Potentials of stable processes

A. E. Kyprianou and A. R. Watson

Abstract For a stable process, we give an explicit formula for the potential measure of the process killed outside a bounded interval and the joint law of the overshoot, undershoot and undershoot from the maximum at exit from a bounded interval. We obtain the equivalent quantities for a stable process reflected in its infimum. The results are obtained by exploiting a simple connection with the Lamperti representation and exit problems of stable processes.

1 Introduction and results

For a Lévy process X, the measure

$$U^A(x, \mathrm{d}y) = \mathrm{E}_x \int_0^\infty \mathbb{1}[X_t \in \mathrm{d}y] \mathbb{1}[\forall s \le t : X_s \in A] \, \mathrm{d}t,$$

called the *potential* (or *resolvent*) *measure of X killed outside A*, is a quantity of great interest, and is related to exit problems.

The main cases where the potential measure can be computed explicitly are as follows. If X is a Lévy process with known Wiener–Hopf factors, it can be obtained when A is half-line or $\mathbb{R}$; see [2, Theorem VI.20]. When X is a totally asymmetric

A. E. Kyprianou
University of Bath, UK, a.kyprianou@bath.ac.uk

A. R. Watson
University of Zürich, Switzerland, alexander.watson@math.uzh.ch. Part of this work was done while the second author was at the University of Bath, UK, and at CIMAT, Mexico.

AMS 2000: 60G52, 60G18, 60G51.

Keywords and phrases: Lévy processes, stable processes, reflected stable processes, hitting times, positive self-similar Markov processes, Lamperti representation, potential measures, resolvent measures

Lévy process with known scale functions, it can be obtained for A a bounded interval, a half-line or $\mathbb{R}$; see [10, Section 8.4]. Finally, [1] details a technique to obtain a potential measure for a reflected Lévy process killed outside a bounded interval from the same quantity for the unreflected process.

In this note, we consider the case where X is a stable process and A is a bounded interval. We compute the measure $U^{[0,1]}$, from which U^A may be obtained for any bounded interval A via spatial homogeneity and scaling; and from this we compute the joint law at first exit of $[0,1]$ of the overshoot, undershoot and undershoot from the maximum. Furthermore, we give the potential measure and triple law also for the process reflected in its infimum.

The potential measure has been previously been computed when X is symmetric; see Blumenthal et al. [4, Corollary 4] and references therein, as well as Baurdoux [1]. We extend these results to asymmetric stable processes with jumps on both sides. The essential observation is that a potential for X with killing outside a bounded interval may be converted into a potential for the *Lamperti transform of* X, say ξ, with killing outside a half-line. To compute this potential in a half-line, it is enough to know the killing rate of ξ and the solution of certain exit problems for X. The results for the reflected process are obtained via the work of Baurdoux [1].

We now give our results. Some facts we will rely on are summarised in section 2, and proofs are given in section 3.

We work with the (strictly) stable process X with scaling parameter α and positivity parameter ρ, which is defined as follows. For (α,ρ) in the set

$$\begin{aligned}\mathcal{A} &= \{(\alpha,\rho) : \alpha \in (0,1),\ \rho \in (0,1)\} \cup \{(\alpha,\rho) = (1,1/2)\} \\ &\quad \cup \{(\alpha,\rho) : \alpha \in (1,2),\ \rho \in (1-1/\alpha, 1/\alpha)\},\end{aligned}$$

let X, with probability laws $(\mathrm{P}_x)_{x\in\mathbb{R}}$, be the Lévy process with characteristic exponent

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha (1 - \mathrm{i}\beta \tan\frac{\pi\alpha}{2}\operatorname{sgn}\theta) & \alpha \in (0,2)\setminus\{1\}, \\ c|\theta| & \alpha = 1, \end{cases} \qquad \theta \in \mathbb{R},$$

where $c = \cos(\pi\alpha(\rho - 1/2))$ and $\beta = \tan(\pi\alpha(\rho-1/2))/\tan(\pi\alpha/2)$. This Lévy process has absolutely continuous Lévy measure with density

$$c_+ x^{-(\alpha+1)} \mathbb{1}[x>0] + c_-|x|^{-(\alpha+1)}\mathbb{1}[x<0], \qquad x \in \mathbb{R},$$

where

$$c_+ = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}, \qquad c_- = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$$

and $\hat{\rho} = 1-\rho$.

The parameter set $\mathcal{A}$ and the characteristic exponent Ψ represent, up a multiplicative constant in Ψ, all (strictly) stable processes which jump in both directions, except for Brownian motion and the symmetric Cauchy processes with non-zero drift. The normalisation is the same as that in [8], and when X is symmetric, that is when $\rho = 1/2$, the normalisation agrees with that of [4]. We remark that the quantities we are interested in can also be derived in cases of one-sided jumps: either X is a subordinator, in which case the results are trivial, or X is a spectrally one-sided Lévy process, in which case the potentials in question may be assembled using the theory of scale functions; see [10, Theorem 8.7 and Exercise 8.2].

The choice α and ρ as parameters is explained as follows. X satisfies the α-*scaling property*, that

$$\text{under } \mathrm{P}_x, \text{ the law of } (cX_{tc^{-\alpha}})_{t\ge 0} \text{ is } \mathrm{P}_{cx}, \tag{1}$$

for all $x \in \mathbb{R}$, $c > 0$. The second parameter satisfies $\rho = \mathrm{P}_0(X_t > 0)$.

Having defined the stable process, we proceed to our results. Let

$$\sigma^{[0,1]} = \inf\{t \ge 0 : X_t \notin [0,1]\},$$

and define the killed potential measure and potential density

$$U_1(x,\mathrm{d}y) := U^{[0,1]}(x,\mathrm{d}y) = \mathrm{E}_x \int_0^{\sigma^{[0,1]}} \mathbb{1}[X_t \in \mathrm{d}y]\,\mathrm{d}t = u_1(x,y)\,\mathrm{d}y,$$

provided the density u_1 exists.

Theorem 1. *For* $0 < x, y < 1$,

$$u_1(x,y) = \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(x-y)^{\alpha-1}\displaystyle\int_0^{\frac{y(1-x)}{x-y}} s^{\alpha\rho-1}(s+1)^{\alpha\hat\rho-1}\,\mathrm{d}s, & y < x,\\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y-x)^{\alpha-1}\displaystyle\int_0^{\frac{x(1-y)}{y-x}} s^{\alpha\hat\rho-1}(s+1)^{\alpha\rho-1}\,\mathrm{d}s, & x < y. \end{cases}$$

Part of the claim of this theorem is that $u_1(x,y)$ exists and is finite on the domain given; this will also be the case in the coming results, and so we will not remark on it again. When X is symmetric, the theorem reduces, by spatial homogeneity and scaling of X and substituting in the integral, to [4, Corollary 4].

With very little extra work, Theorem 1 yields an apparently stronger result. Let

$$\tau_0^- = \inf\{t \ge 0 : X_t < 0\}; \qquad \overline{X}_t = \sup_{s\le t} X_s, \qquad t \ge 0,$$

and write

$$\mathrm{E}_x \int_0^{\tau_0^-} \mathbb{1}[X_t \in \mathrm{d}y, \overline{X}_t \in \mathrm{d}z]\,\mathrm{d}t = u(x,y,z)\,\mathrm{d}y\,\mathrm{d}z,$$

if the right-hand side exists. Then we have the following.

Corollary 2. *For* $x > 0$, $y \in [0,z)$, $z > x$,

$$u(x,y,z) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)} x^{\alpha\hat\rho} y^{\alpha\rho} \frac{(z-x)^{\alpha\rho-1}(z-y)^{\alpha\hat\rho-1}}{z^{\alpha}}\,\mathrm{d}y\,\mathrm{d}z. \tag{2}$$

Proof. Rescaling, we obtain

$$\mathrm{E}_x \int_0^{\tau_0^-} \mathbb{1}[X_t \in \mathrm{d}y,\, \overline{X}_t \leq z]\,\mathrm{d}t = z^{\alpha-1} u_1(x/z, y/z),$$

and the density is found by differentiating the right-hand side in z. ◻

From this density, one may recover the following hitting distribution, which originally appeared in Kyprianou et al. [9, Corollary 15]. Let

$$\tau_1^+ = \inf\{t \geq 0 : X_t > 1\}.$$

Corollary 3. *For* $u \in [0, 1-x)$*,* $v \in (u, 1]$*,* $y \geq 0$*,*

$$\begin{aligned}
&\mathrm{P}_x(1 - \overline{X}_{\tau_1^+-} \in \mathrm{d}u,\, 1 - X_{\tau_1^+-} \in \mathrm{d}v,\, X_{\tau_1^+} - 1 \in \mathrm{d}y,\, \tau_1^+ < \tau_0^-) \\
&\quad = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \frac{x^{\alpha\hat{\rho}}(1-v)^{\alpha\rho}(1-u-x)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(1-u)^{\alpha}(v+y)^{\alpha+1}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y. \qquad (3)
\end{aligned}$$

Proof. Following the proof of [2, Proposition III.2], one may show that the left-hand side of (3) is equal to $u(x, 1-v, 1-u)\pi(v+y)$, where π is the Lévy density of X. ◻

Remark 4. The proof of Corollary 3 suggests an alternative derivation of Theorem 1. Since the identity (3) is known, one may deduce $u(x, y, z)$ from it by following the proof backwards. The potential $u_1(x, y)$ without $\overline{X}$ may then be obtained via integration. However, in section 3 we offer instead a self-contained proof based on well-known hitting distributions for the stable process.

Now let Y denote the stable process X reflected in its infimum, that is,

$$Y_t = X_t - \underline{X}_t, \qquad t \geq 0,$$

where $\underline{X}_t = \inf\{X_s, 0 \leq s \leq t\} \wedge 0$ for $t \geq 0$. Y is a self-similar Markov process.

Let $T_1^+ = \inf\{t > 0 : Y_t > 1\}$ denote the first passage time of Y above the level 1, and define

$$R_1(x, \mathrm{d}y) = \mathrm{E}_x \int_0^{T_1^+} \mathbb{1}[Y_t \in \mathrm{d}y]\,\mathrm{d}t = r_1(x, y)\,\mathrm{d}y,$$

where the density r_1 exists by [1, Theorem 4.1]. Note that, as Y is self-similar, R_1 suffices to deduce the potential of Y killed at first passage above any level.

Theorem 5. *For* $0 < y < 1$*,*

$$r_1(0, y) = \frac{1}{\Gamma(\alpha)} y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}}.$$

Hence, for $0 < x, y < 1$*,*

$$r_1(x,y) = \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\Big[(x-y)^{\alpha-1}\displaystyle\int_0^{\frac{y(1-x)}{x-y}} s^{\alpha\rho-1}(s+1)^{\alpha\hat{\rho}-1}\,\mathrm{d}s \\ \qquad\qquad + y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}}\displaystyle\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\,\mathrm{d}t\Big], \ y < x, \\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\Big[(y-x)^{\alpha-1}\displaystyle\int_0^{\frac{x(1-y)}{y-x}} s^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1}\,\mathrm{d}s \\ \qquad\qquad + y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}}\displaystyle\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\,\mathrm{d}t\Big], \ x < y. \end{cases}$$

Writing

$$\mathrm{E}_x \int_0^\infty \mathbb{1}[Y_t \in \mathrm{d}y, \overline{Y}_t \in \mathrm{d}z]\,\mathrm{d}t = r(x,y,z)\,\mathrm{d}y\,\mathrm{d}z,$$

where $\overline{Y}_t$ is the supremum of Y up to time t, we obtain the following corollary, much as we had for X.

Corollary 6. *For $y \in (0,z)$, $z \geq 0$,*

$$r(0,y,z) = \frac{\alpha\hat{\rho}}{\Gamma(\alpha)} y^{\alpha\rho-1}(z-y)^{\alpha\hat{\rho}-1},$$

and for $x > 0$, $y \in (0,z)$, $z \geq x$,

$$r(x,y,z) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} y^{\alpha\rho-1}(z-y)^{\alpha\hat{\rho}-1}\Big[x^{\alpha\hat{\rho}}(z-x)^{\alpha\rho-1}z^{1-\alpha} + \alpha\hat{\rho}\int_0^{1-\frac{x}{z}} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\,\mathrm{d}t\Big].$$

We also have the following corollary, which is the analogue of Corollary 3.

Corollary 7. *For $u \in (0,1]$, $v \in (u,1)$, $y \geq 0$,*

$$\mathrm{P}_0(1-\overline{Y}_{T_1^+} \in \mathrm{d}u,\, 1-Y_{T_1^+} \in \mathrm{d}v,\, Y_{T_1^+}-1 \in \mathrm{d}y) = \frac{\alpha\cdot\alpha\hat{\rho}}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \frac{(1-v)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(v+y)^{\alpha+1}}\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y,$$

and for $x \geq 0$, $u \in [0,1-x)$, $v \in (u,1)$, $y \geq 0$,

$$\begin{aligned}\mathrm{P}_x&(1-\overline{Y}_{T_1^+} \in \mathrm{d}u,\, 1-Y_{T_1^+} \in \mathrm{d}v,\, Y_{T_1^+}-1 \in \mathrm{d}y) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \frac{(1-v)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(v+y)^{\alpha+1}} \\ &\quad\times\Big[x^{\alpha\hat{\rho}}(1-u-x)^{\alpha\rho-1}(1-u)^{1-\alpha} + \alpha\hat{\rho}\int_0^{1-\frac{x}{1-u}} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\,\mathrm{d}t\Big]\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y.\end{aligned}$$

The marginal in $\mathrm{d}v\,\mathrm{d}y$ appears in Baurdoux [1, Corollary 3.5] for the case where X is symmetric and $x = 0$. The marginal in $\mathrm{d}y$ is given in Kyprianou [11] for the process reflected in the supremum; this corresponds to swapping ρ and $\hat\rho$. However, unless $x = 0$, it appears to be difficult to integrate in Corollary 7 and obtain the expression found in [11].

Finally, one may integrate in Theorem 5 and obtain the expected first passage time for the reflected process.

Corollary 8. *For $x \ge 0$,*

$$\mathrm{E}_x[T_1^+] = \frac{1}{\Gamma(\alpha+1)}\left[x^{\alpha\hat\rho}(1-x)^{\alpha\rho} + \alpha\hat\rho\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat\rho-1}\,\mathrm{d}t\right].$$

In particular,

$$\mathrm{E}_0[T_1^+] = \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho+1)}{\Gamma(\alpha+1)}.$$

2 The Lamperti representation

We will calculate potentials related to X by appealing to the Lamperti transform [12, 15]. Recall that a process Y with probability measures $(\mathrm{P}_x)_{x>0}$ is a *positive self-similar Markov process (pssMp)* if it is a standard Markov process (in the sense of [3]) with state space $[0,\infty)$ which has zero as an absorbing state and satisfies the scaling property:

$$\text{under } \mathrm{P}_x, \text{ the law of } (cY_{tc^{-\alpha}})_{t\ge 0} \text{ is } \mathrm{P}_{cx},$$

for all x, $c > 0$.

The Lamperti transform gives a correspondence between pssMps and killed Lévy processes, as follows. Let $S(t) = \int_0^t (Y_u)^{-\alpha}\,\mathrm{d}u$; this process is continuous and strictly increasing until Y reaches zero. Let T be its inverse. Then, the process

$$\xi_s = \log Y_{T(s)}, \qquad s \ge 0$$

is a Lévy process, possibly killed at an independent exponential time, and termed the *Lamperti transform* of Y. Note that $\xi_0 = \log x$ when $Y_0 = x$, and one may easily see from the definition of S that $e^{\alpha\xi_{S(t)}}\,\mathrm{d}S(t) = \mathrm{d}t$.

A simple example of the Lamperti transform in action is given by considering the process X. Define

$$\tau_0^- = \inf\{t \ge 0 : X_t < 0\},$$

and let

$$\mathrm{P}_x^*(X_t \in \cdot) = \mathrm{P}_x(X_t \in \cdot,\, t < \tau_0^-), \qquad t \ge 0,\ x > 0.$$

The process X with laws $(\mathrm{P}_x^*)_{x>0}$ is a pssMp. Caballero and Chaumont [5] gives explicitly the generator of its Lamperti transform, whose laws we denote $(\mathbb{P}_y^*)_{y\in\mathbb{R}}$, finding in particular that it is killed at rate

$$q := c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}. \tag{4}$$

3 Proofs

To avoid the proliferation of symbols, we generally distinguish processes only by the measures associated with them; the exception is that self-similar processes will be distinguished from processes obtained by Lamperti transform. Thus, the time

$$\tau_1^+ = \inf\{t \geq 0 : X_t > 0\}$$

always refers to the canonical process of the measure it appears under, and will be used for self-similar processes; and

$$S_0^+ = \inf\{s \geq 0 : \xi_s > 0\}, \quad \text{and} \quad S_0^- = \inf\{s \geq 0 : \xi_s < 0\}$$

will likewise be used for processes obtained by Lamperti transform.

Proof of Theorem 1. Our proof makes use of the pssMp (X, P^*) and its Lamperti transform $(\xi, \mathbb{P}^*)$, both defined in section 2. Let $0 < x,\ y < 1$. Then

$$\begin{aligned} U_1(x, \mathrm{d}y) &= \mathrm{E}_x \int_0^{\sigma^{[0,1]}} \mathbb{1}[X_t \in \mathrm{d}y]\,\mathrm{d}t \\ &= \mathrm{E}_x^* \int_0^{\tau_1^+} \mathbb{1}[X_t \in \mathrm{d}y]\,\mathrm{d}t, \end{aligned}$$

using nothing more than the definition of (X, P^*). We now use the Lamperti representation to relate this to $(\xi, \mathbb{P}^*)$. This process is killed at the rate q given in (4), and so it may be represented as an unkilled Lévy process $(\xi, \mathbb{P})$ which is sent to some cemetery state at the independent exponental time $\mathbf{e}_q$. The time-changes in this representation are denoted S and T, and we recall that $\mathrm{d}t = e^{\alpha S(t)}\mathrm{d}S(t)$. In the following calculation, we first use the fact that τ_1^+ (the first passage of X) under P_x is equal to $T(S_0^+)$ (the time-change of the first-passage of ξ) under $\mathbb{P}_{\log x}$. The second line follows by a time substitution in the integral (see, for example, [14, §A4].)

$$\begin{aligned} U_1(x, \mathrm{d}y) &= \mathbb{E}^*_{\log(x)} \int_0^{T(S_0^+)} \mathbb{1}\left[e^{\xi_{S(t)}} \in \mathrm{d}y\right] e^{\alpha \xi_{S(t)}}\,\mathrm{d}S(t) \\ &= y^\alpha \mathbb{E}_{\log(x)} \int_0^{S_0^+} \mathbb{1}\left[e^{\xi_s} \in \mathrm{d}y\right] \mathbb{1}[\mathbf{e}_q > s]\,\mathrm{d}s \\ &= y^\alpha \hat{\mathbb{E}}_{\log(1/x)} \int_0^{S_0^-} \mathbb{1}[\xi_s \in \log(1/\mathrm{d}y)] e^{-qs}\,\mathrm{d}s, \end{aligned}$$

where $\hat{\mathbb{E}}$ refers to the dual Lévy process $-\xi$. Examining the proof of Theorem VI.20 in Bertoin [2] reveals that, for any $a > 0$,

$$\hat{\mathbb{E}}_a \int_0^{S_0^-} \mathbb{1}[\xi_s \in \cdot\,]\, e^{-qs}\, \mathrm{d}s$$
$$= \frac{1}{q} \int_{[0,\infty)} \hat{\mathbb{P}}_0\big(\overline{\xi}_{\mathbf{e}_q} \in \mathrm{d}w\big) \int_{[0,a]} \hat{\mathbb{P}}_0\big(-\underline{\xi}_{\mathbf{e}_q} \in \mathrm{d}z\big) \mathbb{1}[a + w - z \in \cdot\,],$$

where for each $t \geq 0$, $\overline{\xi}_t = \sup\{\xi_s : s \leq t\}$ and $\underline{\xi}_t = \inf\{\xi_s : s \leq t\}$. Then, provided that the measures $\hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot)$ and $\hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot)$ possess respective densities g_S and g_I (as we will shortly see they do), it follows that for $a > 0$,

$$\hat{\mathbb{E}}_a \int_0^{S_0^-} \mathbb{1}[\xi_s \in \mathrm{d}v]\, e^{-qs}\, \mathrm{d}s = \frac{\mathrm{d}v}{q} \int_{(a-v)\vee 0}^{a} \mathrm{d}z\, g_I(-z) g_S(v - a + z).$$

We may apply this result to our potential measure U_1 in order to find its density, giving

$$u_1(x, y) = \frac{1}{q} y^{\alpha-1} \int_{\frac{y}{x}\vee 1}^{\frac{1}{x}} t^{-1} g_I(\log t^{-1}) g_S(\log(tx/y))\, \mathrm{d}t. \tag{5}$$

It remains to determine the densities g_S and g_I of the measures $\hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot)$ and $\hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot)$. These can be related to functionals of X by the Lamperti transform:

$$\begin{aligned} \hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot) &= \mathbb{P}_0(-\underline{\xi}_{\mathbf{e}_q} \in \cdot) = \mathrm{P}_1(-\log \underline{X}_{\tau_0^- -} \in \cdot) \\ \hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot) &= \mathbb{P}_0(-\overline{\xi}_{\mathbf{e}_q} \in \cdot) = \mathrm{P}_1(-\log \overline{X}_{\tau_0^-} \in \cdot). \end{aligned} \tag{6}$$

The laws of the rightmost random variables in (6) are available explicitly, as we now show. For the law of $\underline{X}_{\tau_0^- -}$, we transform it into an overshoot problem and make use of Example 7 in Doney and Kyprianou [6], as follows. We omit the calculation of the integral, which uses [7, 8.380.1].

$$\begin{aligned} \mathrm{P}_1(\underline{X}_{\tau_0^- -} \in \mathrm{d}y) &= \hat{\mathrm{P}}_0(1 - \overline{X}_{\tau_1^+ -} \in \mathrm{d}y) \\ &= K \int_y^\infty \mathrm{d}v \int_0^\infty \mathrm{d}u\, (v - y)^{\alpha\rho - 1} (v + u)^{-(\alpha+1)} (1 - y)^{\alpha\hat{\rho} - 1}\, \mathrm{d}y \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi} y^{-\alpha\hat{\rho}} (1 - y)^{\alpha\hat{\rho} - 1}\, \mathrm{d}y, \qquad y \in [0, 1]. \end{aligned} \tag{7}$$

For the law of $\overline{X}_{\tau_0^-}$, consider the following calculation.

$$\mathrm{P}_1(\overline{X}_{\tau_0^-} \geq y) = \mathrm{P}_1(\tau_y^+ < \tau_0^-) = \mathrm{P}_{1/y}(\tau_1^+ < \tau_0^-).$$

This final quantity depends on the solution of the two-sided exit problem for the stable process; it is computed in Rogozin [13], where it is denoted $f_1(1/y, \infty)$. Note that [13] contains a typographical error: in Lemma 3 of that work and the discussion after it, the roles of q (which is ρ in our notation) and $1 - q$ should be swapped. In the corrected form, we have

$$\begin{aligned}\mathrm{P}_1(\overline{X}_{\tau_0^-} \geq y) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^{1/y} u^{\alpha\hat{\rho}-1}(1-u)^{\alpha\rho-1}\,\mathrm{d}u \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_y^{\infty} t^{-\alpha}(t-1)^{\alpha\rho-1}\,\mathrm{d}t, \end{aligned} \tag{8}$$

which gives us the density for $y \geq 1$.

Since g_S and g_I possess densities on their whole support, we may substitute (7) and (8) into (5) and obtain

$$u_1(x,y) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} x^{\alpha\hat{\rho}-1} y^{\alpha\rho} \int_{\frac{y}{x}\vee 1}^{\frac{1}{x}} t^{-\alpha}(t-1)^{\alpha\rho-1}\Big(t-\frac{y}{x}\Big)^{\alpha\hat{\rho}-1}\,\mathrm{d}t,$$

for $x, y \in (0,1)$. The expression in the statement follows by a short manipulation of this integral. $\square$

Proof of Theorem 5. According to Baurdoux [1, Theorem 4.1], since X is regular upwards, we have the following formula for $r_1(0,y)$:

$$r_1(0,y) = \lim_{z\downarrow 0} \frac{u_1(z,y)}{\mathrm{P}_z(\tau_1^+ < \tau_0^-)}.$$

We have found u_1 above, and as we already mentioned, we have from Rogozin [13] that

$$\mathrm{P}_x(\tau_1^+ < \tau_0^-) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^x t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}\,\mathrm{d}t.$$

We may then make the following calculation, using l'Hôpital's rule on the second line since the integrals converge,

$$\begin{aligned} r_1(0,y) &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \lim_{z\downarrow 0} \frac{\displaystyle\int_0^{\frac{z(1-y)}{y-z}} s^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1}\,\mathrm{d}s}{\displaystyle\int_0^z t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}\,\mathrm{d}t} \\ &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \lim_{z\downarrow 0} \frac{z^{\alpha\hat{\rho}-1}(1-y)^{\alpha\hat{\rho}-1}(y-z)^{1-\alpha\hat{\rho}} \frac{\partial}{\partial z}\big[\frac{z(1-y)}{y-z}\big]}{z^{\alpha\hat{\rho}-1}\frac{\partial}{\partial z}[z]} \\ &= \frac{1}{\Gamma(\alpha)} y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}}. \end{aligned}$$

Finally, the full potential density $r_1(x,y)$ follows simply by substituting in the following formula, from the same theorem in [1]:

$$r_1(x,y) = u_1(x,y) + \mathrm{P}_x(\tau_0^- < \tau_1^+) r_1(0,y).$$

$\square$

Acknowledgements We would like to thank the referee for their careful reading of this paper.

References

[1] Baurdoux, E.J.: Some excursion calculations for reflected Lévy processes. ALEA Lat. Am. J. Probab. Math. Stat. **6**, 149–162 (2009)

[2] Bertoin, J.: Lévy processes, *Cambridge Tracts in Mathematics*, vol. 121. Cambridge University Press, Cambridge (1996)

[3] Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory. Pure and Applied Mathematics, Vol. 29. Academic Press, New York (1968)

[4] Blumenthal, R.M., Getoor, R.K., Ray, D.B.: On the distribution of first hits for the symmetric stable processes. Trans. Amer. Math. Soc. **99**, 540–554 (1961)

[5] Caballero, M.E., Chaumont, L.: Conditioned stable Lévy processes and the Lamperti representation. J. Appl. Probab. **43**(4), 967–983 (2006). doi:10.1239/jap/1165505201

[6] Doney, R.A., Kyprianou, A.E.: Overshoots and undershoots of Lévy processes. The Annals of Applied Probability **16**(1), 91–106 (2006). doi:10.1214/105051605000000647

[7] Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products, seventh edn. Elsevier/Academic Press, Amsterdam (2007). Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger

[8] Kuznetsov, A., Pardo, J.C.: Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. Acta Appl. Math. **123**, 113–139 (2013). doi:10.1007/s10440-012-9718-y

[9] Kyprianou, A.E., Pardo, J.C., Rivero, V.: Exact and asymptotic n-tuple laws at first and last passage. Ann. Appl. Probab. **20**(2), 522–564 (2010). doi:10.1214/09-AAP626

[10] Kyprianou, A.E.: Introductory lectures on fluctuations of Lévy processes with applications. Universitext. Springer-Verlag, Berlin (2006)

[11] Kyprianou, A.E.: First passage of reflected strictly stable processes. ALEA Lat. Am. J. Probab. Math. Stat. **2**, 119–123 (2006)

[12] Lamperti, J.: Semi-stable Markov processes. I. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **22**, 205–225 (1972)

[13] Rogozin, B.A.: The distribution of the first hit for stable and asymptotically stable walks on an interval. Theory of Probability and its Applications **17**(2), 332–338 (1972). doi:10.1137/1117035

[14] Sharpe, M.: General theory of Markov processes, *Pure and Applied Mathematics*, vol. 133. Academic Press Inc., Boston, MA (1988)

[15] Vuolle-Apiala, J.: Itô excursion theory for self-similar Markov processes. Ann. Probab. **22**(2), 546–565 (1994)