Lévy processes

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Lévy processes are closely related to Random walks

Let $S_n, n \geq 0$ be a random walk

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} Y_k, \qquad n \geq 1,$$

with $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued r.v.

- $\{S_n, n \geq 0\}$ has independent increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n$ is independent of $(S_0, S_1, \ldots, S_n)$.

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- $\{S_n, n \geq 0\}$ has homogeneous increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n = \sum_{i=n+1}^{n+k} Y_i$ has the same law as $\sum_{i=1}^{k} Y_i = S_k$, that is

$$\mathbb{P}\left(S_{n+k} - S_n \in dy\right) = \mathbb{P}(S_k \in dy), \qquad \text{on } \mathbb{R}^d.$$

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- $\{S_n, n \geq 0\}$ has homogeneous increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n = \sum_{i=n+1}^{n+k} Y_i$ has the same law as $\sum_{i=1}^{k} Y_i = S_k$, that is

$$\mathbb{P}\left(S_{n+k} - S_n \in dy\right) = \mathbb{P}(S_k \in dy), \qquad \text{on } \mathbb{R}^d.$$

$\Rightarrow$ S is Markov chain, its law is totally characterised by the law of Y_1, central in the theory of stochastic processes...

Definition

A $\mathbb{R}^d$-valued stochastic process $\{X_t, t \geq 0\}$ is called a Lévy process if

- it has right continuous left limited paths,
- it has independent increments, i.e. for any $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ the random variables

  $$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

  are independent
- has stationary increment, i.e. for every $s, t \geq 0$ the law of $X_{t+s} - X_t$ is equal to that of X_s.

- Drift process: the deterministic process $\{X_t = at,\, t \geq 0\}$, its characteristic function given by

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-(-i\lambda a t)\}, \qquad \lambda \in \mathbb{R}\,.$$

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- Poisson process: let $\{\mathbb{e}_i, i \geq 0\}$ i.i.d.r.v. exponential r.v. $c > 0$, and $S_n = \sum_{i=0}^{n} \mathbb{e}_i, n \geq$ the random walk associated to it. The counting process $\{N_t, t \geq 0\}$ defined by

$$N_t = n \text{ if and only if } S_n \leq t < S_{n+1}, \qquad t \geq 0.$$

The independence and loss of memory imply $\{N_t, t \geq 0\}$ is a Lévy process and N_t follows a Poisson law with parameter tc for $t > 0$.

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The independence and loss of memory imply $\{N_t, t \geq 0\}$ is a Lévy process and N_t follows a Poisson law with parameter tc for $t > 0$. An useful fact. For $b \neq 0$, the following processes

$$bN_t - bct, \quad t \geq 0,$$

and

$$(bN_t - bct)^2 - bct, \quad t \geq 0,$$

are martingales. (Use $\mathbb{E}(N_t) = ct$, $Var(N_t) = ct$.)

- Compound Poisson process: $\{Y_i, i \geq 0\}$ i.i.d. $\mathbb{R}^d$-valued r.v. with common distribution F, $\{Z_n, n \geq 0\}$ the random walk associated to it, and $\{N_t, t \geq 0\}$ an independent Poisson process. The process

$$X_t = Z_{N_t}, \qquad t \geq 0,$$

is a Lévy process called *Compound Poisson process*. The uni-dimensional law of X is the so called compound Poisson law with parameters (tc, F). The characteristic function of X_t is given by

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t\int_{\mathbb{R}}(1-e^{i<\lambda,x>})cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}\,.$$

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$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t\int_{\mathbb{R}}(1 - e^{i<\lambda,x>})cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}\,.$$

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \sum_{n\geq 0} \mathbb{P}(N_t = n)\, \mathbf{E}(e^{i<\lambda, S_n>} \mid N_t = n)$$

$$= \sum_{n\geq 0} \frac{(ct)^n}{n!} e^{-ct}\, \mathbf{E}(e^{i<\lambda, Y_1>})^n = e^{-ct} \sum_{n\geq 0} \frac{1}{n!} \left(\int_{\mathbb{R}^d \setminus \{0\}} e^{i<\lambda,x>} ctF(dx) \right)^n$$

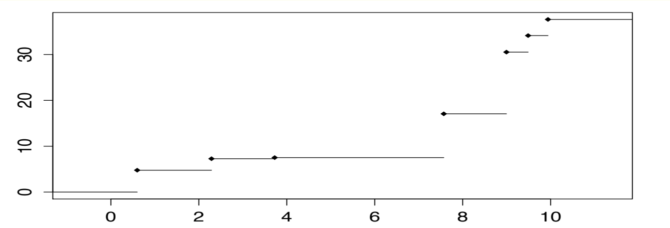


Figure: Monotone Compound Poisson

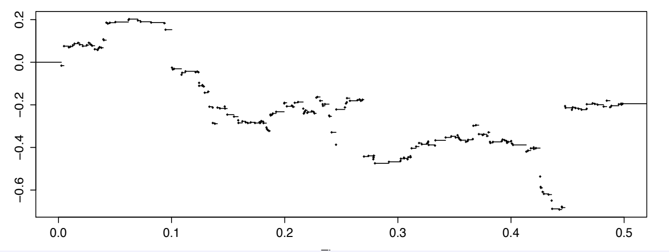


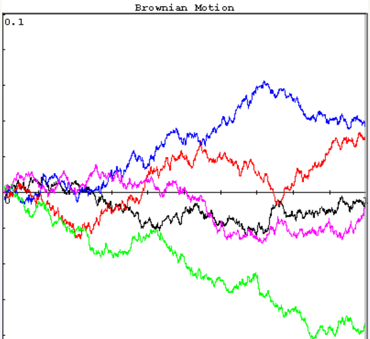
Figure: Non-monotone Compound Poisson

- Standard Brownian Motion: A real valued Lévy process $\{B_t, t \geq 0\}$ is called a standard Brownian motion if for any $t > 0$, B_t follows a Normal law with mean 0 variance t,

$$\mathbb{P}(B_t \in \mathrm{d}x) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{x^2}{2t}\}\mathrm{d}x, \quad x \in \mathbb{R};$$

and its characteristic law is given by

$$\mathbb{E}(e^{i\lambda B_t}) = \exp\{-t\frac{\lambda^2}{2}\}, \qquad \lambda \in \mathbb{R}\,.$$



- Linear Brownian Motion Let $\mathbf{B} = (B^1, B^2, \ldots, B^d)^T$ be d-independent standard Brownian motions, $a \in \mathbb{R}^d$, and σ a $d \times d$-matrix. The process $X_t = at + \sigma \mathbf{B}_t$, $t \geq 0$, is a Lévy process. For each $t \geq 0$, X_t is a Gaussian vector with mean, $\mathbb{E}(X_t^i) = a^{(i)}t$, and covariance matrix

$$\mathbb{E}\left((X_t^{(i)} - a^{(i)}t)(X_t^{(j)} - a^{(j)}t)\right) = \sigma\sigma^T_{i,j}, \qquad i, j \in \{1, \ldots, d\}.$$

Its Fourier transform $\mathbb{E}\left(\exp\{i < \lambda, X_t >\}\right) = \exp\{-t\Psi(\lambda)\}$, is

$$\Psi(\lambda) = - < a, \lambda > + \frac{1}{2}\lambda^T \Sigma \lambda = - < a, \lambda > + \frac{1}{2}||\sigma\lambda||^2, \qquad \lambda \in \mathbb{R}^d.$$

With $\Sigma = \sigma\sigma^T$ the covariance matrix; it is positive definite ($x^T \Sigma x \geq 0$, $x \in \mathbb{R}^d$) and symmetric. $X_t \sim N(at, t\Sigma)$.

Exercise

- Any linear transformations of a Lévy process is a Lévy process.

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- Any linear combination of independent Lévy processes is a Lévy process.

Lemma

The finite dimensional distributions of a Lévy process are totally characterised by the one dimensional distributions.

Proof.

Let $n \geq 1$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^d$ we have that

$$\begin{aligned}
&\mathbf{E}\left(\exp\left\{i < \lambda_1, X_{t_1} > + i < \lambda_2, X_{t_2} > + \cdots + i < \lambda_n, X_{t_n} >\right\}\right) \\
&= \mathbf{E}\left(\exp\left\{i < \overline{\lambda}_1, (X_{t_1} - X_{t_0}) > + \cdots + i < \overline{\lambda}_n, (X_{t_n} - X_{t_{n-1}}) >\right\}\right) \\
&= \prod_{j=1}^{n} \mathbf{E}\left(\exp\left\{i < \overline{\lambda}_j, (X_{t_j} - X_{t_{j-1}}) >\right\}\right) \\
&= \prod_{j=1}^{n} \mathbf{E}\left(\exp\left\{i < \overline{\lambda}_j, (X_{t_j - t_{j-1}}) >\right\}\right)
\end{aligned}$$

with $t_0 = 0$ and $\overline{\lambda}_j = \sum_{k=j}^{n} \lambda_k$. This is true for all $n \geq 1$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. This is enough since the Fourier transform characterises the law of the vectors $(X_{t_1}, \ldots, X_{t_n})$. □

Lemma

For every $t > 0$ we have

$$\mathbb{E}(\exp\{i < \lambda, X_t >\}) = \mathbb{E}(\exp\{i < \lambda, X_1 >\})^t$$

Proof.

If t is an integer n, by independence and stationarity of the increments

$$\mathbb{E}(e^{i<\lambda,X_t>}) = \Big(\mathbb{E}(e^{i<\lambda,X_1>})\Big)^n, \qquad \lambda \in \mathbb{R}^d.$$

Now, if t is a rational, say $t = p/q$, we get

$$\mathbb{E}(e^{i<\lambda,X_{p/q}>}) = \mathbb{E}(e^{i<\lambda,X_{1/q}>})^p, \qquad \mathbb{E}(e^{i<\lambda,X_{1/q}>})^q = \mathbb{E}(e^{i<\lambda,X_1>}),$$

$$\mathbb{E}(e^{i<\lambda,X_{p/q}>}) = \mathbb{E}(e^{i<\lambda,X_1>})^{p/q}.$$

For $t > 0$, take $\{t_n, n \geq 1\}$ rationals $\downarrow t$, the right continuity of X imply

$$\Big(\mathbb{E}(e^{i<\lambda,X_1>})\Big)^t = \lim_{k\to\infty} \Big(\mathbb{E}(e^{i<\lambda,X_1>})\Big)^{t_k} = \lim_{t_k \downarrow t} \mathbb{E}(e^{i<\lambda,X_{t_k}>}) = \mathbb{E}(e^{i<\lambda,X_t>})$$

. $\square$

Recall that a $\mathbb{R}^d$-valued r.v. Z, equivalently its law, is called infinitely divisible is for every $n \geq 1$ there exists $\{Y_{1,n}, Y_{2,n}, \ldots, Y_{n,n}\}$ i.i.d. such that Z has the same law as $\{Y_{1,n} + Y_{2,n} + \ldots + Y_{n,n}\}$.

$$Z \overset{\text{Law}}{=} Y_{1,n} + Y_{2,n} + \ldots + Y_{n,n}.$$

- For $r > 0$, $p \in (0,1)$, Z is negative binomial-(r,p) distributed if

$$\mathbb{P}(Z = k) = \binom{n+r-1}{k}(1-p)^k p^r, \quad k = 0, 1, 2, \ldots .$$

If r integer, X is the number of Bernoulli experiments necessary to make r success.

$$\mathbb{E}\left(e^{-\lambda Z}\right) = \left(\frac{p}{1-(1-p)e^{-\lambda}}\right)^r, \qquad \lambda \geq 0.$$

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For $n\geq 1$ $Z_{i,n}\sim$ negative binomial-$(r/n,p)$.

- Gamma r.v. $Z_{p,\theta}\sim$ Gamma(p,θ), with $p>0$, $\theta>0$,

$$\mathbb{P}(Z_{p,\theta}\in dz)=\frac{\theta^p}{\Gamma(p)}z^{p-1}\exp\{-\theta z\}dz,$$

$$\mathbb{E}(\exp\{-\lambda Z_{p,\theta}\})=\left(\frac{\theta}{\theta+\lambda}\right)^p=\mathbb{E}(\exp\{-\lambda Z_{p/n,\theta}\})^n,$$

- Gaussian, $Z \sim N(a, \Sigma)$, then $Z_{i,n} \sim N(\frac{a}{n}, \frac{1}{\sqrt{n}}\Sigma)$.

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- It is not always so easy to verify that a r.v. is infinitely divisible. *Student t-distribution, Pareto distribution, F-distribution, Gumbel distribution, Weibull, log-normal distribution, logistic distribution, half-Cauchy distribution*, are all i.d. See Sato's book on *Lévy processes and infinitely divisible distributions* or Van Harn and Steutel *Infinite divisibility on the real line.* This an active topic of research.

Infinitely divisible laws as limits

A double sequence of r.v. $(Z_{n,k}, k \in \{1, \dots, r_n\}, n \geq 1)$ on $\mathbb{R}^d$ is a null array if for each n the r.v.

$$(Z_{n,k}, k \in \{1, \dots, r_n\})$$

are independent, and

$$\lim_{n\to\infty} \max_{1\leq j\leq r_n} \mathbb{P}(|Z_{n,k}| > \epsilon) = 0, \qquad \epsilon > 0.$$

Fact Let $S_n = \sum_{k=1}^{r_n} Z_{n,k}$, $n \geq 1$. If for some $b_n \in \mathbb{R}^d$, $S_n - b_n$ converges in distribution towards a r.v. with law μ, then μ is an infinitely divisible law. (Khintchine)

Fact The class of infinitely divisible laws is closed under linear transformations, convolutions and weak convergence. Every infinitely divisible law can be obtained as a weak limit of infinitely divisible laws.

Lemma

Let X be a Lévy process. For $t > 0$, X_t is an infinitely divisible distribution.

Proof.

The property of independent and stationary increments implies that for $n \geq 1$ the random variables $(X_{\frac{t}{n}}, X_{\frac{2t}{n}} - X_{\frac{t}{n}}, \ldots, X_{\frac{nt}{n}} - X_{(n-1)\frac{t}{n}})$, are independent and identically distributed. The claim follows from the observation:

$$X_t = \sum_{k=1}^{n} (X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}})$$

$\square$

Theorem (Lévy–Khintchine's formula)

Let $\{X_t, t \geq 0\}$ *be a* $\mathbb{R}^d$ *valued Lévy process. For* $t > 0$*, the law of* X_t *is infinitely divisible. Furthermore*

$$\mathbb{E}(e^{i<\lambda, X_t>}) = e^{-t\Psi(\lambda)}, \qquad \lambda \in \mathbb{R}^d,$$

where $\Psi : \mathbb{R}^d \to \mathbb{C}$ *is the characteristic exponent of* X *and*

$$\begin{aligned} \Psi(\lambda) &= -i < a, \lambda > + \|Q\lambda^T\|^2/2 \\ &+ \int_{\{x \in \mathbb{R}^d, |x| \in (-1,1) \setminus \{0\}\}} (1 - e^{i<\lambda, x>} + i < \lambda, x >)\Pi(dx) \\ &+ \int_{\{x \in \mathbb{R}^d, |x| \in (-1,1)^c\}} (1 - e^{i<\lambda, x>})\Pi(dx) \end{aligned}$$

with $a \in \mathbb{R}^d$*,* Q *a* $d \times d$ *matrix, and* Π *is a measure on* $\mathbb{R}^d \setminus \{0\}$ *such that* $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge \|x\|^2)\Pi(dx) < \infty$. a, Q, Π *are the linear term, Gaussian term and* Π *is the Lévy measure, respectively. The matrix* $\Sigma = Q^T Q$ *is the covariance matrix. The triplet* (a, Σ, Π) *characterizes the law of* X *under* $\mathbb{P}$*.*

Remark

Π is only required to be σ-finite, nevertheless it is necessarily finite over any set that does not contain a ball of radius $r > 0$ around 0.

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Indeed, the fact that the function $x \mapsto \frac{x^2}{1+x^2}$ is increasing implies that

$$\begin{aligned}\Pi(z \in \mathbb{R}^d : ||z|| > r) &\leq \frac{1+r^2}{r^2} \int_{\mathbb{R}^d} \frac{||z||^2}{1+||z||^2} \Pi(dz) \\ &\leq \frac{1+r^2}{r^2} \int_{\mathbb{R}^d} 1 \wedge ||z||^2 \Pi(dz) < \infty\end{aligned}$$

Remark

The term $< \lambda, z > 1_{\{||z||<1\}}$ ensures

$$1 - e^{i<\lambda,z>} + i < \lambda, z > 1_{\{||z||<1\}} = O(||z||^2),$$

and that it remains bounded for $||z|| \geq 1$. Then the integral

$$\int_{\{x\in\mathbb{R}^d, |x|>0\}} (1 - e^{i<\lambda,x>} + i < \lambda, x >)\Pi(dx) < \infty.$$

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$$\int_{\{x\in\mathbb{R}^d, |x|>0\}} (1 - e^{i<\lambda,x>} + i < \lambda, x >)\Pi(dx) < \infty.$$

Change $h : \mathbb{R}^d \to \mathbb{R}^d$, so that

$$1 - e^{i<1,z>} + i < 1, h(z) >= O(||z||^2) \wedge 1.$$

Same Π and Q, but a changes to $\widetilde{a}$

$$-i < \widetilde{a}, \lambda > + \int_{\{x\in\mathbb{R}^d, |x|>0\}} (1 - e^{i<\lambda,x>} + i < \lambda, h(x) >)\Pi(dx)$$

with $\widetilde{a} = a - \int_{\{x\in\mathbb{R}^d, ||x||<1\}} (x - h(x))\Pi(dx)$.

Some other choices for h are

- $h(x) = \dfrac{1}{1+||x||^2}$
- When $d = 1$, $h(x) = \dfrac{\sin x}{x}$
- $h(x) = |e^x - 1| \wedge 1.$

- Compound Poisson process

$$\Pi(dx) = cF(dx),$$

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t \int_{\mathbb{R}} (1 - e^{i<\lambda, x>}) cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}.$$

- Lineal Brownian motion $\Pi \equiv 0$, $\mathbb{E}\left(\exp\{i < \lambda, X_t >\}\right) = \exp\{-t\Psi(\lambda)\}$,

$$\Psi(\lambda) = - < a, \lambda > + \frac{1}{2}\lambda^T \Sigma \lambda = - < a, \lambda > + \frac{1}{2}||\sigma\lambda||^2, \qquad \lambda \in \mathbb{R}^d.$$

With $a \in \mathbb{R}^d$ and $\Sigma = \sigma\sigma^T$ the covariance matrix, which is positive definite and symmetric.

Stable Lévy processes

Lévy processes with the scaling property: $\exists\alpha > 0$ such that $\forall c > 0$

$$(cX_{tc^{-\alpha}},\ t \geq 0) \overset{\text{Law}}{=} (X_t, t \geq 0).$$

In this case we X is said α-stable. This is equivalent to require that the characteristic exponent Ψ, satisfy

$$\Psi(k\lambda) = k^{\alpha}\Psi(\lambda),$$

for all $k > 0$ and for all $\lambda \in \mathbb{R}^d$. Then

$$\Psi(\lambda) = ||\lambda||^{\alpha}\Psi\left(\frac{\lambda}{||\lambda||}\right), \quad \lambda \in \mathbb{R}^d.$$

When $d = 1$

$$\Psi(\lambda) = |\lambda|^{\alpha}(\mathrm{e}^{\pi i u \alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda>0)} + \mathrm{e}^{-\pi i u \alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda<0)}), \qquad \lambda \in \mathbb{R}.$$

for $\lambda \in \mathbb{R}$, where $\rho = \mathbb{P}(X_1 > 0)$.

$\rho = \mathbb{P}(X_1 > 0)$ and the Lévy measure

When $\alpha = 2$, X is a Brownian motion and hence $\rho = 1/2$. When $\alpha = 1$ the self-similarity holds only when $\rho = 1/2$.
The parameter ρ is bound to $0 < \alpha\rho, \alpha(1-\rho) \leq 1$.
For $0 < \alpha < 1$, $\rho \in [0,1]$ and for $1 < \alpha < 2$, $\rho \in [1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$, and $\rho = \alpha^{-1}$, if the process X has no positive jumps, and $\rho = 1 - 1/\alpha$ if it has no negative jumps.
When $0 < \alpha < 2$, we have that $Q = 0$ and its Lévy measure is given by

$$\Pi(\mathrm{d}x) = \begin{cases} \frac{c_+ \mathrm{d}x}{x^{1+\alpha}} & \text{if } x > 0,\ \alpha \neq 1 \\ \frac{c_- \mathrm{d}x}{|x|^{1+\alpha}} & \text{if } x < 0,\ \alpha \neq 1 \\ \frac{c \mathrm{d}x}{|x|^2}, & \text{if } \alpha = 1,\ \rho = 1/2; \end{cases}$$

with

$$c_+ = \Gamma(1+\alpha)\frac{\sin(\alpha\pi\rho)}{\pi}, \qquad c_- = \Gamma(1+\alpha)\frac{\sin(\alpha\pi(1-\rho))}{\pi},$$

for some $\rho \in [0,1]$,
Necessarily $0 < \alpha < 2$ because of the integrability condition on Π.

Stable processes as limits of centered renormalized r.w.

- Assume $(S_n, n \geq 0)$ is a r.w. for which there exists sequences a_n, b_n such that
$$Y_t^n = \frac{Y_{nt} - ta_n}{b_n}, \qquad t \geq 0,$$
converges weakly in the sense of finite dimensional distributions towards a non-degenerated process X.

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- Then X is an α-stable process and $b_n = n^\alpha \ell(n)$, with ℓ a slowly varying function, viz. $\ell(tc)/\ell(t) \to 1$ as $t \to \infty$.

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- In fact it is enough to verify the convergence of the one dimensional distributions.
- The convergence holds also in Skorohod's topology.
- For $0 < \alpha < 2$ a NASC is that $1 - F(x) + F(-x) \sim \dfrac{2-\alpha}{\alpha} x^{-\alpha} L(x)$, with L an slowly varying function, and
$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p, \qquad \frac{F(-x)}{1 - F(x) + F(-x)} \to q,$$
as $x \to \infty$.

Stable processes as limits of centered renormalized r.w.

- Assume $(S_n, n \geq 0)$ is a r.w. for which there exists sequences a_n, b_n such that
$$Y_t^n = \frac{Y_{nt} - ta_n}{b_n}, \qquad t \geq 0,$$
converges weakly in the sense of finite dimensional distributions towards a non-degenerated process X.
- Then X is an α-stable process and $b_n = n^\alpha \ell(n)$, with ℓ a slowly varying function, viz. $\ell(tc)/\ell(t) \to 1$ as $t \to \infty$.
- In fact it is enough to verify the convergence of the one dimensional distributions.
- The convergence holds also in Skorohod's topology.
- For $0 < \alpha < 2$ a NASC is that $1 - F(x) + F(-x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha} L(x)$, with L an slowly varying function, and
$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p, \qquad \frac{F(-x)}{1 - F(x) + F(-x)} \to q,$$
as $x \to \infty$.
- For $\alpha = 2$, a NASC is that $\int_{-x}^{x} y^2 F(dy) \sim L(x)$ as $x \to \infty$, with L slowly varying.

Subordinators

Definition

A Lévy process is a **subordinator** if it has non-decreasing paths.

Lemma

If X is a subordinator the characteristic exponent Ψ, can be extended analytically to the semi-plan $\Im(z) \in [0, \infty[$. Then the law of a subordinator characterized by the Laplace exponent $\phi(\lambda) : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\mathbf{E}(e^{-\lambda X_1}) = e^{-\phi(\lambda)}, \qquad \lambda \geq 0,$$

where $\phi(\lambda) = \Psi(i\lambda)$. Moreover $Q = 0$, $\Pi(-\infty, 0) = 0$, $\int_0^\infty 1 \wedge x \Pi(dx) < \infty$, and there is $a \geq 0$ s.t.

$$\phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda x}) \Pi(dx), \qquad \lambda \geq 0.$$

Example (Gamma Subordinator)

$(X_t, t \geq 0)$ is a Gamma(b, a) subordinator if its one dimensional law is Gamma(at, b), that is

$$\mathbb{P}(X_t \in dx) = \frac{b^{at}}{\Gamma(at)} x^{at} e^{-bx} dx, \qquad x \geq 0.$$

Its Laplace transform takes the form

$$\mathbb{E}(e^{-\lambda X_t}) = \left(\frac{b}{b+\lambda}\right)^{at} = \exp\left\{-t(a \log((b+\lambda)/b)\right\}, \qquad \lambda \geq 0.$$

Frullani's formula, establishes

$$\log(x/y) = \int_0^\infty (e^{-yt} - e^{-xt}) \frac{dt}{t}, \qquad x, y > 0.$$

Then

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{ae^{-bt}}{t} dt, \qquad \lambda \geq 0.$$

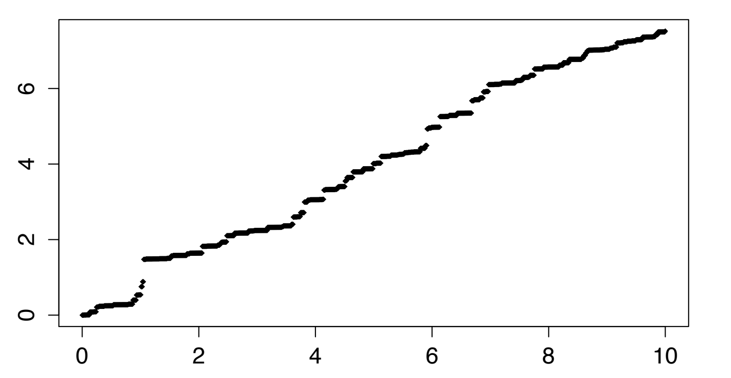

Figure: A Gamma subordinator

Example

Let X be an α-stable subordinator with index $\alpha \in (0,1)$. Its Laplace exponent is

$$\phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x^{1+\alpha}}, \qquad \lambda \geq 0.$$

(Integration by parts)

Lemma

For $t > 0$, $\mathbb{E}(X_t) = \infty$. Furthermore,

$$\mathbb{E}(X_t^\beta) < \infty \quad \textit{if and only if } \beta < \alpha,$$

if and only if

$$\int_1^\infty x^\beta \frac{dx}{x^{1+\alpha}} < \infty.$$

$$
\begin{aligned}
\mathbb{E}(X_1^\beta) &= \frac{\beta}{\Gamma(1-\beta)}\, \mathbb{E}\left(\int_0^\infty (1-e^{-yX_1})\frac{dy}{y^{1+\beta}}\right) \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1-\mathbb{E}\left(e^{-yX_1}\right))\frac{dy}{y^{1+\beta}} \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1-e^{-y^\alpha})\frac{dy}{y^{1+\beta}} \\
&= \frac{\beta}{\alpha\Gamma(1-\beta)} \int_0^\infty (1-e^{-z})\frac{dz}{z^{1+\frac{\beta}{\alpha}}} \\
&= \begin{cases} \frac{\alpha}{\beta}\Gamma\left(1-\frac{\beta}{\alpha}\right), & \text{if } \beta < \alpha, \\ \infty, & \text{if } \beta \geq \alpha. \end{cases}
\end{aligned}
\tag{1}
$$

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\end{aligned} \tag{1}
$$

Although the calculation is illustrative, this is a consequence of a more general fact.

A criteria for the moments under $\mathbb{E}$ in term of Π.

A function $g : \mathbb{R}^d \to \mathbb{R}^+$ is sub-multiplicative if $\exists C > 0$, s.t.

$$g(x+y) \leq C g(x) g(y), \qquad x, y \in \mathbb{R}^d.$$

The functions $g(x) = |x|^\beta$, $\beta > 0$ $g(x) = \exp \delta x$, are sub-multiplicative.

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The functions $g(x) = |x|^\beta$, $\beta > 0$ $g(x) = \exp \delta x$, are sub-multiplicative.

Theorem

Let g be a measurable function, sub-multiplicative, and bounded over compact intervals. The following are equivalent

- $\mathbb{E}(g(X_t)) < \infty$, *for some* $t > 0$,
- $\mathbb{E}(g(X_t)) < \infty$, *for all* $t > 0$,
- $\displaystyle\int_{\{|x|>1\}} g(x)\Pi(dx) < \infty$.

Three important martingales

- If $\mathbb{E}(|X_1|) < \infty$, then $\mathbb{E}(X_t) = t(a + \int_{|x|>1} x\Pi(dx))$, and the process

$$M_t^{(1)} = X_t - t\,\mathbb{E}(X_1), \qquad t \geq 0,$$

is a Martingale.

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  is a Martingale.
- If $\mathbb{E}(|X_1|^2) < \infty$, then $\mathbb{E}(|(X_t - t\,\mathbb{E}(X_1))|^2) = t(\sigma^2 + \int_{|x|>0} x^2\Pi(dx))$, and the process

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  is a Martingale.
- If $\beta \in \mathbb{C}$ is such that $\mathbb{E}(e^{<\beta,X_1>}) < \infty$ then the process

  $$M_t^{(\beta)} = \frac{e^{<\beta,X_t>}}{\mathbb{E}(e^{<\beta,X_1>})^t}, \qquad t \geq 0,$$

  is a (complex)-Martingale. When $\beta \in \mathbb{R}$, this is the so-called *Wald Martingale*.

The strong Markov property

We will denote by $\mathcal{F}_t = \sigma(X_s, s \leq t) \vee \mathcal{N}$, for $t \geq 0$, with $\mathcal{N}$ the null-sets of $\mathbb{P}$.

Lemma

A Lévy process is a strong Markov process. We have that for every T finite stopping time the pre-T-process $(X_s, s \leq T)$ is independent of the post-T-process, $(\widetilde{X}_s = X_{s+T} - X_T, s \geq 0)$, and the latter has the same law as $(X_u, u \geq 0)$.

Remark

For $x \in \mathbb{R}$, we will denote by $\mathbb{P}_x$ the push forward measure of the transform $x + X$. This is the law of X started at $x_0 = x$.

Idea of Proof.

For T deterministic, it is enough to show that for for $m \geq 1$, and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$ and $0 \leq s_1 \leq \cdots \leq s_m$ the vectors

$$(X_{t_1}, \ldots, X_{t_n}) \quad \text{y} \; (\widetilde{X}_{s_1}, \ldots, \widetilde{X}_{s_m})$$

are independent and the second has the same law as $(X_{s_1}, \ldots, X_{s_m})$.

$\square$

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are independent and the second has the same law as $(X_{s_1}, \ldots, X_{s_m})$. For consider the Fourier transforms and show that

$$\begin{aligned} &\mathbb{E}\left(\exp\left\{i\left(\sum_{j=1}^{n}\lambda_j X_{t_j} + \sum_{k=1}^{m}\beta_k \widetilde{X}_{s_k}\right)\right\}\right) \\ &= \mathbb{E}\left(\exp\left\{i\left(\sum_{j=1}^{n}\lambda_j X_{t_j}\right)\right\}\right)\mathbb{E}\left(\exp\left\{i\left(\sum_{k=1}^{m}\beta_k X_{s_k}\right)\right\}\right), \end{aligned} \tag{2}$$

for any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $n, m \geq 1$.

$\square$

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for any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $n, m \geq 1$.
The argument for T taking countably many values is done by considering the events $\{T = a_i\}$. General T by approximation. $\square$

Definition

Let $(\Theta, \mathcal{B}, \rho)$ a space of σ-finite measure. A family of $\mathbb{N} \cup \{\infty\}$-valued random variables $(N(B), B \in \mathcal{B})$ is called a Poisson measure with intensity measure ρ, if

(i) $N(B) \sim$Poisson $\rho(B)$, with the assumption that $\rho(B) = 0$ iff $N(B) = 0$ a.s. and $\rho(B) = \infty$ iff $N(B) = \infty$.

(ii) if $B_j \in \mathcal{B}$, $j \in \{1, \dots, n\}$ are disjoint sets then $N(B_1), \dots, N(B_k)$ are independent.

(iii) For $\omega \in \Theta$ the set function $B \mapsto N(B)(\omega)$ is a measure on $(\Theta, \mathcal{B})$.

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- **(iii)** For $\omega \in \Theta$ the set function $B \mapsto N(B)(\omega)$ is a measure on $(\Theta, \mathcal{B})$.

As a consequence:

$$\mathbb{E}(N(B)) = \rho(B) = Var(N(B)), \qquad B \in \mathcal{B}.$$

For a $t > 0$ we denote $X_{t-} = \lim_{s \uparrow t} X_s$, which exists and is finite by the assumption of having càdlàg paths, and $\Delta X_t = X_t - X_{t-}$.
For $B \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ we define

$$J(B, \omega) = \#\{s > 0 : (s, \Delta X_s) \in B\}, \qquad \omega \in \Omega.$$

Theorem (Lévy-Itô decomposition-I)

Let X be a $\mathbb{R}^d$ valued Lévy process with characteristics (a, Σ, Π) and Λ denote the Lebesgue measure on $[0, \infty)$. We have :

(i) *The familly $(J(B), B \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is a Poisson random measure with intensity measure $\Lambda \otimes \Pi$.*

Theorem (Lévy-Itô decomposition-II)

(ii) *There is a set* Ω_1, *w.p.1, such that for* $\omega \in \Omega_1$ *the limit*

$$X_t^{(0)}(\omega) = \lim_{\epsilon \downarrow 0} \int_{(0,t)\times D_{\epsilon,1}} x\left(J(dsdx,\omega) - \Lambda \otimes \Pi(dsdx)\right)$$

with $D_{\epsilon,1} = \{x : ||x|| \in (\epsilon, 1)\}$ *and* $D_1 = \{x : ||x|| \geq 1\}$, *is well defined, and the convergence holds uniformly over bounded intervals. The process* $X^{(0)}$ *is a Lévy process with characteristics* $(0, 0, \Pi 1_{\{0<|x|<1\}})$, *and its characteristic exponent is*

$$\Psi^{(0)}(\lambda) = \int_{0<|x|<1} (1 - e^{i<\lambda,x>} + i < \lambda, x >)\Pi(dx), \qquad \lambda \in \mathbb{R}^d$$

This process is a martingale and has exponential moments of any order.

Theorem (Lévy-Itô decomposition-III)

(iii) *The process* $X_t^1 = \int_{(0,t)\times D_1} xJ(dsdx,\omega), t \geq 0$, *is a compound Poisson process with intensity* $c = \Pi\{x : ||x|| > 1\}$ *and jump distribution*

$$F(dx) = \frac{1}{c}\Pi(dx)1_{\{||x||\geq 1\}}.$$

Its characteristic exponent is

$$\Psi^{(1)}(\lambda) = \int_{\{x\in\mathbb{R}^d, |x|\in(-1,1)^c\}} (1 - e^{i<\lambda,x>})\Pi(dx), \qquad \lambda \in \mathbb{R}^d.$$

(iii) *The process* $(X_t^{(2)} = X_t - X_t^{(1)} - X_t^{(0)},\ t \geq 0)$ *has continuous paths a.s. and its characteristic exponent is*

$$\Psi^{(2)}(\lambda) = -i < a, \lambda > +\frac{1}{2}\lambda^T\Sigma\lambda, \qquad \lambda \in \mathbb{R}^d.$$

(iv) *The processes* $X^{(0)}$, $X^{(1)}$ *and* $X^{(2)}$ *are independent.*

Corollary

Every Lévy process can be written as

$$\textit{linear Brownian motion} + \textit{Compound Poisson with } |\textit{jumps}| \geq 1$$
$$+ \textit{Square integrable Martingale with } |\textit{jumps}| < 1.$$

Every Lévy process is a semi-martingale.

Corollary

- *A Lévy process has countably many discontinuities a.s.*
- *When $\Pi(\mathbb{R}^d \setminus \{0\}) < \infty$, the first jump time, T_1, follows and exponential distribution of parameter $\Pi(\mathbb{R}^d \setminus \{0\})$.*

Proof.

For $t > 0$, $n > 1$,

$$\#\{s \ : \ (s, |\Delta X_s|) \in (0,t] \times (\frac{1}{n}, \infty)\} \sim \text{Poisson}(t\Pi\{x : ||x|| > n^{-1}\}) < \infty.$$

$$\mathbb{P}(T_1 > t) = \mathbb{P}(J(0,t] \times \mathbb{R}^d = 0) = \exp\{-t\Pi(\mathbb{R}^d \setminus \{0\})\}.$$

$\square$

For $\omega \in \Omega$ the measure $J()(\omega)$ can be written as

$$J(B)(\omega) = \sum_{t \geq 0} 1_{\{(t, \Delta_t(\omega)) \in B\}},$$

where $\Delta_t(\omega)$ are the spatial coordinates in $\mathbb{R}^d \setminus \{0\}$ of the points t for which $J(\{t\} \times \mathbb{R}^d \setminus \{0\})(\omega) = 1$, that is those t for which $\Delta_t(\omega) = X_t(\omega) - X_{t-}(\omega) \neq 0$. We will call

$$((t, \Delta_t), t \geq 0)$$

the Poisson point process of jumps of X.

For $A \subset \mathbb{R}^d \setminus \{0\}$ measurable and such that $\Pi(A) < \infty$, the process $N_t(A) = J((0,t] \times A)$ is a Poisson process. Moreover,

$$J((0,t] \times A) - t\Pi(A) = \int_{(0,t] \times \mathbb{R}^d \setminus \{0\}} x 1_{\{x \in A\}} \left(J(dsdx) - ds\Pi(dx) \right), \qquad t \geq$$

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is a Martingale.

Assume $d = 1$. Let $n \geq 1$, $1 \leq j \leq n$, $A_j \subset \mathbb{R}\setminus\{0\}$ measurable, such that $\Pi(A_j) < \infty$, and disjoints; and $c_j \in \mathbb{R}$ the process

$$M_t^n = \int_{(0,t]\times\mathbb{R}^d\setminus\{0\}} \sum_{1\leq j\leq n} c_j x 1_{\{x\in A_j\}} \left(J(dsdx) - ds\Pi(dx)\right), \qquad t \geq 0$$

is a Martingale. In fact, the process

$$(M_t^n)^2 - t\int_{\mathbb{R}\setminus\{0\}} \sum_{1\leq j\leq n} c_j^2 x^2 1_{\{x\in A_j\}} \Pi(dx),$$

is a Martingale. (Similar result for general d.)

Lemma (Campbell's formula)

For $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ *we have that for* $t > 0$

$$\sum_{s \leq t} |f(s, \Delta_s)|,$$

is finite a.s. if and only if $\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |f(s,y)| \Pi(dy) < \infty$. *In that case*

$$\mathbb{E}\left(\sum_{s \leq t} f(s, \Delta_s) \right) = \int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} f(s,y) \Pi(dy),$$

and the exponential formula holds

$$\mathbb{E}\left(\exp\{i\lambda \sum_{s \leq t} f(s, \Delta_s)\} \right) = \exp\{-\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\lambda f(s,y)}\right) \Pi(dy)\}.$$

If f *is positive the above formula remains valid if* $i\lambda$ *is replaced by* $-\lambda$.

A first consequence

If Π is a a Lévy measure on $(0,\infty)$ such that

$$\int_{(0,\infty)} 1 \wedge x\Pi(dx) < \infty,$$

then for any $a \geq 0$ the process

$$X_t = at + \sum_{s \leq t} \Delta_s, \qquad t \geq 0,$$

is finite a.s., has independent and stationary increments, the paths are non decreasing and according to the exponential formula its Laplace transform is given by

$$\mathbb{E}(e^{-\lambda X_t}) = \exp\{-at - t\int_0^\infty (1 - e^{-\lambda x})\Pi(dx)\}, \qquad \lambda \geq 0.$$

Every subordinator can be build in this way.

Lemma (Compensation or Master formula)

Let $(t, \Delta_t, t \geq 0)$ *the Poisson point process of jumps of* X. *For* H *measurable, left continuous and positive valued functional, the identity*

$$\mathbb{E}\left(\sum_{t>0} H((X_u, u<t), \Delta_t)\right)$$
$$= \mathbb{E}\left(\int_0^\infty dt \int_{\mathbb{R}^d\setminus\{0\}} \Pi(dy) H((X_u, u<t), y)\right),$$

holds. If $\mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^d\setminus\{0\}} \Pi(dy) H\ ((X_u, u<s), y)\right) < \infty$, $\forall t>0$, *the process*

$$\int_{s\in(0,t]} \int_{\mathbb{R}^d\setminus\{0\}} H((X_u, u<s), y)(J(dsdy) - ds\Pi(dy)), \qquad t \geq 0,$$

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holds. If $\mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} \Pi(dy) H^2((X_u, u < s), y)\right) < \infty, \forall t > 0$, *the process*

$$\int_{s\in(0,t]} \int_{\mathbb{R}^d \setminus \{0\}} H((X_u, u < s), y)(J(dsdy) - ds\Pi(dy)), \qquad t \geq 0,$$

is a square integrable Martingale, with quadratic variation $\int_{s\in(0,t]} \int_{\mathbb{R}^d \setminus \{0\}} H^2((X_u, u < s), y) ds\Pi(dy)$

An application to first passage

We will assume that X is a subordinator with characteristics (b, Π).
Let $x > 0$ and $\tau_x^+ = \inf\{t > 0 : X_t > x\}$, the first passage time above level x for X and (U_x, O_x) be the undershoot and overshoot of X at level x,

$$O_x = X_{\tau_x^+} - x, \qquad U_x = x - X_{\tau_x^+ -}.$$

We are interested by the distribution of the random variables (τ_x, U_x, O_x).
The potential measure of X is defined as the measure

$$V(dy) := \mathbb{E}\left(\int_0^\infty ds 1_{\{X_s \in dy\}}\right), \qquad y \geq 0.$$

This measure is characterised by its Laplace transform, which is given by

$$\int_{[0,\infty)} V(dy) e^{-\lambda y} = \frac{1}{\phi(\lambda)}, \qquad \lambda > 0.$$

Theorem

For any $f : \mathbb{R}^2 \to \mathbb{R}^+$ *measurable*

$$\mathbb{E}(f(U_x, O_x)1_{\{U_x>0\}}) = \int_0^x V(dy) \int_{(0,\infty)} \Pi(dz) f(x-y, y+z-x) 1_{\{z>x-y\}}$$

For every $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$\mathbb{E}(f(\tau_x^+) 1_{\{U_x>0\}}) = \int_0^\infty f(t)\, \mathbb{E}\left(\overline{\Pi}(x - X_t), X_t < x\right).$$

Proof.

On $X_{\tau_x^+} > x$, τ_x^+ is the unique instant where $X_{t-} < t$ and $X_t > x$, hence

$$\begin{aligned}&\mathbb{E}\left(f(\tau_x, U_x, O_x)1_{\{U_x>0\}}\right)\\&= \mathbb{E}\left(\sum_{t>0} f(t, x - X_{t-}, (X_t - X_{t-}) + X_{t-} - x)1_{\{X_t>x>X_{t-}\}}\right)\end{aligned}$$

Now, we apply the compensation formula to get

$$= \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \Pi(dy) f(t, x - X_{t-}, y + X_{t-} - x)1_{\{y>x-X_{t-}>0\}}\right).$$

The set of discontinuities has zero Lebesgue measure

$$= \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \Pi(dy) f(t, x - X_t, y + X_t - x)1_{\{y>x-X_t>0\}}\right).$$

Specialize to time or space. □

The creeping case

Theorem

X creeps, viz. $\mathbb{P}(X_{\tau_x^+} = x) > 0$ for some, and hence for all, $x > 0$, if and only if $b > 0$. In that case, for any $0 < t \leq \infty$, the occupation measure

$$U_t(dy) := \mathbb{E}\left(\int_0^t ds 1_{\{X_s \in dy\}}\right), \qquad y \geq 0,$$

has a continuous and bounded density on $(0, \infty)$, $u_t(y), y > 0$. The formula

$$\mathbb{P}(\tau_x^+ \in (t, t+\Delta], X_{T_x} = x) = b \int_{[0,x)} \mathbb{P}(X_t \in dy) u_\Delta(x - y), \tag{3}$$

holds for $x > 0$, $t \geq 0$, $\Delta > 0$.

Corollary

Assume $b = 0$. *The r.v.*

$$\int_0^{\tau_x^+} \overline{\Pi}(x - X_t)dt$$

follows an exponential distribution of parameter 1.

Proof.

Recall that

$$1 = \mathbb{E}(1_{\{U_x > 0\}}) = \int_0^\infty \mathbb{E}\left(\overline{\Pi}(x - X_t), X_t < x\right).$$

Also, for $y > 0$, $\mathbb{E}_y(1_{\{U_x > 0\}}) = \mathbb{E}(1_{\{U_{x-y} > 0\}}) = 1, \qquad \forall y \geq 0$, and thus that

$$\mathbb{E}_y\left(\int_0^{\tau_x^+} \overline{\Pi}(x - X_t)\right) = \int_0^\infty \mathbb{E}\left(\overline{\Pi}((x - y) - X_t), X_t < x - y\right).$$

By iteration, the following expression equals $n!$

$$\mathbb{E}\left(\int_0^{\tau_x^+} \overline{\Pi}(x - X_t)\right)^n = n!\, \mathbb{E}\left(\int 1_{\{0 < s_1 \ldots < s_n < \tau_x^+\}} \prod_{i=1}^n \overline{\Pi}(x - X_{s_i}) ds_1 \cdots ds_n\right).$$

If X is α-stable subordinator

$$\phi(\lambda) = \lambda^{\alpha},$$

the Lévy measure is $\Pi(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$. The renewal measure has Laplace transform

$$\int_{[0,\infty)} V(dy) e^{-\lambda y} = \frac{1}{\phi(\lambda)} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda y} y^{\alpha-1} dy.$$

Thus $V(dy) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} dy$, $y \geq 0$.

Corollary

For any $x > 0$ the random variables U_x/x and O_x/U_x are independent, its law do not depend of x, the former has a $Beta(1-\alpha, \alpha)$ distribution and the latter has a Pareto distribution on $(0, \infty)$ of parameter α.

$$\mathbb{E}\left(f\left(\frac{U_x}{x}\right) g\left(O_x/U_x\right)\right) \propto \int_0^1 du u^{-\alpha} (1-u)^{\alpha-1} f(u) \int_0^{\infty} \frac{dv}{(1+v)^{1+\alpha}} g(v),$$

the normalising constant is $c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)}$.

Itô's formula

We know that X is a semi-martingale.

Theorem (Itô's formula for Lévy processes)

Let $F : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $F \in C^{2,1}$, $F(X_t, t)$ $t \geq 0$ *is a semi-martingale and*

$$F(X_t, t) = F(X_0, 0) + \int_s^t \frac{\partial F}{\partial t}(X_{s-}, s)ds + \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s)dX_s$$
$$+ \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_{s-}, s)ds$$
$$+ \sum_{s \leq t} \left(F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s)\Delta X_s \right).$$

$$F(X_t, t) = F(X_0, 0) + M_t + V_t,$$

where M is a local martingale and V is a bounded variation process, given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s$$
$$+ \int_0^t \int_{|y|<1} F(X_{s-} + y) - F(X_{s-})(J(dsdy) - ds\Pi(dy)),$$

and

$$V_t = \sum_{s \leq t} (F(X_s, s) - F(X_{s-}, s)) 1_{\{|\Delta X_s| \geq 1\}} + \int_0^t \mathcal{L}F(X_s, s) ds,$$

with $\mathcal{L}$ the infinitesimal generator of (t, X_t),

$$\mathcal{L}F(x, s) = \frac{\partial F}{\partial t} F(x, s) + a \frac{\partial F}{\partial x}(x, s) + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(x, s)$$
$$+ \int_{\mathbb{R} \setminus \{0\}} \left(F(x + y, s) - F(x, s) - y 1_{\{|y<1|\}} \frac{\partial F}{\partial x}(x, s) \right) \Pi(dy)$$

Killed Lévy processes

Let $\widetilde{X}$ be Lévy process and $\mathbb{e}_q$ an independent exponential time of parameter $q \geq 0$, where we understand $\mathbb{e}_0 = \infty$ a.s. We consider the **killed Lévy process** as the $\mathbb{R}^d \cup \{-\infty\}$-valued process defined as

$$X_t = \begin{cases} \widetilde{X}_t, & \text{if } t < \mathbb{e}_q \\ -\infty, & \text{if } t \geq \mathbb{e}_q, \end{cases} \qquad t \geq 0.$$

$-\infty$ is a cemetery state and we denote ζ the lifetime of X, $\zeta = \inf\{t > 0 : X_t = -\infty\}$.

Lemma

The process X, while alive, has independent and stationary increments. The lifetime ζ follows an exponential distribution of parameter q.

For, it suffices to verify that for any $t, s \geq 0$, $0 < t_1 < \ldots < t_n \leq t$,

$$\begin{aligned} &\mathbb{E}\left(\left[\prod_{j=1}^{n} e^{i\lambda_j X_{t_j}}\right] e^{i\lambda(X_{t+s}-X_t)}, t+s<\zeta\right) \\ &= \mathbb{E}\left(\left[\prod_{j=1}^{n} e^{i\lambda_j X_{t_j}}\right], t<\zeta\right) e^{-s(q+\Psi(\lambda))}. \end{aligned} \tag{4}$$

We have

$$\mathbb{E}\left(e^{i\lambda(X_{t+s}-X_t)}, t+s<\zeta | t<\zeta\right) = e^{-s(q+\Psi(\lambda))}, \quad t, s \geq 0,\ \lambda \in \mathbb{R}\,.$$

The Lévy-Khinthchine formula can be written as

$$\begin{aligned} q+\Psi(\lambda) &= -i < a, \lambda > + ||Q\lambda^T||^2/2 \\ &\quad + \int_{\{x\in\mathbb{R}^d, |x|\in]-1,1[\setminus\{0\}\}} (1 - e^{i<\lambda,x>} + i < \lambda, x >)\Pi(dx) \\ &\quad + \int_{\{x\in\mathbb{R}^d, |x|\in]-1,1[^c\}\cup\{-\infty\}} (1 - e^{i<\lambda,x>})\left(\Pi(dx) + q\delta_{-\infty}(dx)\right). \end{aligned}$$

Strong law of large numbers

Theorem

Let X be a Lévy process on $\mathbb{R}^d$, which is no equal to zero everywhere. If $\mathbb{E}\left(|X_1|\right) < \infty$, and $\mathbb{E}(X_1) = \gamma$, then

$$\lim_{t\to\infty} \frac{1}{t} X_t = \gamma, \qquad a.s.$$

If $\mathbb{E}(|X_1|) = \infty$, then

$$\limsup_{t\to\infty} \frac{1}{t} |X_t| = \infty, \qquad a.s.$$

When $d = 1$, if $\mathbb{E}(X_1) = \infty$, then

$$\lim_{t\to\infty} \frac{1}{t} X_t = \infty, \qquad a.s.$$

while, if $\mathbb{E}(X_1) = -\infty$, then

$$\lim_{t\to\infty} \frac{1}{t} X_t = -\infty, \qquad a.s.$$

Proof.

It is essentially a consequence of the Strong law of large numbers for sums of i.i.d.r.v. This implies the result along t integer. Then consider $Y_n = \sup_{t\in[n,n+1]} |X_t - X_n|$, $n \geq 1$. The result will follow from the discrete version if $\frac{1}{n}Y_n \to 0$, a.s. For this end, notice these are i.i.d. and it can be verified that $\mathbb{E}(Y_1) < \infty$. The SLLN applied to Y imply

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow[n\to\infty]{} \mathbb{E}(Y_1).$$

It follows $\frac{1}{n}Y_n \to 0$ a.s. $\square$

Corollary

For $d = 1$, we have one and only one of the following

- $\limsup_{t\to\infty} X_t = \infty$ *and* $\liminf_{t\to\infty} X_t = -\infty$ *a.s.*
- $\lim_{t\to\infty} X_t = \infty$ *a.s.*
- $\lim_{t\to\infty} X_t = -\infty$ *a.s.*

- There are many other results that describe the asymptotic behaviour of a Lévy process at infinity that can be inferred from its analogue for random walks.

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- There are many other results that describe the asymptotic behaviour of a Lévy process at infinity that can be inferred from its analogue for random walks.
- It is not always the case that they are obtained by a direct application of its analogue for random walks. The main complication comes from the *infinitely many small jumps.*
- The behaviour of a Lévy process at 0 has no analogue in random walks. This is an active area of research.

From now on $d = 1$

For $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the push forward measure of the transformation $x + X$ under $\mathbb{P}$. Let $\widehat{X}$ be the *dual Lévy process*, defined by

$$\widehat{X}_t = -X_t, \quad t \geq 0, \qquad \text{under } \mathbb{P} .$$

The process $\widehat{X}$ is a Lévy process with characteristic exponent

$$\widehat{\Psi}(\lambda) = \Psi(-\lambda), \quad \lambda \in \mathbb{R} .$$

Lemma

For each $t > 0$, fixed, the time reversed process $\{X_{(t-s)-} - X_t, \ 0 \leq s \leq t\}$, has the same law as the dual process $\{\widehat{X}_s, \ s \leq t\}$ under $\mathbb{P}$.

Proof.

The time reversed process has independent increments, has cádlág paths. The law of $\{X_{(t-s)-} - X_t$ equals that of $-X_s$, for any $0 \leq s \leq t$, under $\mathbb{P}$. □

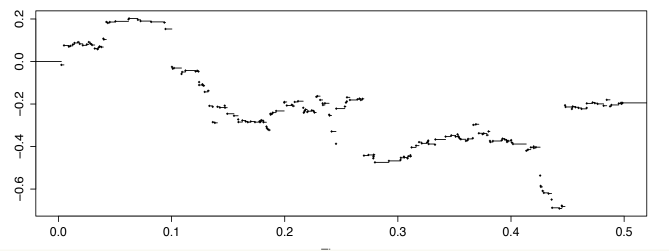

Figure: Seen from right to left the jumps change of sign, the increments are still independent and stationary.

For $t > 0$, denote the running supremum by $S_t = \sup\{0 \vee X_s, s \leq t\}$ and the running infimum by $I_t = \inf\{0 \wedge X_s, s \leq t\}$, for $t > 0$.

Lemma

For each $t > 0$ fixed, the pairs of variables $(S_t, S_t - X_t)$ and $(X_t - I_t, -I_t)$

Proof.

Take $\widetilde{X}_t = X_t$ and $\widetilde{X}_s = X_t - X_{(t-s)-}$ $0 \leq s < t$. Notice $(S_t, S_t - X_t) = (\widetilde{X}_t - \widetilde{I}_t, -\widetilde{I}_t)$ a.s. By the duality lemma X and $\widetilde{X}$ have the same law. $\square$

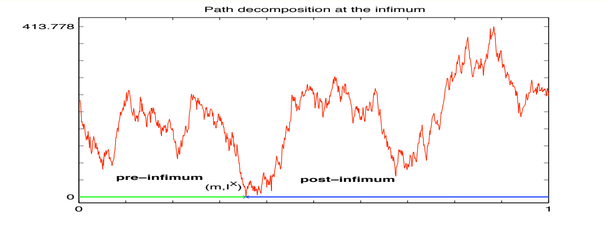

Figure: $(X_t - I_t, -I_t)$

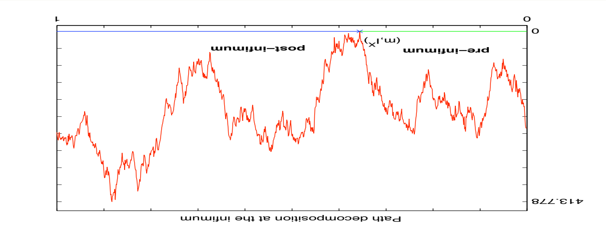

Figure: $(\widetilde{S}_t, \widetilde{S}_t - \widetilde{X}_t)$

The Wiener-Hopf factorisation-I

Let $\tau = \mathbb{e}$ be an exponential time of parameter q, and independent of X. Recall $S_t = \sup\{0 \vee X_s, s \leq t\}$ and $I_t = \inf\{0 \wedge X_s, s \leq t\}$,

$$g_t = \sup\{s < t : X_s = S_s\}, \qquad t > 0.$$

The Wiener-Hopf factorisation states

$$(\tau, X_\tau) = (g_\tau, S_\tau) \underbrace{+}_{independent} (\tau - g_\tau, X_\tau - S_\tau),$$

$$(\tau - g_\tau, X_\tau - S_\tau) \overset{\text{Law}}{=} (\widehat{g}_\tau, -\widehat{S}_\tau),$$

and provides a characterisation of the law of these r.v.

Theorem

The joint law of $(g_\tau, \tau - g_\tau, S_\tau, S_\tau - X_\tau)$ *is determined by*

(i) *The pairs* (g_τ, S_τ) *and* $(\tau - g_\tau, S_\tau - X_\tau)$ *are independent and infinitely divisible*

(ii) *For all* $\alpha, \beta > 0$,

$$\begin{aligned}&\mathbb{E}\left(\exp\{-\alpha g_\tau - \beta S_\tau\}\right)\\&= \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t}\int_{[0,\infty[}(e^{-\alpha t - \beta x} - 1)e^{-qt}\,\mathbb{P}(X_t \in \mathrm{d}x)\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\exp\{-\alpha(\tau - g_\tau) - \beta(S_\tau - X_\tau)\}\right)\\&= \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t}\int_{]-\infty,0]}(e^{-\alpha t - \beta x} - 1)e^{-qt}\,\mathbb{P}(X_t \in \mathrm{d}x)\right).\end{aligned}$$

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The proof is based in excursion theory

The characteristic function of X_τ can be written as

$$\mathbb{E}(\exp\{i\lambda X_\tau\}) = \frac{q}{q+\Psi(\lambda)} = \Psi_+(\lambda)\Psi_-(\lambda),$$

where

$$\Psi_+(\lambda) = \mathbb{E}(\exp\{i\lambda S_\tau\}), \qquad \Psi_-(\lambda) = \mathbb{E}(\exp\{-i\lambda(S_\tau - X_\tau)\}), \qquad \lambda \in \mathbb{R}\,.$$

- If X is a Brownian motion

$$\frac{q}{q+\frac{\lambda^2}{2}} = \frac{\sqrt{q}}{\sqrt{q}-i\lambda}\frac{\sqrt{q}}{\sqrt{q}+i\lambda},$$

- In Kyprianou's course other explicit factorisations will be given for particular values of q.

Lemma

Assume X is a real valued Lévy process. The process X reflected in the supremum $R_t = S_t - X_t$, $t \geq 0$, is a Markov process in the filtration $\mathcal{F}_t, t \geq 0$ and it has the Feller property.

Proof.

Let T be a finite stopping time and $s \geq 0$. We have the identity

$$\begin{aligned} S_{T+s} &= S_T \vee \sup\{X_{T+u}, 0 \leq u \leq s\} \\ &= X_T + (S_T - X_T) \vee \sup\{X_{T+u} - X_T, 0 \leq u \leq s\}. \end{aligned} \tag{5}$$

We can write

$$S_{T+s} - X_{T+s} = (S_T - X_T) \vee \sup\{X_{T+u} - X_T, 0 \leq u \leq s\} - (X_{T+s} - X_T).$$

The Markov property of X implies that the conditional law of $S_{T+s} - X_{T+s}$ given $\mathcal{F}_T$ is the same as that of $(x \vee S_s) - X_s$ under $\mathbb{P}$ with $x = S_T - X_T \geq 0$, which is the law of $S_s - X_s$ under $\mathbb{P}_{-x}$. $\square$

Local time

For simplicity we will assume that 0 is regular upwards and downwards i.e. $\tau_0^+ = \inf\{t > 0 : X_t > 0\}$ is such that $\mathbb{P}(\tau_0^+ > 0) = 0 = \widehat{\mathbb{P}}(\tau_0^+ > 0)$. Equivalently the first return to 0 for R is zero a.s.

Local time

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Equivalently the first return to 0 for R is zero a.s.
General theory on Markov processes establishes that there exists a local time at 0 for R, i.e. a non-decreasing adapted process $(L_t, t \geq 0)$ such that:

- $L_0 = 0$ when $R_0 = 0$;
- $L_{t+s} = L_t + \underbrace{L_s \circ \theta_t}_{\text{shift at } t}$, for $s, t \geq 0$;
- L is the unique, up to multiplicative constants, functional that grows at the times where $R = 0$;

$$\int_0^\infty 1_{\{R_s \neq 0\}} dL_s = 0, \quad \text{a.s.}$$

- if T is a random time such that on $\{T < \infty\}$, $R_T = 0$, a.s. and the conditional law of $\{(L_{T+t} - L_T, R_t), t \geq 0\}$, given $\{T < \infty\}$, is the same as that of $\{(L_t, R_t), t \geq 0\}$ under $\mathbf{P}(|R_0 = 0)$.

- If X is a Brownian motion

$$L_t = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_0^t 1_{\{S_s - X_s \leq \epsilon\}} ds, \qquad t \geq 0,$$

  the limit holds uniformly over bounded intervals in probability.
- The same holds if X has no-negative jumps.
- In general there exists a function $\widehat{V}$, s.t.

$$L_t = \lim_{\epsilon \to 0+} \frac{1}{\widehat{V}(\epsilon)} \int_0^t 1_{\{S_s - X_s \leq \epsilon\}} ds, \qquad t \geq 0,$$

  the limit holds uniformly over bounded intervals in probability.
- There exists a constant $\delta \geq 0$ such that

$$L_t = \delta \int_0^t 1_{\{R_s = 0\}} ds, \qquad t \geq 0.$$

Theorem (Itô's excursion theory for R)

- *The process of excursions* $(\mathbf{e}_t, t \geq 0)$

$$\mathbf{e}_t = \begin{cases} X_{L^{-1}_{t-}} - X_{L^{-1}_{t-}+s}, & 0 \leq s \leq L^{-1}_t - L^{-1}_{t-}, \quad \textit{if } L^{-1}_t - L^{-1}_{t-} > 0 \\ \Delta, & \textit{if } L^{-1}_t - L^{-1}_{t-} = 0, \end{cases}$$

is a Poisson point process with values in $\mathbb{D}^{\dagger}$, *and characteristic measure* $\overline{n}$. *(Itô, 1971)*

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is a Poisson point process with values in $\mathbb{D}^\dagger$, *and characteristic measure* $\overline{n}$. *(Itô, 1971)*

- *Under* $\overline{n}$ *the process of coordinates has lifetime* ζ, *bears the Markov property with the same semigroup as* $\widehat{X}$ *killed at the time* $\tau_0^- = \inf\{t > 0 : X_t < 0\}$, *that is*

$$\begin{aligned} &\overline{n}\left(F(\mathbf{e}_u, u \leq t) f(\mathbf{e}_{t+s}), t+s < \zeta\right) \\ &= \overline{n}\left(F(\mathbf{e}_u, u \leq t)\widehat{\mathbb{E}}_{\mathbf{e}_t}\left(f(X_s), s < \tau_0^-\right), t < \zeta\right), \end{aligned}$$

for any F, f *measurable bounded functionals.*

Chaumont's construction of the Normalized excursion

Assume X is a stable process, and define $d_1 = \inf\{t > 1 : S_t - X_t = 0\}$, and $g_1 = \sup\{t < 1 : S_s = X_t\}$. The scaling property implies that the process

$$\frac{1}{(d_1 - g_1)^{1/\alpha}} R_{g_1 + (d_1 - g_1)s}, \quad 0 \leq s \leq 1,$$

has the same law as the excursion process under $\overline{n}\left(\cdot|\zeta = 1\right)$, and this is independent of $d_1 - g_1$. This is the normalised stable excursion.

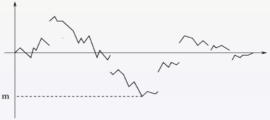
For a Brownian motion the normalized excursion (length one) is obtained from the brownian bridge using the Vervaat transform.

For a Brownian motion the normalized excursion (length one) is obtained from the brownian bridge using the Vervaat transform.
Let B be a standard Brownian motion and $X_t, 0 \leq t \leq 1$, the process

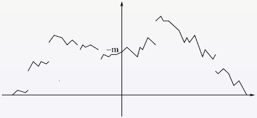
$$X_t = B_t - tB_1, \qquad 0 \leq t \leq 1,$$

is the Brownian Bridge. Let $\rho = \inf\{t > 0 : X_t = m =: \min_{\{0 \leq s \leq 1\}} X_s\}$. The Vervaat transform inverts the path of X after and before the time ρ. The resulting process is the normalised excursion.

$$X_t(\phi) = X_{\zeta+t} - m \quad \text{si } \rho - \zeta < t < 0,$$
$$X_t(\phi) = X_t - m \quad \text{si } 0 < t < \rho,$$
$$X_t(\phi) = \delta \quad \text{si } t \notin]\rho - \zeta, \rho[.$$



X_t



$X_t(\phi)$

This holds if X is a bridge from 0 to 0 of length 1 of a stable process.

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For a stable process we have $\mathbb{P}_x(X_s \in dy) = p_s(y-x)dy$, with $s \geq 0$ $x, y \in \mathbb{R}^d$, can be constructed by taking the time inhomogeneous process with semigroup

$$P^{0,0}_{u,s}(x, dy) = \frac{p_{s-u}(y-x)p_{1-s}(-y)}{p_{1-u}(-x)}dy, \qquad 0 \leq u \leq s \leq 1, x, v \in \mathbb{R},$$

under $\mathbb{P}$. (Fitzsimmons-Pitman-Yor, 1995).

Theorem (Master formula)

Let $\mathcal{G}$ denote the left extrema of the excursion intervals, and for $g \in \mathcal{G}$,
$d_g = \inf\{t > 0 : R_t = 0\}$.

$$\mathbb{E}\left(\sum_{g\in\mathcal{G}} F(X_s, s<g) H \underbrace{\left(S_g - X_{g+u}, u \le d_g - g\right)}_{\text{excursion at time } g}\right)$$

$$= \mathbb{E}\left(\int_0^\infty dL_t F(X_s, s<t)\overline{n}\left(H(\epsilon_u, u \le \zeta)\right)\right);$$

and

$$\mathbb{E}\left(\int_0^\infty dt F(X_s, s<t) f(X_t) 1_{\{X_t = S_t\}}\right) = \delta\, \mathbb{E}\left(\int_0^\infty dL_t F(X_s, s<t) f(X_t)\right),$$

where F, G, f are test functionals, and the stochastic process
$(\omega, t) \mapsto F(X_s(\omega), s<t)$, *is adapted and left continuous.*

Lemma

The processes $(X_t, 0 \leq t < g_\tau)$ *and* $(X_{g_\tau + t} - X_{g_\tau}, 0 \leq t \leq \tau - g_\tau)$ *are independent.*

Proof.

$\square$

Lemma

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Proof.

By the compensation formula, for $s > 0$,

$$\mathbb{E}\left(F\left(X_t, 0 \leq t < g_s\right) H(X_{g_s+t} - X_{g_s}, 0 \leq t \leq s - g_s)\right)$$

$$= \mathbb{E}\left(\sum_{g \in \mathcal{G}} F\left(X_t, 0 \leq t < g\right) H(X_{g+t} - X_g, 0 \leq t \leq s - g) 1_{\{0 \leq g < s < d_g\}}\right) =$$

$$\mathbb{E}\left(\int_0^s \mathrm{d}L_u F\left(X_t, 0 \leq t < u\right) \int_{\mathbb{D}} \overline{n}(\mathrm{de}) H(-e(t), 0 \leq t \leq s - u) 1_{\{s - u < \zeta\}}\right),$$

Notice that for s fixed there is no independence. $\square$

Integrate w.r.t. $qe^{-q}ds$ to get

$$\mathbb{E}\left(F\left(X_t, 0 \le t < G_\tau\right) H(X_{G_\tau+t} - X_{G_\tau}, 0 \le t \le \tau - G_\tau)\right) =$$

$$\mathbb{E}\left(\int_0^\infty \mathrm{d}L_u e^{-qu} F\left(X_t, 0 \le t < u\right)\right)\left(\int_{\mathbb{D}} \overline{n}(\mathrm{d}e) H(-e(t), 0 \le t \le \tau) 1_{\{\tau<\zeta\}}\right),$$

Conclude by normalising to get probability measures.

$(L_t, t \geq 0)$, the local time at 0 of the reflected process $S - \xi = (S_t - X_t,\ t \geq 0)$. We define the right continuous inverse of L by

$$L_t^{-1} = \inf\{s > 0 : L_s > t\}, \quad t \geq 0.$$

- upward ladder time process $(L_t^{-1}, t \geq 0)$,
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Fristedt's formula

for $\lambda, \mu \geq 0$,

$$\begin{aligned}\kappa(\lambda, \mu) &= -\log \mathbf{E}(\exp\{-\lambda L_1^{-1} - \mu H_1\}) \\ &= c \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} (e^{-t} - e^{-\lambda t - \mu x}) \, \mathbf{P}(\xi_t \in \mathrm{d}x)\right).\end{aligned}$$

Draw the ladder height process

Lemma

The r.v. (g_τ, S_τ) *is infinitely divisible and its Laplace transform is*

$$\mathbb{E}\left(\exp\{-\alpha g_\tau - \beta S_\tau\}\right) = \kappa(q,0)/\kappa(\alpha+q,\beta), \qquad \alpha, \beta > 0.$$

Proof.

Master Formula! $\square$

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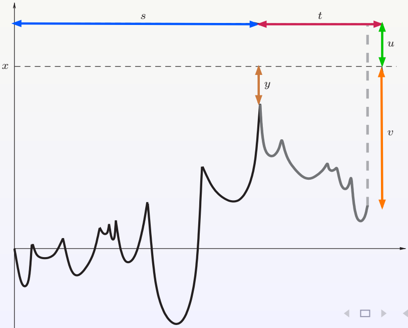
- The ladder processes has (amongst other things) hidden information about the distribution of $\overline{X}_t$, τ_x^+ and

$$g_t = \sup\{s < t : X_s = \overline{X}_s\}.$$

The quintuple law at first passage



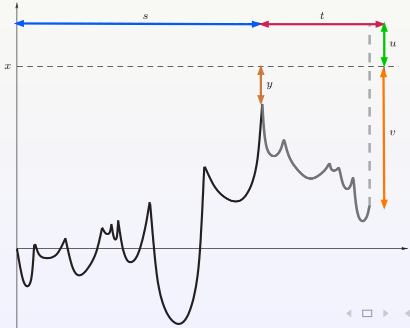
The quintuple law at first passage

Theorem (Doney and Kyprianou 2006)

For each $x > 0$ *we have on* $u > 0$, $v \geq y$, $y \in [0, x]$, $s, t \geq 0$,

$$\mathbb{P}(\tau_x^+ - g_{\tau_x^+-} \in dt,\ g_{\tau_x^+-} \in ds,\ X_{\tau_x^+} - x \in du,\ x - X_{\tau_x^+-} \in dv,\ x - \overline{X}_{\tau_x^+-} \in dy)$$
$$= V(ds, x - dy)\widehat{V}(dt, dv - y)\Pi(du + v)$$

where the equality holds up to a normalising multiplicative constant.



Suppose that X is a two-sided strictly stable process with index $\alpha \in (1,2)$ and positivity parameter $\rho = \mathbb{P}(X_t \geq 0) \in (0,1)$, then the following facts are known:

- Its jump measure is given by

$$\Pi(dx) = 1_{(x>0)} \frac{c_+}{x^{1+\alpha}} dx + 1_{(x<0)} \frac{c_-}{|x|^{1+\alpha}} dx$$

- Its renewal measures $V(dx) := V(\mathbb{R}_+, dx)$ and $\widehat{V}(x) := \widehat{V}(\mathbb{R}_+, dx)$ are known

$$V(dx) = \frac{x^{\alpha\rho-1}}{\Gamma(\alpha\rho)} dx \text{ and } \widehat{V}(dx) = \frac{x^{\alpha(1-\rho)-1}}{\Gamma(\alpha(1-\rho))} dx.$$

Corollary

The random variables $r^{-1}(U(r), O(r))$ have a joint p.d.f.

$$p_{\alpha\rho}(u,v) = \frac{\alpha\rho \sin \alpha\rho\pi}{\pi} (1-u)^{\alpha\rho-1}(u+v)^{-1-\alpha\rho},$$

for $0 < u < 1, v > 0$, if $\alpha\rho \in (0,1)$; and is the Dirac mass at $(0,0)$ if $\alpha\rho = 1$.

Lemma (Vigon 2002, Équations amicales)

Let $\widehat{V}(\mathrm{d}y) = \widehat{V}([0,\infty) \times \mathrm{d}y)$

$$\overline{\Pi}_H(x) = \int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}^+(x+y),$$

$$\overline{\Pi}^+(x) = \int_{]x,\infty[} \Pi_H(\mathrm{d}y)\overline{\Pi}_{\widehat{H}}(y-x) + \widehat{d}\overline{p}(x) + \widehat{k}\overline{\Pi}_H(x),$$

where $\overline{p}(x)$ *is the density of the measure* Π_H*, which exists if* $\widehat{d} > 0$.

Proof.

Notice that

$$\overline{\Pi}_H(x) = \overline{n}\left(\epsilon_\zeta < -x, \zeta < \infty\right).$$

In the event where the excursion ends by a jump, ζ is the unique time where $\epsilon_{t-} > 0 > \epsilon_t$, this equals

$$\overline{n}\left(\sum_{0<t} 1_{\{\epsilon_{t-}>0>-x>\epsilon_{t-}+\epsilon_t-\epsilon_{t-}\}}\right),$$

By the Poissonian structure of the jumps and the compensation formula

$$\overline{n}\left(\int_0^\zeta \mathrm{d}t 1_{\{\epsilon_{t-}>0\}} \overline{\Pi}^+(x+\epsilon_{t-})\right) = \int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}^+(x+y).$$

$\square$

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- Work of Bertoin, Chaumont, Doney and others help us justify the claim that $(X, \mathbb{P}_x^{\uparrow})$ as a Doob h-transform is the result of "conditioning" X to stay non-negative. Their final conclusion is

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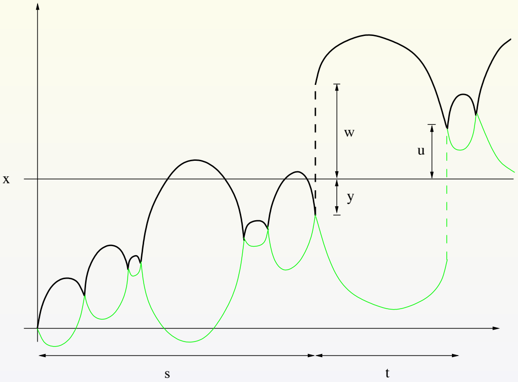
- Moreover, in the sense of weak convergence with respect to the Skorohod topology, they have also shown that $\mathbb{P}^{\uparrow} := \lim_{x \downarrow 0} \mathbb{P}_x$ is well defined.

- The Tanaka-Doney pathwise construction of $(X, \mathbb{P}^{\uparrow})$ from $(X, \mathbb{P})$ replaces excursions of X from $\overline{X}$ by their time-reversed dual.



- We have also that

$$\mathbb{P}^{\uparrow}|_{\mathcal{F}_t} = \frac{1}{\widehat{V}(X_t)} \underline{n}|_{\mathcal{F}_t}, \quad t > 0.$$

Where $\underline{n}$ is the excursion measure for X reflected in the infimum.

The quintuple law at last passage

Let

$$\underset{\rightarrow t}{X} = \inf\{X_s : s \geq t\}$$

be the future infimum of X,

$$\underset{\rightarrow t}{D} = \inf\{s > t : X_s - \underset{\rightarrow t}{X} = 0\}$$

is the right end point of the excursion of X from its future infimum straddling time t. Now define the last passage time

$$U_x = \sup\{s \geq 0 : X_t \leq x\}$$

Theorem

Suppose that X is a Lévy process which does not drift to $-\infty$. For $s, t \geq 0$, $0 < y \leq x$, $w \geq u > 0$,

$$\mathbb{P}^{\uparrow}\left(\underrightarrow{D}_{U_x} - U_x \in dt,\ U_x \in ds,\ \underrightarrow{X}_{U_x} - x \in du,\ x - X_{U_x -} \in dy,\ X_{U_x} - x \in dw\right)$$
$$= V(ds, x - dy)\widehat{V}(dt, w - du)\Pi(dw + y)$$

where the equality hold up to a multiplicative constant.

Spectrally negative LP

We will assume $\Pi(0,\infty)=0$, and that X is not monotone.

- $\mathbb{E}(e^{\beta X_1})<\infty$ because $\underbrace{\int_1^\infty e^{\beta x}\Pi(dx)}_{=0}+\int_{-\infty}^{-1} e^{\beta x}\Pi(dx)<\infty$
- Ψ is well defined and analytical on $\{\Im(z)\le 0\}$, $\mathbb{E}(\exp\{\lambda X_1\})=e^{\psi(\lambda)}$,

$$\psi(\lambda)=-\Psi(-i\lambda)=a\lambda+\frac{\sigma^2}{2}\lambda^2+\int_{(-\infty,0)} e^{\lambda x}-1-\lambda x 1_{\{x<-1\}}\Pi(dx).$$

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- By Hölder's inequality ψ is convex on $[0,\infty)$, $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\mathbb{E}_0(X_1) = \psi'(0+)$.

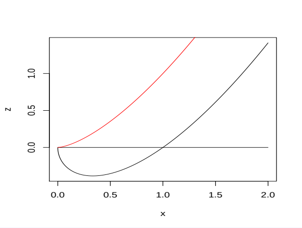

Figure: Typical shape of ψ. Black $\psi'(0+) < 0$, Red $\psi'(0+) \geq 0$.

Lemma

For $q \geq 0$*, let* $\Phi(q)$ *be the largest solution to* $\psi(\lambda) = q$*. The continuous increasing process* $S_t = \sup\{X_s, s \leq t\}$ *is the local time at* 0 *for the process reflected* R*. Its right continuous inverse*

$$\tau_x^+ = \inf\{t > 0 : X_t > x\}, \qquad x \geq 0,$$

is subordinator with Laplace exponent Φ*,*

$$\mathbb{E}\left(\exp\{-\beta\tau_x^+\}\right) = \exp\{-x\Phi(\beta)\}, \qquad \beta \geq 0.$$

If X *drifts towards* $-\infty$*,* τ^+ *is killed with rate* $\Phi(0)$*.*

Proof.

The process $M_t = \exp\{\Phi(\beta)X_t - t\beta\}$ is a Martingale (the Wald martingale of $\Phi(\beta)$). So is the process $M_{t\wedge\tau_x^+}$, and it is bounded by $e^{\Phi(\beta)x}$. By a Dominated convergence argument we get

$$1 = \mathbb{E}\left(e^{\beta x}e^{-\beta\tau_x^+}\right), \qquad x \geq 0.$$

□

When X is Brownian motion it is a consequence of the reflection principle that τ^+ is an $1/2$-stable subordinator.

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  for all $\alpha, \beta \geq 0$.
- The downward ladder heigh process has Laplace exponent

$$\widehat{\kappa}(\alpha,\beta) = \frac{\alpha - \Psi(\beta)}{\Phi(\alpha) - \beta}, \qquad \alpha, \beta > 0.$$

Scale functions

For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and elsewhere continuous and increasing satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

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Scale functions are fundamental to virtually **all** fluctuation identities concerning spectrally negative Lévy processes.
Let $\tau_a^- = \inf\{t > 0 : X_t < a\}$, $\tau_b^+ = \inf\{t > 0 : X_t > b\}$, $a, b \in \mathbb{R}$. We have the classical identity

$$\mathbb{E}_x(e^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

for $q \geq 0$, $0 \leq x \leq a$.

Applications in:

- ruin theory (first appearance in Tackács (1966), Zolotarev (1964)),

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \frac{W(x)}{W(\infty)}, \qquad W(\infty) = 1/\psi'(0+) \in (0, \infty).$$

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- fluctuation theory of Lévy processes,
- optimal stopping,
- optimal control,
- queuing and storage models,
- branching processes,
- insurance risk and ruin,
- credit risk,
- fragmentation.

Proof of the two sided exit formula $q = 0$ and $X_t \to \infty$ a.s.

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}.$$

For $y \geq 0$ let $h_y = \sup\{(S - X)\tau_{y-}^+ + t, 0 \leq t < \tau_y^+ - \tau_{y-}^+\}$,

$$\mathbb{P}(\tau_{a-x}^+ < \tau_{-x}^-) = \mathbb{P}\left(\#\{h_y > y + x,\ y \in [0, a - x]\} = 0\right),$$

By the Poissonnian structure of the excursions this is equal to

$$\exp\left\{-\int_{[0,\infty)} dy 1_{\{y\in[0,a-x]\}} \int_{\mathbb{D}} \overline{n}(de) 1_{\{h(e)>y+x\}}\right\}$$
$$= \exp\{-\int_x^{a+x} dy \overline{n}(h > y)\}.$$

Make $a \to \infty$, to get that

$$\mathbb{P}(-I_\infty \leq x) = \mathbb{P}(\tau_{-x}^- = \infty) = \exp\{-\int_x^\infty dy \overline{n}(h > y)\},$$

and verify that this has the right Laplace transform.
For a general X and q use a change of measure.

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