Long Horizon Stock Price Models or Stochastic Perpetuities

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Outline of Presentation

- Remarks on long horizon modeling issues.

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- Modeling balanced markets.

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- Valuation of coupon for stock above a barrier.

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- We infer from such observations that markets are comfortable in valuing claims with no clear distribution dates or explicit payoffs.
- You get some random payoffs sometime and possibly well past your remaining lifetime.

Discounted stock prices in financial engineering models

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- We ask here what kind of martingale should one be considering.
- In particular should it be uniformly integrable.

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- Imagine all gains, be they dividends, free cash flows or abnormal earnings, being invested at market rates, to engineer no distributions, with the resulting accumulation being embedded into the final stock price at time t.

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- Now discount back to time zero and let t go to infinity.
- There will be no future left after time t but the present value computed will converge to the present value of all gains over all time as evaluated at infinity.
- The resulting discounted gains martingale converges at infinity to the present value of all gains through time and is thus a uniformly integrable martingale.

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- In fact it converges to zero with probability one at infinity.
- We wish to consider uniformly integrable martingale models for the discounted stock price.

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- The martingale $m(t)$ represents the shocks to free cash flows or abnormal earnings.
- The discounted stock is the stochastic exponential of discounted shocks to earnings or free cash flows.

Explicit representation of positive martingale

- Suppose $m(t)$ has a continuous martingale component $m^{(c)}(t)$ and finite variation jump compensator $\nu(dx, dt)$ with

$$m(t) = m^{(c)}(t) + \int_0^t \int_{-\infty}^{\infty} x\left(\mu(dx, du) - \nu(dx, du)\right)$$

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$$m(t) = m^{(c)}(t) + \int_0^t \int_{-\infty}^{\infty} x\left(\mu(dx, du) - \nu(dx, du)\right)$$

- Then

$$M(t) = \exp\left(\begin{array}{c} \int_0^t Y(u)dm(u) - \frac{1}{2}\int_0^t Y^2(u)d\langle m^c\rangle_u \\ -\int_0^t \int_{-\infty}^{\infty} \left(\exp\left(Y(u)x\right) - 1\right)\nu(dx, du) \end{array} \right).$$

Balanced Market Condition

- For a stock in a balanced market we ask that M be uniformly integrable or equivalently that $M(\infty)$ is a well defined terminal random variable with

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- It is consistently valued through time in the market in line with its final resolution occuring at infinity.

Balanced Geometric Brownian Motion

- In the case of geometric Brownian motion with $m^c = \sigma W$ for a standard Brownian motion $W = (W(t), t \geq 0)$ and $Y(t) = e^{-rt}$ we have, $\langle m^c \rangle = \sigma^2 t$ and

$$M(t) = \exp\left(\int_0^t e^{-ru} \sigma dW(u) - \frac{\sigma^2}{2} \int_0^t e^{-2ru} du \right)$$

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- For $t < \infty$, $\log(M(t))$ is normal with mean $-(\sigma^2/2)(1 - \exp(-2rt))/(2r)$ and variance $\sigma^2 (1 - \exp(-2rt))/(2r)$.

Discounted Variance Gamma Model

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- This yields

$$m_Y(t) = \int_0^t e^{-ru} d\gamma_p(u) - \int_0^t e^{-ru} d\gamma_n(u).$$

Discounted Variance Gamma (DVG) with Parameters

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- This is accommodated by space scaling and a time change.
- Hence for parameters b_p, c_p, b_n, c_n we write for $X = (X(t), t \geq 0)$ the process

$$
\begin{aligned}
X(t) &= \int_0^t b_p e^{-ru} d\gamma_p(c_p u) - \int_0^t b_n e^{-ru} d\gamma_n(c_n u) \\
&= \int_0^{c_p t} b_p e^{-ru/c_p} d\gamma_p(u) - \int_0^{c_n t} b_n e^{-ru/c_n} d\gamma_n(u) \\
&= H_p(t) - H_n(t)
\end{aligned}
$$

Compensator for DVG

- The process $m(t)$ is given by the jump compensator

$$\nu(dx, dt) = \left(\mathbf{1}_{x<0} c_n \frac{\exp(-|x|/b_n)}{|x|} + \mathbf{1}_{x>0} c_p \frac{\exp(-x/b_p)}{x} \right) dxdt.$$

Characteristic function for DVG

- The characteristic function for $X(t)$ is given by

$$\phi_{X(t)}(u) = \exp\left(\begin{array}{c} \frac{c_p}{r}\left(dilog\left(iub_p\right) - dilog\left(iub_p e^{-rt}\right)\right) \\ +\frac{c_n}{r}\left(dilog\left(-iub_n\right) - dilog\left(-iub_n e^{-rt}\right)\right) \end{array} \right)$$

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- The *dilog* function is given by

$$dilog(x) = -\int_0^x \frac{\ln(1-t)}{t} dt.$$

Characteristic function for limiting random variable

- The limiting random variable $X(\infty)$ has characteristic function

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- The distribution of the random variable $X(\infty)$ is the stationary distribution for $G(t)$.

Exponential DVG

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- Given that $X(t)$ is a process of independent but inhomogeneous increments we get that $M = (M(t), t \geq 0)$ is a martingale where

$$M(t) = \exp\left(X(t) + \omega(t)\right).$$

Calibration of Exponential DVG to option data

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- $\widetilde{r}(t)$ and $q(t)$ are the annualized discount rates and dividend yields for maturity t consistent with the forward price for maturity t.

Unbalanced Markets

- For generality consider separate discounting functions for abnormal gains and losses whereby we write for the positive and negative components of the log price the processes

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\begin{aligned}
X_p(t) &= \int_0^t h_p(u)d\gamma^p(u) \\
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- We may lose uniformly integrability by the stochastic exponential of compensated jumps failing to be uniformly integrable on either the positive or negative side or both.
- For the positive side we have a boom market and for the negative side we have a crashing market or one in a bust.

Conditions for uniform integrability

- In the context of Wiener gamma integrals the required integrability conditions may be shown to be

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\begin{aligned}
\int_0^\infty \log\left(1 + h_i(u)\right) du &< \infty, \quad i = p, n. \\
-\int_0^\infty \log\left(1 - h_p(u)\right) du &< \infty.
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- The last condition is needed to ensure a finite exponential moment for X_p.

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- It follows also that

$$\int_0^\infty h_p(u) du = \infty$$

Drift action for booms

- The process $L_p(t)$ for the logarithm of the positive shocks may then be defined as

$$L_p(t) = \int_0^t h_p(u) d\gamma^p(u) + \int_0^t \log(1 - h_p(u)) du.$$

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- The expectation of $L_p(t)$ is

$$E\left[L_p(t)\right] = \int_0^t h_p(u) du + \int_0^t \log(1 - h_p(u)) du,$$

and as

$$-\log(1-x) > x$$

the expectation of $L_p(t)$ is negative and tending to negative infinity.

Summary for booms

- The drift dominates and the logarithm tends to negative infinity.

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- The terminal random variable is then zero.

The structure of busts

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- The process $L_n(t)$ for the logarithm of the negative shocks may be defined as

$$L_n(t) = -\int_0^t h_n(u) d\gamma^n(u) + \int_0^t \log(1 + h_n(u)) du.$$

Jump action for busts

- The expectation of the logarithm of the negative shocks is

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- yields that this expectation is negative and tending to negative infinity.
- However, this time the negative shocks are too large with insufficient compensation to keep the price from dropping.

Differences between booms and busts

- With regard to *booms* and *busts we observe* that a jump of size $x > 0$ is compensated in the exponential by a drift of $-(e^x - 1)$ and this dominates the jump with $(e^x - 1) > x$.

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- Similarly for a negative jump of size $-x < 0$ the compensation in the drift is $(1 - e^{-x}) < x$ and the jumps dominate the drift compensation.
- Hence risk neutrally *booms* are ended by the negative drift while jumps dominate the positive drift in *busts*.

DVGBB generalization for booms and busts

- The *DVG* model is extended to accomodate booms and busts to form the model *DVGBB*.

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- The *DVG* model is extended to accomodate booms and busts to form the model *DVGBB*.
- With a view to accomodating *booms* and *busts we* consider the Wiener-Gamma integral

$$
\begin{aligned}
K(t) &= \int_0^t \left(a + be^{-ru}\right) d\gamma(cu) \\
&= \int_0^{ct} \left(a + be^{-ru/c}\right) d\gamma(u)
\end{aligned}
$$

Laplace transform of K(t)

- The Laplace transform of $K(t)$ is given by

$$E\left[\exp\left(-\lambda K(t)\right)\right] = \exp\left(\frac{c}{r}\left(\Xi\left(\frac{1}{a+b}, a, \lambda\right) - \Xi\left(\frac{1}{a+be^{-rt}}, a, \lambda\right)\right)\right)$$

where

$$\begin{aligned}
\Xi\left(z, a, \lambda\right) &= \int \frac{\log\left(1+\frac{\lambda}{z}\right)}{z-az^2} dz \\
&= -dilog\left(\frac{a(\lambda+z)}{a\lambda+1}\right) \\
&\quad -\log\left(\lambda+z\right)\log\left(1-\frac{a(\lambda+z)}{a\lambda+1}\right) + dilog\left(az\right) \\
&\quad +\log(z)\log(1-az) - dilog\left(-\frac{z}{\lambda}\right) \\
&\quad +\log(z)\log(\lambda) \\
&\quad -\frac{1}{2}\log^2(z).
\end{aligned}$$

DVG calibrated to S&P 500 index options

- The discount rate for the response of the discounted stock price to abnormal earnings as captured in the processes H_p, H_n is taken to be fixed at a long term interest rate and in the reported calibrations this was set at the 30 year discount rate.

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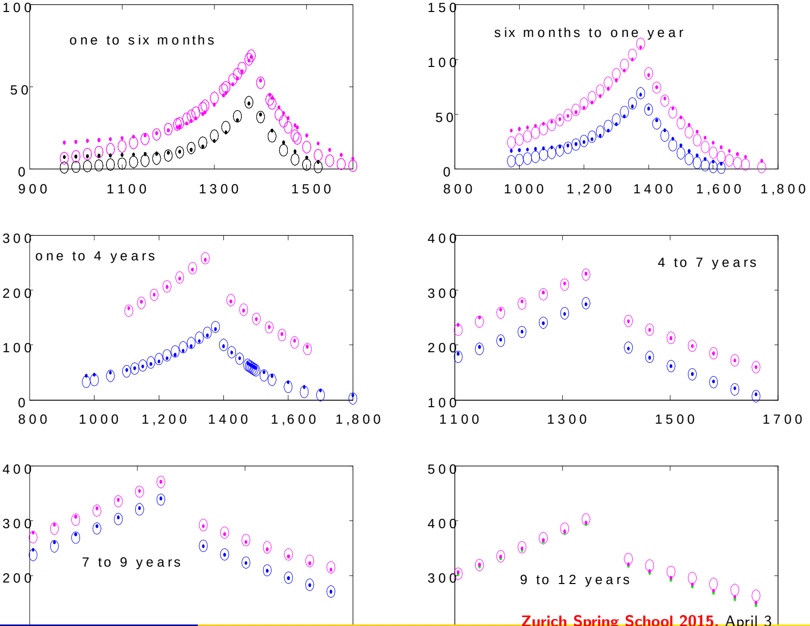
- We first fit the DVG model to 582 options covering 30 maturities for S&P 500 options as at market close on April 18 2012.

Calibration Result

- The result is as follows, where the fit statistics *rmse*, *aae* and *ape* are respectively, the root mean square error, the average absolute error and the average percentage error defined as the *aae* relative to the average option price in the data set.

$$
\begin{aligned}
r &= 0.02966 \\
b_p &= 0.0066 \\
c_p &= 512.9406 \\
b_n &= 122.4987 \\
c_n &= 0.0095 \\
rmse &= 4.9734 \\
aae &= 4.0684 \\
ape &= 0.0311
\end{aligned}
$$



one to six months
100
50
0
900
1100
1300
1500
six months to one year
150
100
50
0
800
1000
1,200
1400
1,600
1,800
one to 4 years
300
200
100
0
800
1000
1,200
1400
1,600
1,800
4 to 7 years
400
300
200
100
1100
1300
1500
1700
7 to 9 years
400
300
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9 to 12 years
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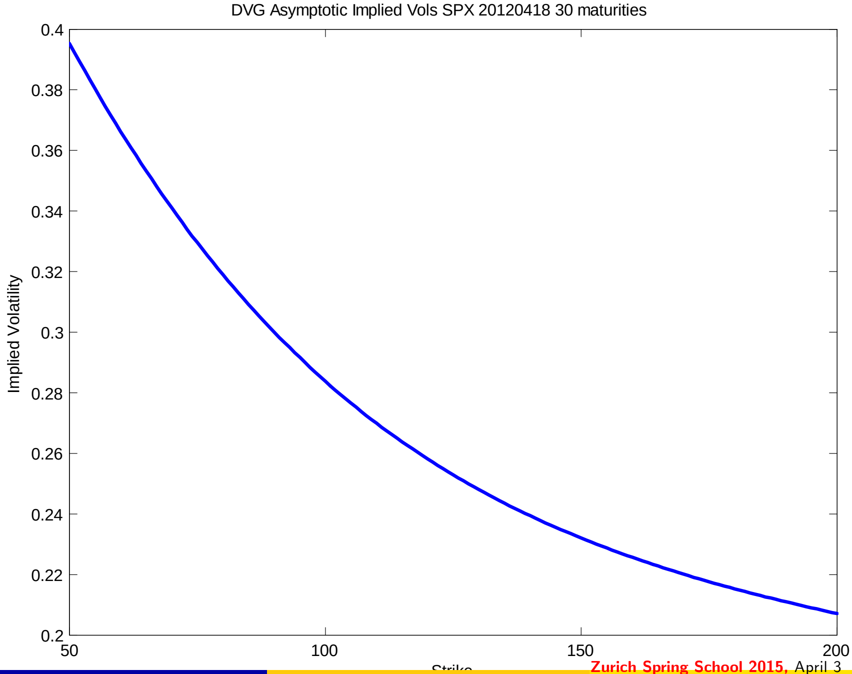
- The four parameter Sato process (Carr, Geman, Madan and Yor (2007)) does fit the option surface but has a time inhomogeneity that is financially not as well motivated as the inhomogeneity for the DVG model.
- Furthermore the Sato process generally cannot be extended to the larger maturities.
- Eventually the time scaling involved will lead to a loss of exponential moments for the scaled unit time random variable.

Asymptotic implied volatilities

- In addition we now have a well defined terminal random variable and we present the asymptotic implied volatility curve for options written on this terminal random variable in a zero rate economy.


DVG Asymptotic Implied Vols SPX 20120418 30 maturities
Implied Volatility
0.4
0.38
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
50
100
150
200
Strike

Calibration of DVGBB

- The *DVGBB* model was calibrated to the 582 options across 30 maturities for the S&P 500 index as at April 18 2012. The results were as follows.

$$
\begin{aligned}
a_p &= .0061415\\
b_p &= .0003486\\
c_p &= 512.9436\\
a_n &= .0112195\\
b_n &= 122.4965\\
c_n &= .0089424\\
rmse &= 4.3626\\
aae &= 3.6366\\
ape &= .02781
\end{aligned}
$$

DVGBB Calibration Remarks

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- In general statistical tests are not available for calibration exercises.
- Recently Madan (2013) has developed such procedures and future research could implement such significance tests for the boom and bust coefficients a_p, a_n.

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- We consider claims that depend on the entire path of our martingale $(M(t), t \geq 0)$.

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$$C(\infty) = (M(\infty) - K)^{+} + N \int_0^{\infty} e^{-rt} \mathbf{1}_{M(t)>a} dt$$

which is a call option on the terminal value of the discounted stock with strike K coupled with a note with notional N that is a perpetual coupon earned while the discounted stock is above a lower barrier.

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- Such products are now popularly issued by numerous entities in the financial sector and are attractive to the investment community seeking principal protected yield enhancement opportunities.

Time zero and time t values

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- The value in time t dollars is then given by
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- Now we know that

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\begin{aligned}
M(\infty) &= \exp\left(X(\infty) + \omega(\infty)\right), \\
X(\infty) &= \int_0^{\infty} b_p e^{-ru} d\gamma_p(c_p u) - \int_0^{\infty} b_n e^{-ru} d\gamma_n(c_n u)
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- By construction we have that

$$
\begin{aligned}
X(\infty) &= X(t) + \int_t^\infty b_p e^{-ru} d\gamma_p(c_p u) - \int_t^\infty b_n e^{-ru} d\gamma_n(c_n u) \\
&= X(t) + e^{-rt} Y
\end{aligned}
$$

where the law of Y is that of $X(\infty)$.

The law of Y in the interim

- Furthermore we also have for $u > t$

$$X(u) = X(t) + e^{-rt} Y(u - t)$$

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- The law of $Y(u)$ is that of

$$\int_0^u b_p e^{-rs} d\gamma_p(c_p s) - \int_0^u b_n e^{-rs} d\gamma_n(c_n s)$$

Evaluating the conditional expectation

- We then have

$$
\begin{aligned}
& E_t\left[C(\infty)\right] \\
= \quad & E_t\left[\begin{array}{c} (\exp\left(X(\infty)+\omega(\infty)\right)-K)^+ \\ +N\int_0^\infty e^{-rs}\mathbf{1}_{X(s)+\omega(s)>\ln(a)}ds \end{array}\right] \\
= \quad & E_t\left[\begin{array}{c} (\exp\left(X(t)+e^{-rt}Y+\omega(\infty)\right)-K)^+ + \\ \left(\begin{array}{c} N\int_0^t e^{-rs}\mathbf{1}_{X(s)>\ln(a)-\omega(s)}ds \\ +N\int_t^\infty e^{-rs}\mathbf{1}_{X(t)+e^{-rt}Y(s-t)+\omega(s)>\ln(a)}ds \end{array}\right) \end{array}\right] \\
= \quad & E_t\left[\begin{array}{c} (\exp\left(X(t)+e^{-rt}Y+\omega(\infty)\right)-K)^+ + \\ \left(\begin{array}{c} N\int_0^t e^{-rs}\mathbf{1}_{X(s)>\ln(a)-\omega(s)}ds \\ +e^{-rt}N\int_0^\infty e^{-ru}\mathbf{1}_{Y(u)>(\ln(a)-\omega(t+u)-X(t))e^{rt}}du \end{array}\right) \end{array}\right]
\end{aligned}
$$

Introduce the path statistic

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$$c(t) = \int_0^t e^{-rs} \mathbf{1}_{X(s) > \ln(a) - \omega(s)} ds.$$

- Given $X(t)$, e^{-rt}, $c(t)$ we may employ the terminal density of Y, $f(y)$ and the densities $g(y,u)$ for the random variables $Y(u)$ to construct the above expectation as a function

$$w(t) = H(X, e^{-rt}) + Nc(t)$$

Martingale property for w(t)

- We also know that $w(t)$ is a martingale and furthermore

$$H(X(\infty),0)+Nc(\infty)=\left(\exp\left(X(\infty)+\omega(\infty)\right)-K\right)^{+}+Nc(\infty).$$

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- The martingale condition yields
$$\begin{aligned} & -re^{-rt}H_u+e^{-rt}\mathbf{1}_{X>\ln(a)-\omega(t)} \\ & +\int_{-\infty}^{\infty}(H(X+y,e^{-rt})-H(X,e^{-rt}))k(y)dy \\ = & \; 0 \end{aligned}$$

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- or in terms of u

$$\begin{aligned} & -ruH_u(X,u)+u\mathbf{1}_{X>\ln(a)-\omega(-\frac{1}{r}\ln(u))} \\ & +\int_{-\infty}^{\infty}\left(H(X+y,u)-H(X,u)\right)k(y)dy \end{aligned}$$

Direct computation of H(X,u)

- We may also directly compute the value as

$$
\begin{aligned}
& H(X,u) \\
= \; & E_Y\left[\left(\exp\left(X + uY + \omega(\infty)\right) - K\right)^+\right] \\
& + E_{(Y(s),s\geq 0)}\left[uN\int_0^\infty e^{-rs}\mathbf{1}_{Y(s)>(\ln(a)-\omega(-\frac{\ln(u)}{r}+s)-X)/u}ds\right]
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$$
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& H(X,u) \\
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& + E_{(Y(s)+\omega(s),s\geq 0)} \left[uN \int_0^\infty e^{-rs} \mathbf{1}_{Y(s)+\omega(s) > (\ln(a) - \omega(-\frac{\ln(u)}{r} + s) + u\omega(s)
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- In this way we may compute the values of claims through time that are defined at infinity and in the interim have no cash flows but just a market value process.

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- Alternatively in terms of the discounted stock martingale we could write

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A GBM computation

- Suppose we take for our martingale the classical geometric Brownian motion model whereby

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- Suppose we take for our martingale the classical geometric Brownian motion model whereby

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- Compare this with the discounted approach which would give a discounted shock integral with

$$M_D(t) = \exp\left(\int_0^t e^{-ru}\sigma dW(u) - \frac{\sigma^2}{2}\int_0^t e^{-2ru}du\right)$$

Perpetuity values

- The values of these perpetuities are respectively w_C, w_D with

$$
\begin{aligned}
w_C &= c\int_0^\infty e^{-rt} N\left(-\frac{\ln(a/S_0)}{\sigma\sqrt{t}} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{t}\right) dt \\
w_D &= c\int_0^\infty e^{-rt} N\left(\begin{array}{c} -\frac{\ln(a/S_0)}{\sigma\sqrt{\frac{1-e^{-2rt}}{2r}}} + \frac{rt}{\sigma\sqrt{\frac{1-e^{-2rt}}{2r}}} \\ -\frac{\sigma}{2}\sqrt{\frac{1-e^{-2rt}}{2r}} \end{array}\right) dt
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$$

Numerical Estimates

- For 70% down or $a/S_0 = .7$ at an interest rate of 3% and a volatility of 25% we have

$$\begin{aligned} w_C &= 21.1327c \\ w_D &= 29.5941c \end{aligned}$$

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- The classical model could then be seriously underpricing such contracts and thereby offering larger coupons than would be justified by a discounted approach.

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- We observe that the *DVG* fit to the surface of option prices is certainly much better.

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w_{DVG} &= -\int_0^\infty \frac{\partial c_{DVG}(.7,t)}{\partial K}dt
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$$

- We perform these calculations for the estimated parameters at $r = .02966$ the rate used in the DVG calibrations.

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- We also present a graph of the integrands employed in the calibrated *VG* and *DVG* models.

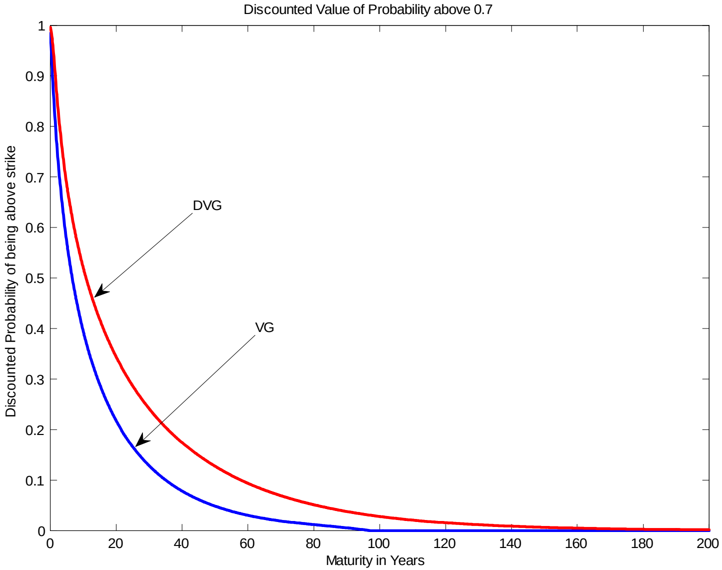

Figure: The discounted probability of being above the strike of 0.7 for the calibrated VG and DVG models as a function of the maturity.

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- This makes the discounted gains martingale the conditional expectation of its limit.
- The same should be true for long horizon modeling of balanced stock price processes in the absence of dividend distributions.
- Out of balance, and then in the absence of a limiting random variable, *booms* or *busts* are distinguished by the manner in which one loses uniform integrability, by either the positive shock or negative shock martingales failing to be uniformly integrable.

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- In addition procedures are presented for the valuation of path dependent stochastic perpetuities that never have a payoff and it is argued that as they may be easily valued theoretically, that markets may value as them as well.
- For the new models it is also observed that implied volatility curves do not flatten out as they have nontrivial non-Gaussian limits at infinity.