_A\,\mathrm{d}u, \quad \text{for all measurable set } A \subset (0, +\infty) \cup \{\partial\}.$$

Defining $h_n(x) = \int_a^x g_n(y)\,\mathrm{d}y$, we observe that $(h_n)_{n\in\mathbb{N}}$ is a bounded sequence in D' such that $h_n(x) \to f(x)$ when $n \to +\infty$, for all $x > 0$. In particular, using the fact that $A \mapsto \int_0^t S_u Q(\cdot, A)(x)\,\mathrm{d}u$ defines a bounded measure and by dominated convergence,

$$\int_0^t S_u Q(\cdot, h_n)(x)\,\mathrm{d}u \xrightarrow[n\to+\infty]{} \int_0^t S_u Q(\cdot, f)(x)\,\mathrm{d}u.$$

Similarly, $S_t h_n(x) \to S_t f(x)$ when $n \to +\infty$, for all $x > 0$. Moreover, $\int_0^t S_u \frac{\partial h_n}{\partial s}(x)\,\mathrm{d}u = \int_0^t S_u g_n(x)\,\mathrm{d}u = \mu_t(g_n)$ converges to $\mu_t(\partial f/\partial s) = \int_0^t S_u \frac{\partial h}{\partial s}(x)\,\mathrm{d}u$ when $n \to +\infty$. Finally, we proved that S satisfies

$$S_t f(x) = f(x) + \int_0^t S_u \frac{\partial f}{\partial s}(x)\,\mathrm{d}u + \int_0^t S_u Q(\cdot, f)(x)\,\mathrm{d}u$$
$$- f(x) \int_0^t S_u Q(\cdot, E)(x)\,\mathrm{d}u, \quad \forall f \in D.$$

This implies that Y satisfies the $(\mathcal{U}, D)$ martingale problem, and hence, according to Proposition 16, that the finite-dimensional laws of Y and X are the same. This concludes the proof of Proposition 17. □

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