A Theory of Risk for Two Price Economies

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The Classical Pricing Problem

- Asset pricing in liquid financial markets has developed the theory of risk neutral valuation.
- Based on principles of no arbitrage discounted prices for claims with no intermediate cash flows are seen to be martingales under a suitably chosen equilibrium pricing probability.
- **The martingale condition in Markovian contexts reduces the pricing problem to an equivalent solution of a linear partial differential or integro-differential equation subject to a boundary condition at maturity.**
- The essential property of market liquidity is the supposition of the law of one price or the ability, on the part of market participants, to trade in both directions at the same price.

The Two Price Economy

- In the absence of such liquidity, the law of one price is abandoned and we get at a minimum a two price economy where the terms of trade depend on the direction of trade.

- Such a two price equilibrium was studied in a static one period context in M. (2012).

- **The two equilibrium prices arise on account of an exposure to residual risk that cannot be eliminated, by construction, and the prices are designed to make this exposure acceptable.**

- Acceptability is modeled by requiring positive expectations under a whole host of test or scenario probabilities as described for example in Artzner, Delbaen, Eber and Heath (1999).

- As a consequence the ask or upper price turns out to be the supremum of test valuations while the bid or lower price is an infimum of the same set of test valuations.

- The resulting pricing operators are now nonlinear on the space of random variables, with the lower price being concave and the upper price convex.

- In particular the upper price of a package of risks is smaller than the sum of component prices while the lower price is similarly above.

The Other Direction

- Suppose securities are being traded at a price spread for the buying price over the selling price.

- **In the absence of arbitrage there must exist test or scenario probabilities that value the securities at a level between the selling and buying price.**

- **One may then define the cone of acceptable risks as those with positive expectation for all valuations conducted using probabilities pricing traded securities in the so-called bid ask interval.**

- Hence given a cone of acceptable risks two prices result.

- Also given two prices and no arbitrage a cone of acceptability results.

- We place our selves in such economies where the law of one price is no longer available for buying or selling anything.

- We develop instead the theory for the two prices in continuous time.

An Important Aside

- What is being developed is not to be confused with transactions costs that get the two prices from the one price by inflating and deflating by a cost factor.

- The two prices we work with satisfy nonlinear equations reflecting charges for risk exposure and are not mere multiples of a one price theory.

- **Also importantly for insurance theoretic applications we get a separation of a theory of liability valuation from a theory asset pricing as in our economy assets are priced at the lower or selling price and liabilities are priced at the upper or buying price.**

- We do mark to market, but ours is a two price market that treats liabilities differently from assets.

The Two Prices Using Probability Distortions

- When the decision of risk acceptability is further modeled as solely depending on the probability distribution of the risk and if in addition we ask for additivity of the two prices for risks that are monotonically related then closed forms for the two prices become available Kusuoka (2001).

- **Specifically, the lower price may be expressed as an expectation computed after distorting the risk distribution function by composing it with a prespecified concave distribution function on the unit interval.**

- Such a formulation was proposed and tested on option market data in Cherny and M. (2010).

- Carr, M. and Vicente Alvarez (2011) employ this approach to define capital requirements and up front profits on trades.

- Eberlein and M. (2012) apply the method to estimating capital requirements for the financial sector during the financial crisis of 2008.

Dynamic Two Price Sequences in Discrete Time

- Dynamically consistent two price sequences based on locally applying probability distortions are examples of non-linear expectations as studied in Cohen and Elliott (2010).

- M. and Schoutens (2012) apply such pricing principles to study the impact of illiquidity on a variety of financial markets.

- The lower price is a submartingale while the upper price is a supermartingale with the two prices converging to each other and the payout at maturity.

- M., Wang and Heckman (2011) apply discrete time distortion based nonlinear expectations to the valuation of insurance liabilities.

Two Price Economies in Continuous Time

- In continuous time the two prices are related to nonlinear expectations seen as the G-expectations introduced by Peng (2004).

- The use of probability distortions to formulate G-expectations for the upper and lower price was developed in Eberlein, M., Pistorius, Schoutens and Yor (2012).

- This paper extends the theory of distortion based G-expectations in two directions.

- **The first is to generalize away from probability distortions to measure distortions as they arose in M., Pistorius and Stadje (2013) where the continuous time limit of discrete time distortion based nonlinear expectations was developed.**

- Here we directly introduce and apply measure distortions.

- The second extension deals with the convergence of bid ask spreads to zero at maturity.

- **Though many contracts have explicit maturities, economic activities of running airlines, insuring losses, selling goods and services need to be, and are valued, in financial markets with no apparent maturity.**

- M. and Yor (2012) introduce valuation models for such claims termed stochastic perpetuities conducted under a liquid, law of one price setting.

- The resulting martingales are uniformly integrable and the explicit maturity is transferred to infinity.

- Here we extend distortion based G-expectations to valuation processes with an infinite maturity.

Illustrative Applications

- The theory is illustrated on the two price valuation of stocks.

- We employ both the quadratic variation based probability distortion introduced in Eberlein, M., Pistorius, Schoutens and Yor (2012) and the new measure based distortion introduced here.

- We also apply measure distortions to value compound Poisson processes of insurance loss liabilities for both the homogeneous and inhomogeneous cases.

Introduction to Measure Distortions

- We introduce the use of measure distortions for defining the acceptability of a set of random variables in the context of a static one period model.

- In continuous time we apply this structure at the instantaneous level to local risk characteristics.

- Consider first the acceptability of a random variable X with distribution function $F(x)$.

- When acceptability is defined just in terms of the distribution function it may be reduced to a positive expectation under a fixed concave distortion.

- More precisely, let Ψ be a fixed concave distribution function defined on the unit interval.

- The random variable X is acceptable just if the expectation of X taken with respect to the distorted distribution function $\Psi(F(x))$ is nonnegative, or

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0,$$

where $\mathcal{E}(X)$ refers to a distorted expectation.

From Distorted Expectation to Choquet Integrals

- After splitting the distorted expectation at zero and integrating by parts we may write the distorted expectation as a Choquet integral in the form

$$\mathcal{E}(X) = \int_0^\infty (1 - \Psi(F(x)))dx - \int_{-\infty}^0 \Psi(F(x))dx.$$

- It is now useful to follow M., Pistorius and Stadje (2012) and introduce

$$\widehat{F}(x) = 1 - F(x)$$

and

$$\widehat{\Psi}(x) = 1 - \Psi(1 - x)$$

and write **the distorted expectation more symmetrically as**

$$\mathcal{E}(X) = \int_0^\infty \widehat{\Psi}(\widehat{F}(x))dx - \int_{-\infty}^0 \Psi(F(x))dx.$$

Connecting to Two Price Economies

- Formally, Artzner, Delbaen, Eber and Heath (1999) show that acceptable random variables form a convex cone and earn their acceptability by having a positive expectation under a set $\mathcal{M}$ of scenario or test measures equivalent to the original probability P.

- Cherny and M (2010) then introduce the bid, $b(X)$, and ask, $a(X)$ or upper or lower prices as

$$
\begin{aligned}
b(X) &= \inf_{Q \in \mathcal{M}} E^Q[X] \\
a(X) &= \sup_{Q \in \mathcal{M}} E^Q[X]
\end{aligned}
$$

  with acceptability being equivalent to $b(X) \geq 0$. As a consequence $b(X) = \mathcal{E}(X)$.

- The set of measures Q supporting acceptability or the set $\mathcal{M}$ is identified in M., Pistorius and Stadje (2012) as all measures Q, absolutely continuous with respect to P, with square integrable densities, that satisfy for all sets A, the condition

$$\widehat{\mathbf{\Psi}}(P(A)) \leq Q(A) \leq \mathbf{\Psi}(P(A)).$$

- The probability bounds may be explained in terms charging prices $Q(A)$ for lotteries $\mathbf{1}_A$ that rule out acceptability for buyers and sellers.

Remarks on the Distortions

- We now remark on the distortion Ψ and its complementary distortion $\widehat{\Psi}$.

- **Both distortions are monotone increasing in their arguments but Ψ is concave and bounded below by the identity function while $\widehat{\Psi}$ is convex and bounded above by the identity function.**

- For the bid price or the distorted expectation one employs the concave distortion on the losses or negative outcomes while one employs the convex distortion on the gains or positive outcomes.

- This is reasonable as distorted expectations are expectations under a change of measure with the measure change being the derivative of the distortion taken at the quantile.

- The concave distortion then reweights upwards the lower quantiles associated with large losses, while the convex distortion reweights downward the upper tail.

Distorting Integrals with respect to a measure

- Consider now in place of an expectation an integration with respect to a positive, possibly infinite measure μ or the measure integral

$$m = \int_{-\infty}^{\infty} v(y)\mu(dy) < \infty$$

- Though the measure may be infinite, we suppose that all the tail measures are finite.

- We may then rewrite the measure integral as

$$\begin{aligned} m &= -\int_{-\infty}^{0} \mu\left((v(y) \leq x)\right) dx \\ &\quad + \int_{0}^{\infty} \mu\left((v(y) \geq x)\right) dx. \end{aligned}$$

Bringing in the measure distortions

- We now consider two functions Γ_+, Γ_- defined on the positive half line that are zero at zero, monotone increasing, respectively concave and convex, and respectively bounded below and above by the identity function.

- These functions will now be used to distort the measure μ and we refer to them as measure distortions.

- **We then define the distorted measure integral as**

$$\begin{aligned} \mathfrak{m} &= -\int_{-\infty}^{0} \Gamma_+ \left(\mu\left((v(y) \leq x)\right)\right) dx \\ &\quad + \int_{0}^{\infty} \Gamma_- \left(\mu\left((v(y) \geq x)\right)\right) dx, \end{aligned}$$

where we assume both integrals are finite.

- For computational purposes we shall employ

$$\begin{aligned}\mathfrak{m} &= \int_{-\infty}^{0} xd\left(\Gamma_{+}\left(\mu\left((v(y) \leq x)\right)\right)\right) \\ &\quad - \int_{0}^{\infty} xd\left(\Gamma_{-}\left(\mu\left((v(y) \geq x)\right)\right)\right)\end{aligned}$$

- Acceptability of a random outcome with respect to a possibly infinite measure with finite tail measures may then be defined by a positive distorted measure integral.

- M., Pistorius and Stadje (2012) identify the set of supporting measures as absolutely continuous with respect to μ with square integrable densities that satisfy for all sets A, for which $\mu(A) < \infty$ the condition

$$\Gamma_{-}(\mu(A)) \leq Q(A) \leq \Gamma_{+}\left(\mu(A)\right).$$

  We shall replace measure integrals by distorted measure integrals in defining G-expectations as solutions of nonlinear partial integro-differential equations.

The discounted variance gamma model

- We now introduce the discounted variance gamma (dvg) model of M. and Yor (2012) as the driving uncertainty for the stock price.

- The discounted stock price is modeled as a positive martingale on the positive half line.

- The discounted stock price responds to positive and negative shocks given by two independent gamma processes.

- The variance gamma model of M. and Seneta (1990), M., Carr and Chang (1998) has such a representation as the difference of two independent gamma processes, but unlike the variance gamma process, as we now consider perpetuities, the shocks are discounted in their effects on the discounted stock price.

- **More specifically, let $\gamma_p(t)$ and $\gamma_n(t)$ be two independent standard gamma processes (with unit scale and shape parameters) and define for an interest rate r the process**

$$X(t) = \int_0^t b_p e^{-rs} d\gamma_p(c_p s) - \int_0^t b_n e^{-rs} d\gamma_n(c_n s).$$

The parameters $b_p > 0, c_p > 0$ and $b_n > 0, c_n > 0$ reflect the scale and shape parameters of the undiscounted gamma processes, however, $X(t)$ accumulates discounted shocks.

- The characteristic function for $X(t)$ is explicitly derived in M. and Yor (2012) and is shown to be

$$E\left[\exp\left(iuX(t)\right)\right] = \exp\left(\begin{array}{c} \frac{c_p}{r}\left(dilog\left(iub_p\right) - dilog\left(iub_p e^{-rt}\right)\right) \\ +\frac{c_n}{r}\left(dilog\left(-iub_n\right) - dilog\left(-iub_n e^{-rt}\right)\right) \end{array}\right)$$

where the $dilog$ function is given by

$$dilog(x) = -\int_0^x \frac{\ln(1-t)}{t} dt.$$

The DVG driven discounted stock price

- The discounted stock price driven by the discounted variance gamma process is given by the positive martingale

$$M(t) = \exp\left(X(t) + \omega(t)\right)$$

where

$$\exp\left(\omega(t)\right) = \frac{1}{E\left[\exp\left(X(t)\right)\right]}.$$

- **Unlike geometric Brownian motion or exponential Lévy models, this martingale is uniformly integrable on the half line and the discounted stock price at infinity is a well defined positive random variable**

$$M(\infty) = \exp\left(X(\infty) + \omega(\infty)\right)$$

where

$$\begin{aligned} X(\infty) &= \int_0^\infty b_p e^{-rs} d\gamma_p(c_p s) \\ &\quad - \int_0^\infty b_n e^{-rs} d\gamma_n(c_n s) \end{aligned}$$

and

$$\begin{aligned} & E\left[\exp\left(iuX(\infty)\right)\right] \\ = \ & \exp\left(\frac{c_p}{r} dilog\left(iub_p\right) + \frac{c_n}{r} dilog\left(-iub_n\right)\right). \end{aligned}$$

Contingent Claims at Infinity

- Consider now any claim promising at infinity the payout in time zero dollars of $F(M(\infty))$.

- Equivalently one may consider the limit as T goes to infinity of the claim paying at T, the sum $e^{rT}F(M(T))$.

- Markets in the future and hence markets at all times t price the claim in time zero dollars at the risk neutral price of

$$w_F(t) = E\left[F(M(\infty))|\mathcal{F}_t\right].$$

- By construction the price process $w_F(t)$ is a martingale.

- Now let Y be an independent random variable with the same law as that of $X(\infty)$.

- To determine the price $w_F(t)$ we note that

$$\begin{aligned} X(\infty) &= X(t) + \int_t^\infty b_p e^{-ru} d\gamma_p(c_p u) \\ &\quad - \int_t^\infty b_n e^{-ru} d\gamma_n(c_n u) \\ &\overset{(d)}{=} X(t) + e^{-rt} Y. \end{aligned}$$

- We thus observe that conditional on t, there is a function $H(X, v)$ such that

$$w_F(t) = H(X(t), e^{-rt}).$$

- **The martingale condition on $w_F(t)$ then implies that**

$$\begin{aligned} & -rvH_v + \\ & \int_{-\infty}^{\infty} \left(H(X+y,v) - H(X,v)\right) k(y,v) dy \\ = \;& 0 \end{aligned}$$

where $k(y, v)$ is the Lévy system associated with the jumps of the process $X(t)$.

- The price process is determined on solving this partial integro-differential equation subject to the boundary condition

$$H(X, 0) = F(\exp(X + \omega(\infty))$$

in the interval $0 \leq v \leq 1$.

Identifying the Lévy system

- For an implementation of the solution we need to identify the Lévy system $k(y,v)$. Define by

$$H(t) = \int_0^t be^{-rs} d\gamma(cs).$$

- From the Laplace transform of $H(t)$ we have

$$\begin{aligned} & E\left[\exp\left(-\lambda H(t)\right)\right] \\ = \; & \exp\left(\int_0^t \int_0^\infty \left(e^{-\lambda b e^{-rs} x} - 1\right) \frac{dx}{x} e^{-x} c\, ds\right) \\ = \; & \exp\begin{pmatrix} \int_0^t \int_0^\infty \left(e^{-\lambda y} - 1\right) \times \\ c \exp\left(-\frac{y}{be^{-rs}}\right) \frac{1}{y} dy\, ds \end{pmatrix} \end{aligned}$$

- It follows that

$$\begin{aligned} k(y,v) \; = \; & \frac{c_p}{y} \exp\left(-\frac{y}{b_p v}\right) \mathbf{1}_{y>0} \\ & + \frac{c_n}{|y|} \exp\left(-\frac{|y|}{b_n v}\right) \mathbf{1}_{y<0}. \end{aligned}$$

Bid and ask prices for DVG driven stock prices using probability distortions based on quadratic variation

- The partial integro-differential equation is transformed into a nonlinear partial integro-differential equation to construct bid and ask prices as G-expectations.

- The first transformation we employ uses probability transformations on introducing a quadratic variation based probability introduced in Eberlein, M., Pistorius, Schoutens and Yor (2012).

- Specifically we rewrite the equation as

$$\begin{aligned} & rvH_v \\ = \ & \int_{-\infty}^{\infty} \frac{(H(X+y,v) - H(X,v)) \int_{-\infty}^{\infty} y^2 k(y,v)dy}{y^2} \\ & \times dF_{QV}(y) \end{aligned}$$

where

$$F_{QV}(a) = \frac{1}{\int_{-\infty}^{\infty} y^2 k(y,v)dy} \int_{-\infty}^{a} y^2 k(y,v)dy.$$

- For the specific case considered here we have

$$
\begin{aligned}
F_{QV}(a) &= \frac{c_n(b_n v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2} \\
&\times \left(\exp\left(-\frac{|a|}{b_n v}\right) + \left(\frac{|a|}{b_n v}\right)\exp\left(-\frac{|a|}{b_n v}\right)\right)\mathbf{1}_{a<0} + \\
&\frac{c_n(b_n v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2} + \\
&\frac{c_p(b_p v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2} \times \\
&\left(1 - \exp\left(-\frac{|a|}{b_p v}\right) - \left(\frac{|a|}{b_p v}\right)\exp\left(-\frac{|a|}{b_p v}\right)\right)\mathbf{1}_{a>0}.
\end{aligned}
$$

Distorting the QVCDF

- We next employ the probability distortion $minmaxvar$ of Cherny and M. (2009) where

$$\Psi^{\gamma}(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}.$$

- **The nonlinear G-expectation for the bid price is then given by the solution of the distorted partial integro-differential equation**

$$\begin{aligned} & rvH_v \\ = \; & \int_{-\infty}^{\infty} \frac{(H(X+y,v) - H(X,v)) \int_{-\infty}^{\infty} y^2 k(y,v)dy}{y^2} \\ & \times d\Psi^{\gamma}(F_{QV}(y)) \end{aligned}$$

- **The ask price is computed as the negative of the bid price of the negative cash flow.**

Properties of the linear expectation equation for the stock price

- For the specific context of the stock price the function F is the identity function.

- In this case the solution of the linear expectation equation can be independently verified.

- Firstly one may solve explicitly for $H(X, v)$ as follows.

- The conditional law of $X(\infty)$ given $X(t) = X$ is that of

$$X + e^{-rt}Y = X + vY$$

where Y is an independent random variable with the same law as

$$\begin{aligned} & X(\infty) \\ = & \int_0^\infty b_p e^{-ru} d\gamma_p(c_p u) \\ & - \int_0^\infty b_n e^{-ru} d\gamma_n(c_n u). \end{aligned}$$

- It follows that

$$H(X, v) = \exp\left(X + \omega(\infty)\right) \phi_Y\left(-iv\right)$$

- From the characteristic function for Y we have that

$$\begin{aligned} & \phi_Y(-iv) \\ = & \exp\left(\frac{c_p}{r} dilog\left(b_p v\right) + \frac{c_n}{r} dilog\left(-b_n v\right)\right). \end{aligned}$$

- It follows that

$$
\begin{aligned}
& H(X, v) \\
= \; & \mathsf{exp}\left(X + \omega(\infty)\right) \times \\
& \mathsf{exp}\left(\frac{c_p}{r} dilog\left(b_p v\right) + \frac{c_n}{r} dilog\left(-b_n v\right)\right).
\end{aligned}
$$

- For numerical solutions it is preferable to have a stationary grid for the space variable and this is expected for the discounted stock price.

- We are therefore led to write

$$
\begin{aligned}
& H(X, v) \\
= \; & \exp\left(X + \omega(t) + \omega(\infty) - \omega(t)\right) \\
& \times \phi_Y(-iv) \\
t = \; & -\frac{\ln v}{r}.
\end{aligned}
$$

- Further observing that

$$M(t) = \exp\left(X(t) + \omega(t)\right)$$

- define

$$
\begin{aligned}
& G(M(v), v) \\
= \; & M(v) \times \\
& \exp\left(\omega(\infty) - \omega(-\frac{\ln v}{r})\right) \\
& \times \phi_Y(-iv).
\end{aligned}
$$

- Dropping for notational convenience the dependence of M on v we write that

$$
\begin{aligned}
& \int_{-\infty}^{\infty} (G(Me^y, v) - G(M, v)) k(y, v) dy \\
= \; & G(M, v) \int_{-\infty}^{\infty} (e^y - 1) \, k(y, v) dy.
\end{aligned}
$$

- Also we have that

$$
\begin{aligned}
& G_v \\
= \; & G(M,v) \\
& \times \left(\frac{\partial \ln \phi_Y(-iv)}{\partial v} + \frac{1}{rv} \omega' \left(-\frac{\ln v}{r} \right) \right) \\
= \; & G(M,v) \\
& \times \left(\begin{array}{c} -\frac{c_p \ln(1-b_p v)}{rv} - \frac{c_n \ln(1+b_n v)}{rv} \\ +\frac{1}{rv} \omega' \left(-\frac{\ln v}{r} \right) \end{array} \right)
\end{aligned}
$$

- It follows that $G(M,v)$ satisfies the differential equation

$$
\begin{aligned}
& rvG_v \\
= \; & \int_{-\infty}^{\infty} (G(Me^y, v) - G(M,v)) k(y,v) dy \\
& + G(M,v) \omega' \left(-\frac{\ln v}{r} \right)
\end{aligned}
$$

One may therefore work on a fixed stock grid centered around unity with the differential equation on applying the desired distortions to the Lévy system $k(y,v)$.

- The function $\omega'(t)$ may be precomputed.

Explicit Solution

- The discretized update for the conditional expectation of $M(\infty)$ is now

$$\begin{aligned} & G(M, v+h) \\ = \; & G(M, v) + \\ & \frac{h}{rv} \times \left(\begin{array}{c} \int_{-\infty}^{\infty} (G(Me^y, v) - G(M, v)) k(y, v) dy \\ + G(M, v) \omega' \left(-\frac{\ln v}{r} \right) \end{array} \right). \end{aligned}$$

- However, it will be useful to incorporate the analytical solution to into the numerical scheme.

- Note that when $X(t) = X$ we have

$$M(t) = \exp\left(X + \omega(t)\right)$$

- but as $M(t)$ is a martingale we must have

$$\begin{aligned} E_t \left[\exp\left(X(\infty) + \omega(\infty) \right) \right] &= M(t) \\ &= \exp\left(X + \omega(t) \right) \end{aligned}$$

- But this conditional expectation is

$$\exp\left(X + \omega(\infty)\right)\phi_Y(-iv).$$

- Hence one has the implication that

$$\phi_Y(-iv) = \exp\left(\omega(-\frac{\ln v}{r}) - \omega(\infty)\right)$$

- It follows that

$$M \exp\left(\omega(\infty) - \omega\left(-\frac{\ln v}{r}\right)\right)\phi_Y(-iv) = M$$

and the solution of the differential equation

$$\begin{aligned} & rvG_v \\ = \; & \int_{-\infty}^{\infty}\left(G(Me^y, v) - G(M, v)\right)k(y, v)dy \\ & + G(M, v)\omega'\left(-\frac{\ln v}{r}\right) \end{aligned}$$

is in fact

$$G(M, v) = M.$$

Implementation details

- The pricing is implemented for risk neutral parameter values for the S&P 500 index taken at their median values as reported in M. and Yor (2012).

- These are

$$\begin{aligned} r &= .02966 \\ b_p &= 0.0145 \\ c_p &= 48.4215 \\ b_n &= 0.5707 \\ c_n &= 0.3493 \end{aligned}$$

- The differential equation solved for the bid price is

$$\begin{aligned} & rvG_v \\ = \; & \int_{-\infty}^{\infty} \frac{\left(G(Me^y, v) - G(M, v)\right) \int_{-\infty}^{\infty} y^2 k(y, v) dy}{y^2} \\ & \times d\Psi^{\gamma}\left(F_{QV}(y)\right) \\ & + G(M, v)\omega'\left(-\frac{\ln v}{r}\right). \end{aligned}$$

- In the absence of a distortion the equation has the solution $G(M, v) = M$.

- In the computations we set ω' to $\widehat{\omega}'$ that forces the expectation equation to solve out at the identity function.

- Hence we set
$$\widehat{\omega}' = -\frac{\int_{-\infty}^{\infty}(G(Me^y, v) - G(M, v))k(y, v)dy}{G(M, v)}$$
in the solution of the expectation equation.

- This value of $\widehat{\omega}'$ is then used in the bid and ask equations.

- It was checked that the values for $\widehat{\omega}'$ and ω' were close.

- For this parameter setting and with the minmaxvar stress level set at 10 basis points the bid and ask prices were solved for as a function of the spot on the initial date.

- We also present a graph of the bid and ask prices as a function of calendar time for different levels of the initial spot.

- The prices converge at infinity to the expected value.

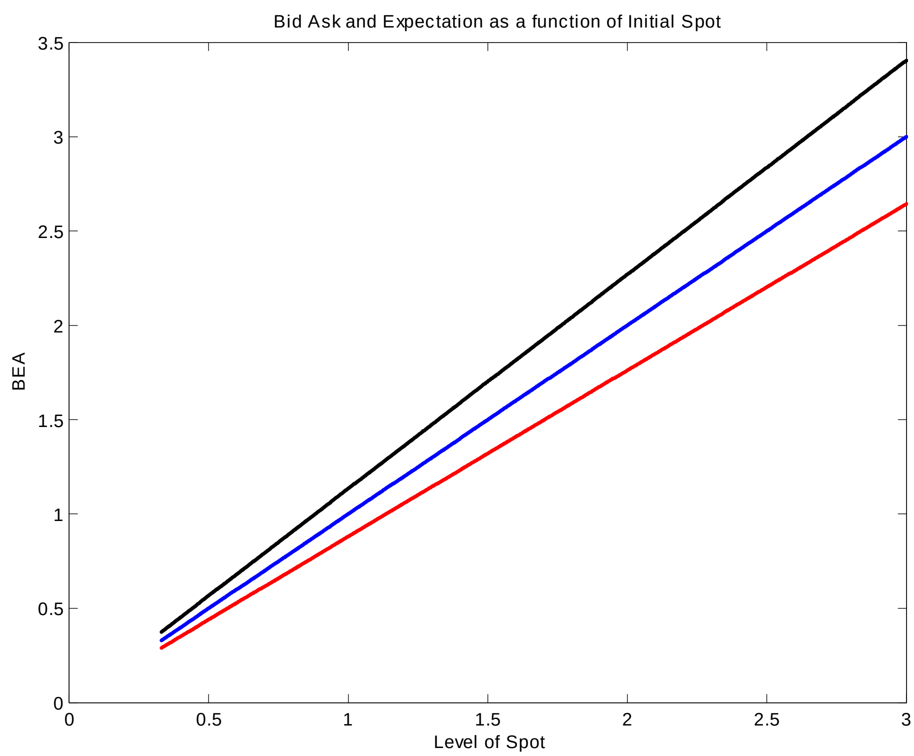

Figure 1: Bid Ask and Expectation as a function of the spot price at time zero.

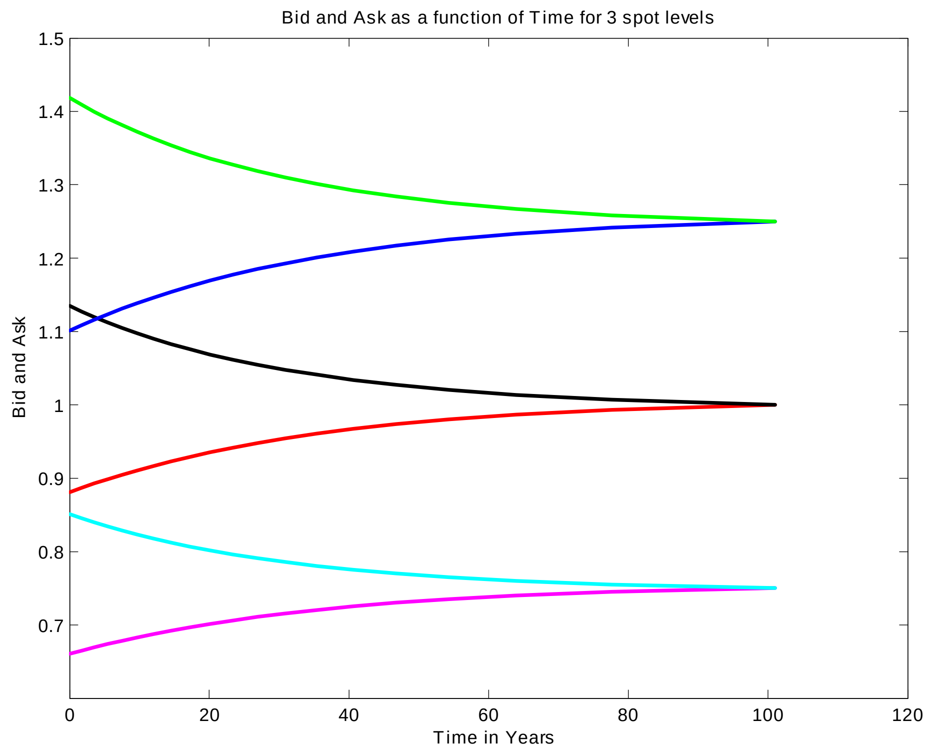

Figure 2: Bid and Ask Prices as functions of calendar time.

Bid and ask for dvg driven stock using measure distortions

- The first step in applying measure distortions is that of choosing specific functional forms for the measure distortions Γ_+, Γ_-.

- Recognizing that Γ_+ lies above the identity and Γ_- lies below we consider functional forms for the positive gap

$$
\begin{aligned}
G_+(x) &= \Gamma_+(x) - x \\
G_-(x) &= x - \Gamma_-(x)
\end{aligned}
$$

- Both these functions are concave and positive.

- If we suppose that for large x associated with large tail measures and therefore events nearer to zero, there need be no reweighting then one has Γ'_+ falling to unity at large x while Γ'_- rises to unity.

- As a result G_+, G_- are increasing concave functions that are eventually constant.

- We may scale by the final constant and model them to be multiples of increasing concave functions that are finally unity.

- We then write

$$\begin{aligned} G_+(x) &= aK_+(x) \\ G_-(x) &= bK_-(x) \end{aligned}$$

where K_+, K_- are unity at infinity.

Selecting Distortions

- Now consider a generic candidate for such a function, say $K(x)$. Suppose the concavity coefficient defined by
$$-\frac{K''}{K'}$$
is bounded below by a constant c.

- Define
$$\Psi(y) = K\left(-\frac{\ln(1-y)}{c}\right), \quad 0 \leq y \leq 1.$$
The function Ψ is zero at zero, unity at unity, and increasing in its domain.

- Furthermore we have
$$\begin{aligned}
\Psi' &= K'\left(\frac{1}{c(1-y)}\right) \\
\Psi'' &= K''\left(\frac{1}{c^2(1-y)^2}\right) + K'\frac{1}{c(1-y)^2}
\end{aligned}$$

and $\Psi'' \leq 0$ just if

$$\frac{K''}{c} + K' \leq 0$$

or

$$-\frac{K''}{K'} \geq c.$$

Measure Distortion via Probability Distortions

- With a lower bound on the concavity coefficient we have that

$$K(x) = \Psi\left(1 - e^{-cx}\right)$$

and Ψ is a probability distortion.

- **Hence we take as models for specific measure distortions**

$$\begin{aligned}\Gamma_+(x) &= x + a\Psi_+\left(1 - e^{-cx}\right)\\ \Gamma_-(x) &= x - b\Psi_-(1 - e^{-cx})\end{aligned}$$

for probability distortions Ψ_+, Ψ_-.

- If one takes maxvar for Ψ_+ to get an infinite reweighting of large losses and minvar for Ψ_- we have the specific formulation

$$\begin{aligned}
\Gamma_+(x) &= x + a\left(1 - e^{-cx}\right)^{\frac{1}{1+\gamma_+}} \\
\Gamma_-(x) &= x - \frac{b}{c}(1 - e^{-c(1+\gamma_-)x})
\end{aligned}$$

- In the calculations reported we set $\gamma_- = 0$ and employed a four parameter specification for the measure distortion. The maximum downward discounting of gains is $\Gamma_-(0) = 1 - b$.

Measure Distortion results for the dvg stock price

- The equation solved in this case is

-

$$
\begin{aligned}
& rvG_v \\
= \; & \int_{-\infty}^{0} x d\left(\Gamma_{+}\left(\int_{(G(Me^y,v)-G(M,v)\leq x)} k(y,v)dy\right)\right) dx \\
& \int_{0}^{\infty} x d\left(\Gamma_{-}\left(\int_{(G(Me^y,v)-G(M,v)\geq x)} k(y,v)dy\right)\right) dx \\
& +G(M,v)\omega'\left(-\frac{\ln v}{r}\right).
\end{aligned}
$$

- The results shown are for

$$
\begin{aligned}
a &= 0.01 \\
b &= 0.05 \\
c &= 1 \\
\gamma &= 0.0010
\end{aligned}
$$

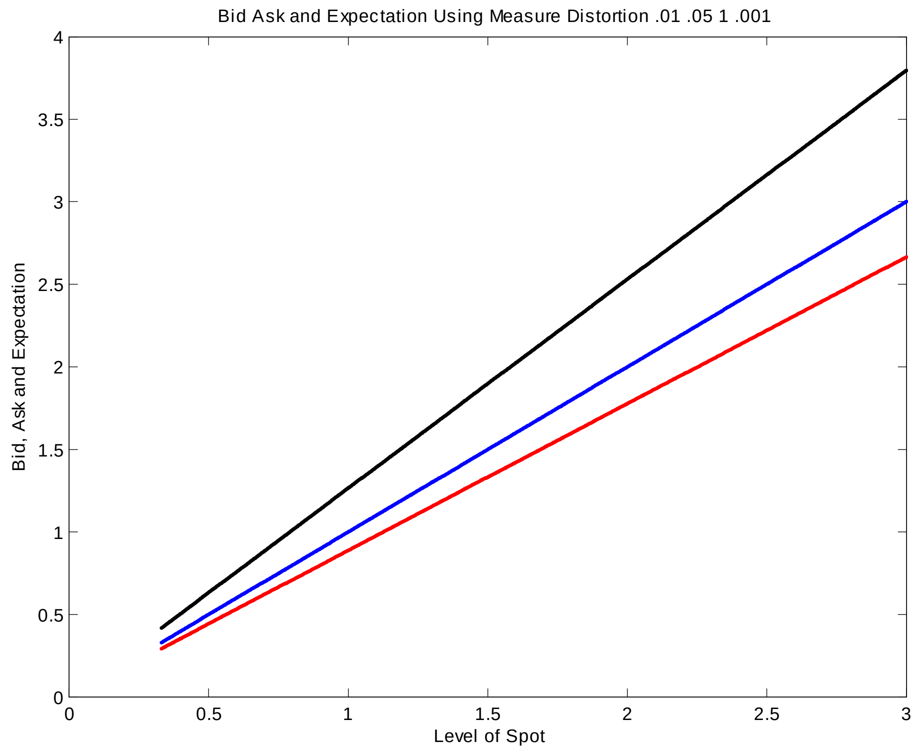

Figure 3: Bid, Ask and Expected Values as a function of the spot at the initial time using measure distortions.

- The Figure presents the bid, ask and expectation as a function of the initial spot.

- We also present the bid and ask as functions of time for three different spot levels in Figure

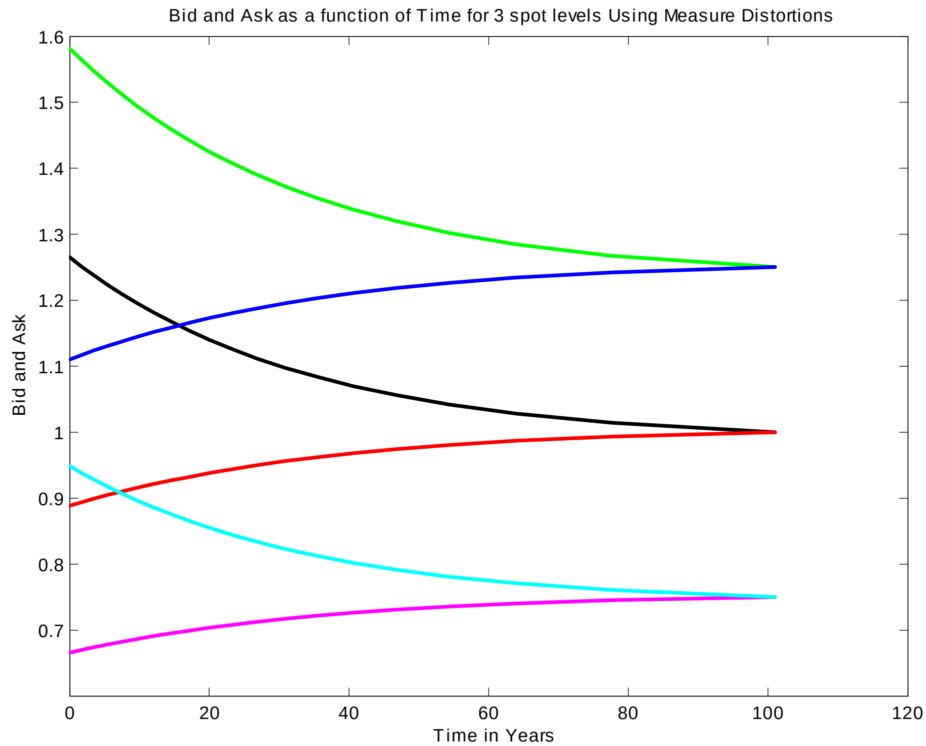

Figure 4: Bid and Ask as functions of time for three different spot levels using measure distortions. The lower curves in magenta and cyan are the bid and ask for the spot level of 0.75. Red and Black are the bid and ask for the level 1.0 and blue and green are for level 1.25.

Two price valuation of insurance loss processes

- We now apply measure distortions to the two price valuation of insurance losses.

- Let $L(t)$ be the process for cumulated losses.

- A discounted expected value may be computed as
$$E\left[\int_0^\infty e^{-rs} dL(s)\right]$$
where $L(t)$ is for example a compound Poisson process with arrival rate λ and loss sizes that are $i.i.d.$ gamma distributed.

- Consider the value process in time zero dollars for these losses,
$$V(t) = E_t\left[\int_0^\infty e^{-rs} dL(s)\right].$$

- Let $X(t)$ be the level of discounted losses to date or

$$X(t) = \int_0^t e^{-rs} dL(s).$$

- We then have that conditional on this realization

$$\begin{aligned}\int_0^\infty e^{-rs} dL(s) &= X + e^{-rt} \int_t^\infty e^{-r(s-t)} dL(s) \\ &\overset{(d)}{=} X + e^{-rt} Y\end{aligned}$$

where Y is an independent copy of the random variable

$$\int_0^\infty e^{-rs} dL(s).$$

- It follows that the conditional expectation is a martingale of the form

$$H(X(t), e^{-rt})$$

- The martingale condition for H once again yields that

$$
\begin{aligned}
& rvH_v \\
= & \int_0^\infty (H(X+w,v) - H(X,v))k(w,v)dw
\end{aligned}
$$

where $k(w,v)$ is related to the Lévy system for $X(t)$.

The Lévy system

- We may derive this Lévy system from the characteristic function for $X(t)$.

- The characteristic function is developed as follows

$$
\begin{aligned}
& E\left[\exp\left(iuX(t)\right)\right] \\
= \; & E\left[\exp\left(iu\int_0^t e^{-rs}dL(s)\right)\right] \\
= \; & \exp\left(\int_0^t\int_0^\infty\left(e^{iue^{-rs}x}-1\right)\frac{\lambda}{\Gamma(\gamma)}c^\gamma x^{\gamma-1}e^{-cx}dxds\right) \\
= \; & \exp\left(\begin{array}{c} \int_0^t\int_0^\infty\left(e^{iuw}-1\right)\times \\ \frac{\lambda}{\Gamma(\gamma)}\left(\frac{c}{e^{-rs}}\right)^\gamma w^{\gamma-1}\exp\left(-\frac{c}{e^{-rs}}w\right)dwds \end{array}\right)
\end{aligned}
$$

- It follows that the Lévy system for $X(t)$ is

$$k(w,t)=\frac{\lambda}{\Gamma(\gamma)}\left(\frac{c}{e^{-rt}}\right)^\gamma w^{\gamma-1}\exp\left(-\frac{c}{e^{-rt}}w\right)$$

Implementing Insurance Loss Valuation

- We consider an arrival rate $\lambda = 10$ with gamma distribution of mean 5 and variance 10.

- Therefore we have $c = .5$, $\gamma = 2.5$ and $\lambda = 100$.

- We take the interest rate at $r = .02$.

- The mean of the final discounted loss is
$$\frac{\lambda\gamma}{rc} = 2500$$

- The differential equation is
$$H_v = \frac{1}{rv}\int_0^\infty (H(X+y,v) - H(X,v))k(y,v)dy$$

with

$$k(y,v) = \frac{\lambda}{\Gamma(\gamma)} \left(\frac{c}{v}\right)^{\gamma} y^{\gamma-1} \exp\left(-\frac{c}{v}y\right)$$

- We fix a grid in X from 0 to 100 measured in thousands for which we take $c = 500$. We take the grid in X to be (.25 : .25 : 100).

One sided measure distortions

- We only have positive outcomes for the cumulated discounted loss process.

- **Our equation for the bid price is therefore**

$$b = -\int_0^\infty x d\Gamma_- \left(\mu \left(\chi > x\right)\right)$$

- **and for ask price we have**

$$a = -\int_0^\infty x d\Gamma_+ \left(\mu \left(\chi > x\right)\right).$$

- We have the same equation but we use Γ_- for the bid and Γ_+ for the ask.

- The measure distortion parameters used were $a = .1$, $b = .2$, $c = 1$ and $\gamma = .02$.

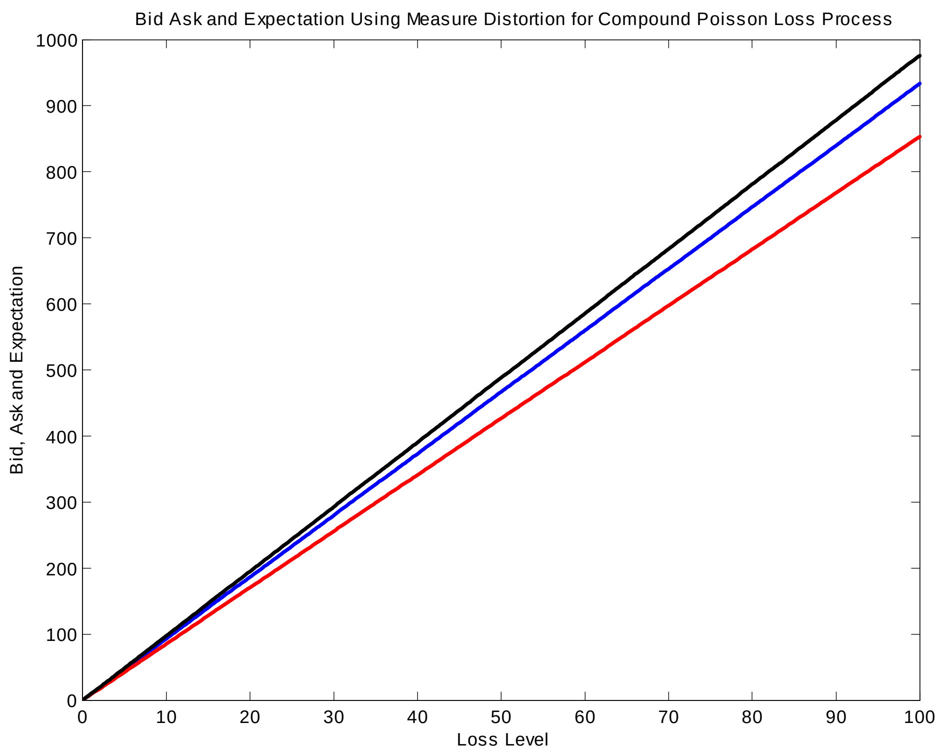

Figure 5: Bid, Ask and Expectation as functions of initial loss level

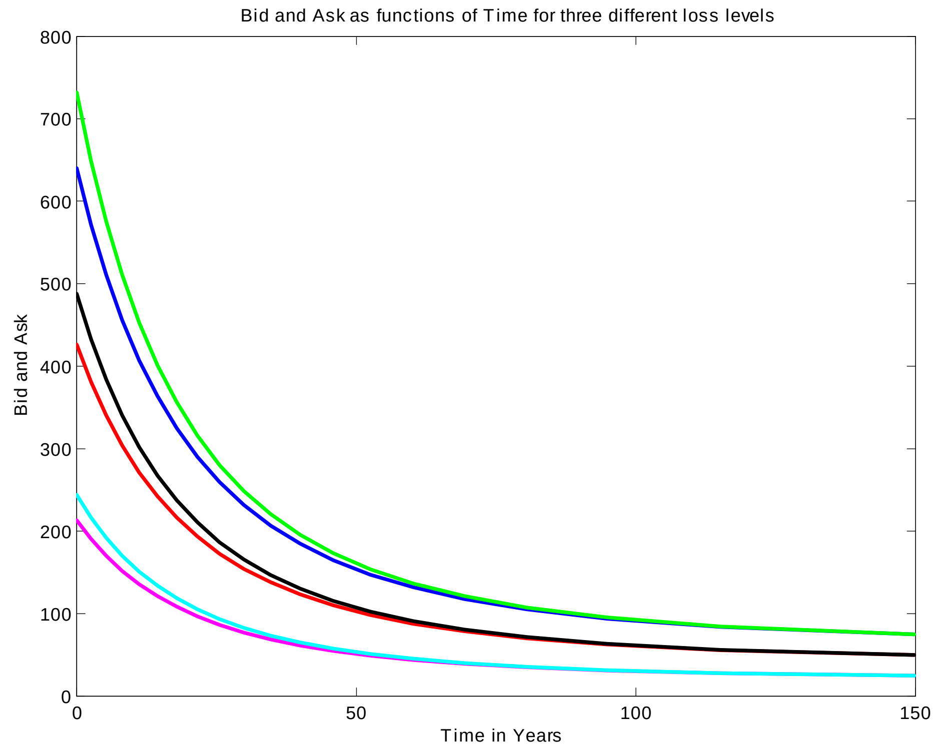

Figure 6: Bid and Ask as functions of Time for three different attained loss levels. The lower curves in magenta and cyan are the bid and ask for the loss level of 25. Red and Black are the bid and ask for the level 50 and blue and green are for level 75.

Remarks on the design of measure distortions

- There are four parameters in the proposed measure distortion Γ_+, Γ_- and they are a, b, c and γ.

- The parameter c may be calibrated by a cutoff on what are viewed as rare events.

- If the exponential of $-cx$ is below $1/2$ then $1 - \exp(-cx) > 1/2$ and these are the likely events.

- Defining

$$x_* = \frac{-\ln(1/2)}{c}$$

we have arrival rates below x_* constituting the rare events.

- Hence for $c = -\ln(1/2)$ arrival rates above one per year are the normal events while arrival rates below one per year are the rare ones.

- If arrival rates below 2 per year are to be the rare ones then $c = -\ln(1/2)/2 = 0.3466$ and if rare is viewed as one every two years then $c = 1.3863$.

- The parameter b sets the discount on gains.

- For $b = 0$ there is no gain discount and Γ_- is the identity function. The highest gain discount is unity. The gain discount should be set below unity.

- Once b and c are set then the choice of η in
$$a = \frac{b}{c}\eta$$
sets the parameter a.

- The choice of $\eta = 1$ provided a balanced treatment of gains and losses as the maximum penalty in the gap functions G_+ and G_- are then equal. The parameter η is then a balance parameter

- The parameter γ is a stress parameter and controls the speed with which losses are reweighted upwards. This is a parameter familiar from the uses of probability distortions $maxvar$ or $minmaxvar$.

Inhomogeneous compound Poisson losses

- Consider an inhomogeneous arrival rate $\lambda(t)$ for losses, a general discount curve $D(t)$ and gamma distributed loss sizes.

- For the loss process $L(t)$ the linear expectation valuation is given by

$$V(t) = E_t\left[\int_0^\infty D(s)dL(s)\right]$$

- Now conditioning on time t we have that

$$\begin{aligned} V(t) &= \int_0^t D(s)dL(s) \\ &\quad + E_t\left[\int_t^\infty D(s)dL(s)\right] \end{aligned}$$

- Define

$$\begin{aligned} & C(t) \\ = & \int_t^\infty D(s) \int_0^\infty \lambda(s) \frac{c^\gamma}{\Gamma(\gamma)} x^\gamma e^{-cx} dx ds \end{aligned}$$

and observe that

$$\begin{aligned} M(t) = & \int_0^t D(s) dL(s) \\ & + C(t) \end{aligned}$$

is a martingale.

- Infact

$$\begin{aligned} dM(t) = & D(t) dL(t) \\ & - D(t) \int_0^\infty \lambda(t) \frac{c^\gamma}{\Gamma(\gamma)} x^\gamma e^{-cx} dx dt \end{aligned}$$

- and as the compensator of $dL(t)$ is

$$\lambda(t) \frac{c^\gamma}{\Gamma(\gamma)} x^\gamma e^{-cx} dx dt$$

we have a martingale.

Nonlinear Valuations

- In general in the current context we have that $V(t) = H(X(t), t)$

  where in fact the function H takes the specific form

  $$H(X(t), t) = X(t) + C(t).$$

- Apply Ito's lemma to the function H and noting that it is a martingale we deduce that

  $$H_t + \int_0^\infty (H(X(t)+y, t) - H(X(t), t))k(y, t)dy = 0$$

  where again $k(y, t)$ is the Lévy system for $X(t)$.

- Equivalently in terms of the compensator for $dL(t)$ we may write

  $$\begin{aligned} H_t \;=\; & -\int_0^\infty (H(X(t) + D(t)x, t) - H(X(t), t)) \times \\ & \lambda(t)\frac{\zeta^\kappa}{\Gamma(\kappa)} x^{\kappa-1} e^{-\zeta x} dx. \end{aligned}$$

- Now substituting the specific form of the function H yields that

$$C_t = -\int_0^\infty D(t)x\lambda(t)\frac{\zeta^\kappa}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dx$$

  or that

$$C(t) = \int_t^\infty \int_0^\infty D(s)x\lambda(s)\frac{\zeta^\kappa}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dxds.$$

- For the nonlinear measure distorted result we write

-
$$\mathcal{E}_t\left[\int_0^\infty D(s)dL(s)\right] = \int_0^t D(s)dL(s) + \widetilde{C}(t)$$
$$\widetilde{C}(t)$$
$$= \int_t^\infty D(s)\int_0^\infty \Gamma_+\left(\int_x^\infty \lambda(t)\frac{c^\gamma}{\Gamma(\gamma)}y^{\gamma-1}e^{-cy}dy\right)dxds$$

- The computation of $\widetilde{C}(t)$ as the ask price valuation is what we implement for the inhomogeneous case as a conjectured solution. For the bid we replace Γ_+ by Γ_-.

Inhomogeneous Example

- We employ for the discount curve a Nelson-Siegel discount curve with yield to maturity $y(t)$ specification

$$
\begin{aligned}
y(t) &= a_1 + (a_2 + a_3 t)e^{-a_4 t} \\
a_1 &= .0424 \\
a_2 &= -.0367 \\
a_3 &= .0034 \\
a_4 &= .0686.
\end{aligned}
$$

The discount curve is graphed in Figure 7.

- For the inhomogeneous arrivar rate we take an exponential model with

$$\lambda(t) = \frac{a}{\tau} \exp\left(-\frac{t}{\tau}\right).$$

The parameters used were $a = 100$ and $\tau = 10$.

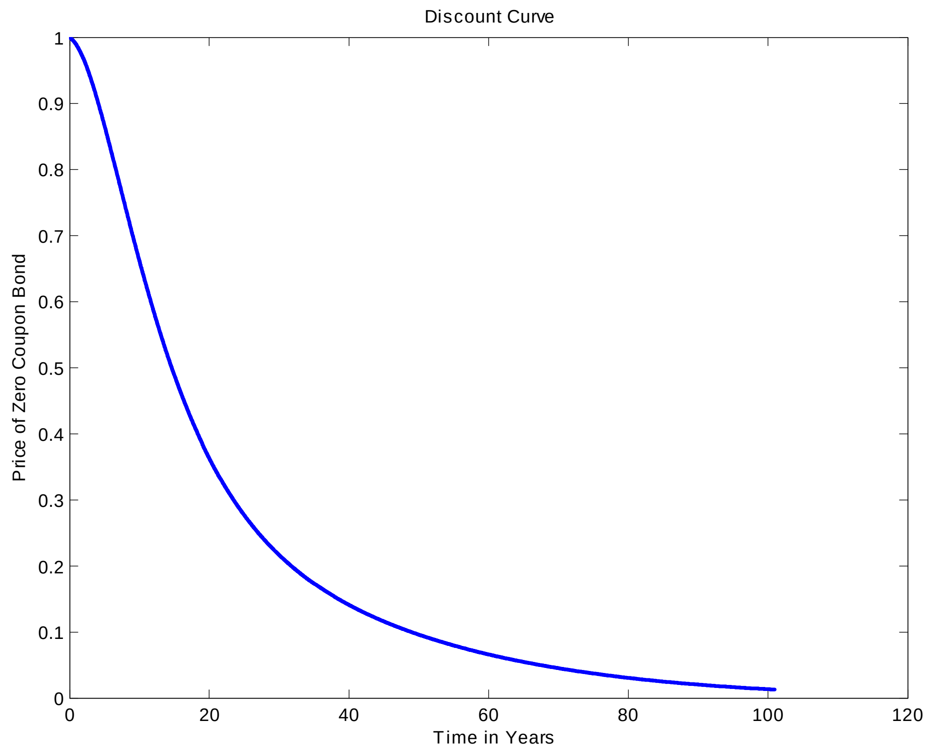

Figure 7: Discount Curve used in Inhomogeneous Compound Poisson Loss Model.

- The distribution for losses are gamma with $c = \gamma = 1.2346$, consistent with a unit mean and a standard deviation of 0.9.

- For the measure distortions parameters we employ $a = .7214$, $b = .5$, $c = .6931$ and $\gamma = 0.25$.

- Presented are the premia of ask over expectation and the shave of bid relative to expectation in basis points, for the function $\widetilde{C}(t)$ when we use Γ_+ for the ask and Γ_- for the bid.

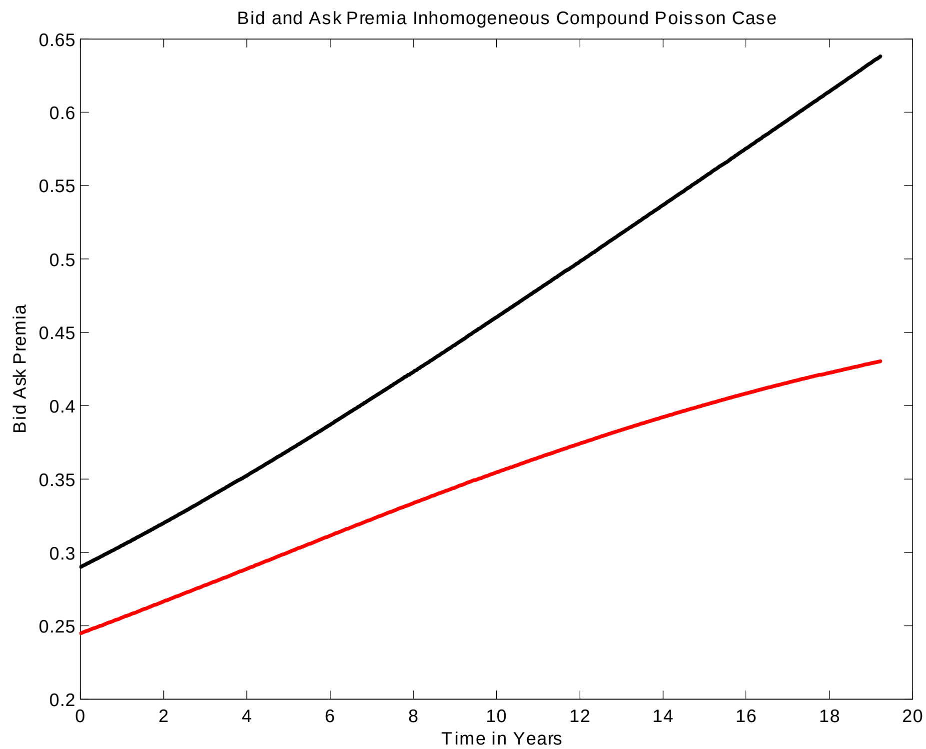

Figure 8: Bid and Ask premia in basis points relative to expected values for the inhomogeneous compound Poisson case.

Conclusion

- The use of probability distortions in constructing nonlinear G-expectations for bid and ask or lower and upper prices in continuous time as introduced in Eberlein, M., Pistorius, Schoutens and Yor (2012) is here extended to the direct use of measure distortions.

- Integrals with respect to a positive and possibly infinite measure with finite measure in the two sided tails on either side of zero are distorted using concave measure distortions for losses and convex measure distortions for gains.

- It is shown that measure distortions can fairly generally be constructed as probability distortions applied to an exponential distribution function on the half line.

- The valuation of economic activities as opposed to contracts places the problem in a context with no apparent maturity.

- The two price continuous time methodologies heretofore available for explicit maturities are extended to infinitely lived economic activities.

- This permits the construction of two prices for stock indices and the coverage of insurance liabilities in perpetuity.

- The methods are illustrated with explicit computations using probability and measure distortions for an infinitely lived stock price model as developed in M. and Yor (2012).

- Measure distortions are applied to infinitely lived compound Poisson insurance loss processes.