Hedging Insurance Books

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Outline of Presentation

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- Design of multiplier underlier simulations

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- We present methodologies for constructing good multi underlier options hedges.
- Hedges are designed not for replication, as this is not possible, but to enhance the value of the hedged position.
- We generalize the static hedging approach of Carr and Wu (2014) to the wider context admitting no replication.

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- The hedging technology can be extended to many other contexts.

Financial path space simulation

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- Especially when it comes to multiple underliers over long periods of time, that are the typical context for insurance products.
- There is a strong temptation to attempt to simulate the physical process as learned from time series data.
- Yet we learn from the market valuation of financial products that rare events with low probability have a much higher price relative to the probability (Bollerslev and Todorov (2011)).

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- It is little comfort that theoretically the two probabilities are equivalent.
- For on the finite sample space on which decisions are to be based, the relevant events are lost.
- They do not occur and measure changes cannot compensate for nonexistence.

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- Simultaneously the probability of smaller moves are also lowered.
- However, the risk neutral process may be estimated well for a single underlier from data on option prices for this underlier.
- The risk neutral process for the joint law on multiple underliers is not that readily available.

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- In the absence of joint information, one may proceed further by making additional modeling assumptions.

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- The mixing matrix is further assumed to be risk neutrally relevant.
- Risk neutral laws for the independent components are sought with a view to repricing all options on all underliers across their strikes and maturities at market close on a single day.

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- For some assessment in this direction one may wish to be explicit about the risks that are being priced and the market prices being assumed.

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- Return distributions over small intervals are Gaussian and locally risk is adequately characterized by covariance.
- Such a paradigm has served us well, as is effectively documented by the large literature based on such assumptions.

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- The modeling of risk by continuous processes unrealistically assumes away such exposures.
- The local motion for continuous processes is however Gaussian.
- A good feature of the Gaussian model has always been the observation that it is a limit law.
- Sums of large numbers of independent effects or shocks tend to a Gaussian distribution.

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- However, no jump diffusions are limit laws as they are not self decomposable.
- Limit laws have infinite arrival rates of jumps and are rich enough to permit dispensing with the diffusion component.

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- This function is clearly decreasing for positive x as it is $\exp(-Mx)$ and increasing for negative x as it is $\exp(Gx)$.

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- The parameters price the overall jump size, the asymmetry between down and up moves and the spread for large over small moves.
- The parameters may be estimated from a combination of time series and option data.

Variance Gamma details

- The variance gamma process $X(t; \sigma, \nu, \theta)$ was originally introduced (M. Carr and Chang (1998)) as Brownian motion with drift θ and volatility σ time changed by a gamma process $g(t; \nu)$ of unit mean rate and variance rate ν.

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- The transformations between σ,ν,θ and C, G, M may be found in Schoutens (2003).

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- Hence the one step simulation is that of correlated normals, transformed to risk neutralized variance gamma marginal distributions.
- Non-Gaussian simulations of correlated random variables along these lines are also used in Bouchaud and Potters (2003).

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- If the account value hits zero before death, the account holder receives a percentage of the base to death and all premiums stop.

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- θ is the vector of account proportions invested in the equity assets and dN is the martingale component of equity returns.

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- η_n is a positive constant, η, for n less than or equal to N and zero thereafter.

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- For an account holder of age α entering the contract at time zero, the present value at time t of the payout until death, if the account value hits zero and death has not occurred is

$$V(t) = xB(t) \int_0^\infty e^{-rh} G(\alpha + t + h) dh.$$

Target cash flow construction

- We simulate the path space of equity underliers $S_{k,t}^{(j)}$ for equity asset k, at time t, for $k = 1, \cdots, K$, $t = 1, \cdots T$ on path j with $j = 1, \cdots, J$.

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- They are positive for the insurance component of withdrawals after the account value has gone to zero and prior to the account holders death.

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- We then simulate the account values of numerous account holders $A_i(t)$ and the associated payouts $C_{i,t}^{(j)}$
- These are negative for the premiums paid both as a percentage of the account value and a percentage of the base.
- They are positive for the insurance component of withdrawals after the account value has gone to zero and prior to the account holders death.
- We record this insurance benefit as a lump sum paid out at the time of the account value going to zero before death.

Target cash flow construction II

- We then form the aggregate present value of all payouts to each account holder on each path by

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- The target cash flow is $\mathcal{C} = (\mathcal{C}^{(j)}, j = 1, \cdots, J)$.

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- We also allow investment in the underliers themselves until the same horizon of H.
- The cash flows to the options and the investment in the underliers with payoffs occuring at time H is given by the $D \times J$ matrix $\mathcal{H}$ where $\mathcal{H}_d^{(j)}$ is the financed payoff to hedging asset d on path j in row d and column j of $\mathcal{H}$ for $d = 1, \cdots D$ and $j = 1, \cdots J$.

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- We thus write
$$\mathcal{C}' = a + \mathcal{H}' x_{LS} + u$$
- The bond position is a, the least squares hedge is x_{LS} and u is the least squares residual.

Building the ask price hedge

- Our second hedge first removes constants from the sample space to define the risk components

$$\widetilde{\mathcal{C}} = \mathcal{C} - \frac{1}{J}\sum_j \widetilde{\mathcal{C}}^{(j)}$$
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- The centered residual cash flow is

$$\widetilde{\mathcal{R}}(x) = x\widetilde{\mathcal{H}} - \widetilde{\mathcal{C}}.$$

The hedged liability

- We may think of $\widetilde{\mathcal{C}}$ as a random variable and of

$$\widetilde{\mathcal{W}}(x) = \widetilde{\mathcal{C}} - x\widetilde{\mathcal{H}}$$

as a hedged position.

Hedge acceptability

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- The bid and ask prices, b, a respectively, for X require that

$$X - b(1+r) \in \mathcal{A},\ a(1+r) - X \in \mathcal{A},$$

(see for example Carr, M. and Vicente Alvarez (2011)).

Bid Ask Functionals

- Equivalently for all $Q \in \mathcal{M}$

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- The best bid and ask prices for X provided by the market, $\widetilde{b}(X)$, $\widetilde{a}(X)$ respectively, are then given by

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\begin{aligned}
\widetilde{b}(X) &= \frac{1}{1+r} \inf_{Q \in \mathcal{M}} E^Q[X] \\
\widetilde{a}(X) &= \frac{1}{1+r} \sup_{Q \in \mathcal{M}} E^Q[X].
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- From the relationship of the sup to the inf we observe that the ask price is the negative of the bid price for the negative.

Computing Bid Ask

- On a finite sample space one may order outcomes in increasing order for $X_{(n)}$ being the n^{th} largest outcome for $n = 1, \cdots, N$ and employ the empirical distribution function to evaluate the bid price as

$$\widetilde{b}(X) = \sum_{n} X_{(n)} \left(\Psi^{\gamma}\left(\frac{n}{N}\right) - \Psi^{\gamma}\left(\frac{n-1}{N}\right)\right).$$

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- In our case this is given by

$$z(x) = -\sum_{j=1}^{J} \widetilde{R}_{(j)}(x) \left(\Psi^{\gamma} \left(\frac{j}{J} \right) - \Psi^{\gamma} \left(\frac{j-1}{J} \right) \right).$$

Ask price as a hedge criterion

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- Our third hedge is chosen to minimize the ask price for the hedged liability under the further constraint that the option positions are kept non-negative.
- We refrain from becoming an option seller with a view towards lowering ask prices computed on a finite sample space.

Physical estimation of variance gamma law

- The estimation was conducted on 501 observations for two years of data on the nine underliers *xlb*, *xle*, *xlf*, *xli*, *xlk*, *xlp*, *xlu*, *xlv* and *xly*.

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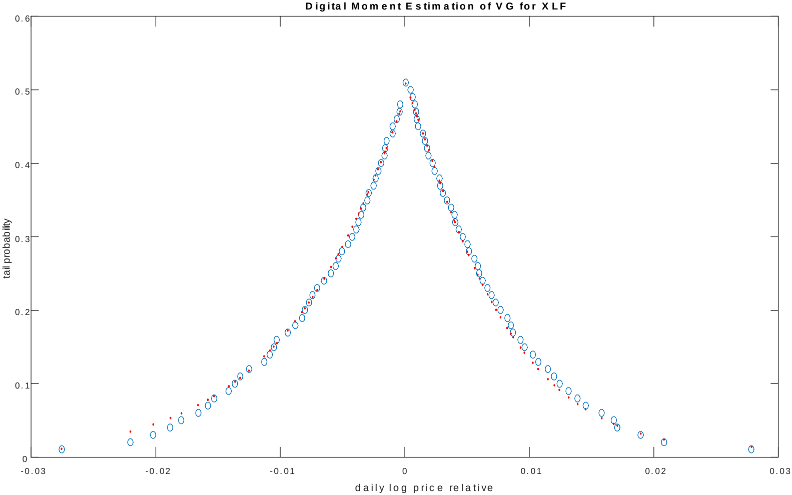
- For positive strikes we use digital calls were priced, as the probability observed for being above the selected strike, while for negative strikes digital puts were priced, at the probability of being below the corresponding strike.
- The variance gamma model was fit by least squares minimization of the percentage squared error between the observed probability of being in the tails and the model probability for the same.

Sample fit for XLF



VG parameter values

		Table 1	
	Digital Moment Estimation		
	of VG Parameters		
ETF	sigma	nu	theta
XLB	0.010608	0.769945	-0.002265
XLE	0.010006	0.362770	-0.005162
XLF	0.010797	0.634793	-0.001855
XLI	0.009021	0.420204	-0.003797
XLK	0.007949	0.239938	-0.005311
XLP	0.006385	0.393545	-0.001448
XLU	0.007222	0.344631	-0.001208
XLV	0.006965	0.165835	-0.003199
XLY	0.008464	0.301066	-0.003845

VG parameter values in CGM format

	Table 2		
	Estimated VG Parameters		
	in CGM format		
ETF	C	G	M
XLB	327.30	133.13	173.39
XLE	694.65	188.70	291.81
XLF	396.98	149.25	181.08
XLI	599.71	199.65	292.95
XLK	1050.27	288.75	456.85
XLP	640.33	319.34	390.37
XLU	731.22	311.21	357.53
XLV	1519.58	437.00	568.87
XLY	837.03	255.53	362.85

Risk Neutralization of Physical Laws

- It is shown in M. and Schoutens (2015) that the market price of jump risk by jump size, $Y(x)$ has the form

$$Y(x) = a \exp\left(bx + c|x|\right)$$

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- For all nine underliers we take the average market prices of risk given by the parameters a, b, and c and this relationship to construct risk neutral VG parameters at the one year point.
- For the nine underliers $k(x)$ is the annualized physical Lévy measure reported above in Table 2.

Market prices of risk employed

		Table 3	
	Market prices of Risk		
ETF	a	b	c
XLB	0.004111	17.7897	146.4575
XLE	0.001861	48.2356	234.1455
XLF	0.002849	14.6195	161.0551
XLI	0.001883	44.4990	240.7047
XLK	0.000924	80.9255	366.7470
XLP	0.001276	36.2157	354.6252
XLU	0.001006	22.2935	331.4028
XLV	0.000435	62.3980	495.7091
XLY	0.001019	54.0275	309.2121

Parameters in

σ, ν, θ format

		Table 4	
	Risk Neutralized VG Parameters		
ETF	sigma	nu	theta
XLB	0.2568	0.7432	-0.1544
XLE	0.3135	0.7733	-0.3262
XLF	0.3859	0.8841	-0.1930
XLI	0.2911	0.8857	-0.1824
XLK	0.2688	1.0303	-0.2260
XLP	0.7172	1.2240	-0.2667
XLU	0.4274	1.3591	-0.1589
XLV	0.1824	1.5111	-0.1177
XLY	0.3127	1.1722	0.0330

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- The one month marginal distributions are those of a *VG* process with these annualized parameters.
- However the simulated underliers are generated as correlated normals that are transformed nonlinearly to have the appropriate marginal *VG* distribution.
- The simulated path space is stored in a three dimensional matrix of size $480 \times 10000 \times 9$.

Correlation matrix employed

				Table 5					
				Correlation Matrix					
	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
XLB	1.0000	0.8076	0.8046	0.8505	0.7638	0.5846	0.5150	0.6595	0.7858
XLE	0.8076	1.0000	0.7897	0.8006	0.7244	0.5909	0.4965	0.6383	0.7466
XLF	0.8046	0.7897	1.0000	0.8412	0.7633	0.6375	0.5071	0.7072	0.8073
XLI	0.8505	0.8006	0.8412	1.0000	0.8054	0.6541	0.5330	0.7380	0.8450
XLK	0.7638	0.7244	0.7633	0.8054	1.0000	0.6059	0.4484	0.6706	0.8061
XLP	0.5846	0.5909	0.6375	0.6541	0.6059	1.0000	0.6628	0.7286	0.7160
XLU	0.5150	0.4965	0.5071	0.5330	0.4484	0.6628	1.0000	0.5404	0.5321
XLV	0.6595	0.6383	0.7072	0.7380	0.6706	0.7286	0.5404	1.0000	0.7759
XLY	0.7858	0.7466	0.8073	0.8450	0.8061	0.7160	0.5321	0.7759	1.0000

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- The increment of the base was at five percent or the improvement in the account value, whichever was greater, for the first ten years.
- Thereafter it went up only on account of the account value.

Account holder details

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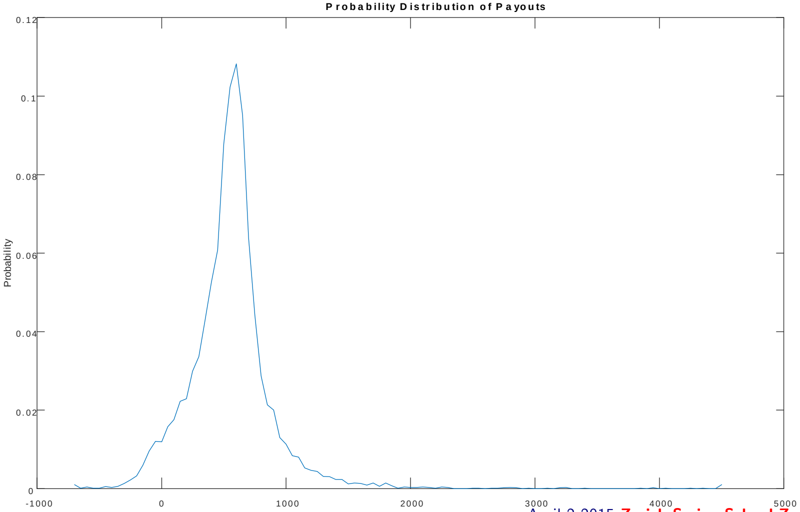
- The account holders received as a lump sum the expected present value of five percent of the base from the time the account value goes to zero to death, when the former was earlier

Target cash flow distribution



The space of hedging assets

- For a maturity close to six months on December 22, 2014, we obtained data on the prices of options on the nine underliers with strikes within 50% of the spot.

					Table 6				
			Number of Options and strike ranges on Underliers						
Variable	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
Number	30	79	14	26	18	20	22	34	33
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- The hedge is constructed on a shaved sample space of 7024 paths.
- We removed paths with the six month outcome below 50 or above 200 for an initial value reset to 100.

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- It appears that the ask price minimizing hedge when unconstrained takes some large long positions in the stocks. It also takes bigger option positions.
- From a cost perspective the constrained ask price minimization appears the most favorable alternative.

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- Of course the smoothed functions would have to be rehedged to determine the required option positions, but this is a simple exercise not conducted here.

Residual Statistics

	Table 7		
	Residual Statistics		
Variable	Least Squares	Ask Price	Ask Price Constrained
std	371.5509	391.4379	378.8404
skew	-3.9751	-2.2213	-4.2935
kurtosis	56.0941	28.2784	63.1922
peakedness	0.8161	0.7977	0.8155
tailweightedness	0.0108	0.0108	0.0111
1%	-1061.3622	-1033.5319	-1106.1660
5%	-501.0562	-520.2784	-501.4194
10%	-314.0071	-347.6303	-316.1633
25%	-136.0459	-180.1364	-128.0253
50%	-0.5341	-14.1656	-3.0193
75%	173.8528	193.7751	169.0452
90%	385.0372	434.0794	389.3928
95%	517.9756	585.9301	528.4485
99%	724.4330	904.5527	715.0523

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- The first two if smoothed would have to be reevaluated as well.

Kernel estimation of conditional expectations

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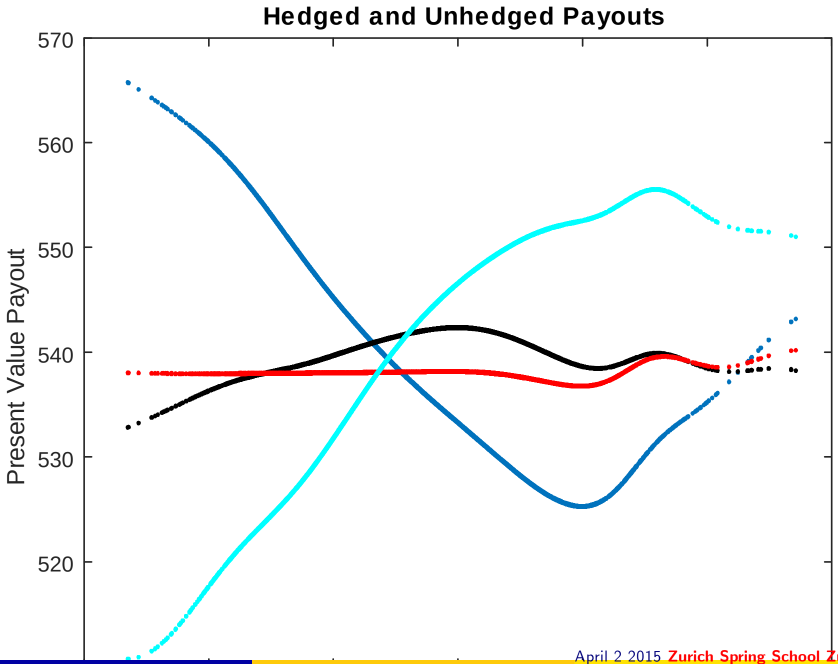
- We observe that least squares is the flatest followed by the constrained ask minimizing hedge.


Hedged and Unhedged Payouts
Present Value Payout
570
560
550
540
530
520

Conditional Ask Prices

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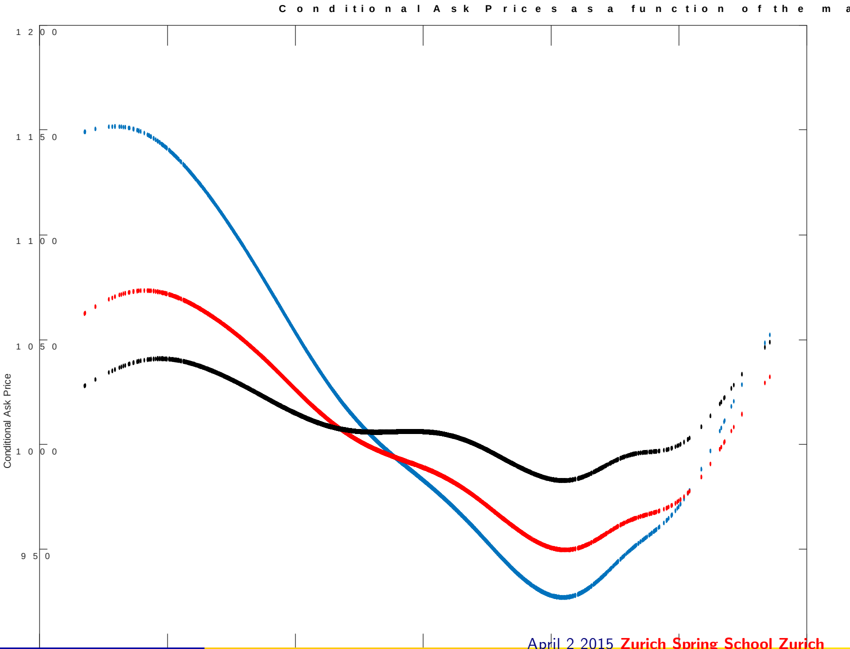
- We observe that the constrained ask has the flatest conditional ask price.


Conditional Ask Prices as a function of the ma
Conditional Ask Price
1200
1150
1100
1050
1000
950

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- The strategy is illustrated by hedging GMWBVA accounts invested in the nine sector ETF's of the US economy.
- The implementation requires a risk neutral simulation of underliers and this accomplished by writing the underliers as transformed correlated normals with the transformation respecting risk neutrality at the simulation horizon.

Concluding Remarks II

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Concluding Remarks II

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- The last of these delivers a least cost and most stable result.