Lévy processes and self-similar Markov processes
Zürich Spring School on Lévy Processes
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Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that:

for all $c > 0$ and $x \in \mathbb{R}^d$

$(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$

is equal in law to

$(Z_t : t \geq 0)$ under $\mathbb{P}_{cx}$.

Some of your best friends are ssMp

- Write $\mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma})$ for the Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and correlation (matrix) $\mathbf{\Sigma}$. The moment generating function of $X_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma}t)$ satisfies, for $\theta \in \mathbb{R}^d$,

$$E[\mathrm{e}^{\theta \cdot X_t}] = \mathrm{e}^{t\theta^{\mathsf{T}}\mathbf{\Sigma}\theta/2} = \mathrm{e}^{(c^{-2}t)(c\theta)^{\mathsf{T}}\mathbf{\Sigma}(c\theta)/2} = E[\mathrm{e}^{\theta \cdot cX_{c^{-2}t}}].$$

- Thinking about the stationary and independent increments of Brownian motion, this can be used to show that $\mathbb{R}^d$-**Brownian motion:** is a ssMp with $\alpha = 2$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}$-Brownian motion:

- Write $\underline{X}_t := \inf_{s\leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.
- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c\inf_{s\leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u\leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.

- $\Rightarrow$ Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ with index 2.
- $\Rightarrow Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process. Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|$, $t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s \to \infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x \left[\frac{B_t}{x} \mathbf{1}_{(\underline{B}_t > 0)} \mathbf{1}_{(A)} \right]$$

where $A \in \sigma\{X_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}_x^{\uparrow}).$$

More examples?

- All of the previous examples have in common that their paths are continuous. Is this a necessary condition?
- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).

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α-stable process

Definition

A Lévy process X is called (strictly) α-stable if it is also a self-similar Markov process.

- Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$

- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi}\frac{1}{|x|^{1+\alpha}}\left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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Your new friends

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$. Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X})$, $x > 0$.
- $Z = X\mathbf{1}_{(\underline{X} > 0)}$
- $Z = |X|$ providing $\rho = 1/2$
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Conditioned α-stable processes

- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1} \log \mathbb{E}[e^{i\theta\xi_t}]$ respects the factorisation

$$\Psi_\xi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \qquad \theta \in \mathbb{R},$$

  where κ and $\hat{\kappa}$ are Bernstein functions. That is e.g. κ takes the form

$$\kappa(\lambda) = q + a\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\nu(dx), \qquad \lambda \geq 0$$

  where ν is a measure satisfying $\int_{(0,\infty)} 1 \wedge x\nu(dx) < \infty$.
- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.
- In the case of α-stable processes, up to a multiplicative constant,

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0.$$

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Conditioned α-stable processes

- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies

$$\int_{[0,\infty)} e^{-\lambda x} \hat{U}(dx) = \frac{1}{\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$

- It can be shown that for a Lévy process which satisfies $\limsup_{t\to\infty} \xi_t = \infty$, for $A \in \sigma(\xi_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A | \underline{X}_{t+s} > 0) = \mathbb{E}_x\left[\frac{\hat{U}(X_t)}{\hat{U}(x)} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)}\right]$$

- In the α-stable case $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$
[Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

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Conditioned α-stable processes

- For $c, x > 0$, $t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

$$\begin{aligned}
&\mathbb{E}^{\uparrow}_x[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\
&\quad = \mathbb{E}\left[f(\{cX^{(x)}_{c^{-\alpha}s} : s \leq t\})\frac{(X^{(x)}_{c^{-\alpha}t})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(x)}_{c^{-\alpha}t} \geq 0)}\right] \\
&\quad = \mathbb{E}\left[f(\{X^{(cx)}_s : s \leq t\}\frac{(X^{(cx)}_t)^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(cx)}_t \geq 0)}\right] \\
&\quad = \mathbb{E}^{\uparrow}_{cx}[f(\{X_s : s \leq t\})].
\end{aligned}$$

- This also makes the process $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process.

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Notation

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s}\mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) *$\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,*

(2) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or*

(3) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

Lamperti transform for POSITIVE ssMp

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}\mathbf{1}_{(t<x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^\alpha I_\infty$, with index α.

Lamperti transform for POSITIVE ssMp

$(Z, \mathrm{P}_x)_{x>0}$ pssMp $\leftrightarrow$ $(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy

$$Z_t = \exp(\xi_{S(t)}),$$

S a random time-change

$$\xi_s = \log(Z_{T(s)}),$$

T a random time-change

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

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Stable process killed on entry to $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z^*_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

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- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

- Given that we know the value of Z_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})Z_{t-}^{-\alpha}.$$

- On the other hand, the Lamperti transform says that on $\{t<\zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*Z_t^{-\alpha}.$$

- Comparing gives us

$$q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

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- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\mathbb{P}_1(\xi^*_0 = 0) = 1$),

$$Z_{\zeta-} = X_{\tau_0^- -} = \mathrm{e}^{\xi^*_{\mathbf{e}_{q^*}}},$$

where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

- This motivates the computation

$$\mathbb{E}_1[(Z_{\zeta-})^{\mathrm{i}\theta}] = \mathbb{E}_1[\mathrm{e}^{\mathrm{i}\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of ξ^*.

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Setting

$$K = \frac{\sin \alpha\hat{\rho}\pi}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})},$$

Remembering the "overshoot-undershoot" distributional law at first passage (see Victor Rivero's notes) and deduce that, for all $v \in [0, 1]$,

$$\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&= \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&= K\left(\int_0^\infty \int_0^\infty \mathbf{1}_{(y \leq 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \mathrm{d}u \mathrm{d}y\right) \mathrm{d}v \\
&= \frac{K}{\alpha}\left(\int_0^1 \mathbf{1}_{(y \leq v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}\mathrm{d}y\right) \mathrm{d}v,
\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

Stable process killed on entry to $(-\infty, 0)$

We are led to the conclusion that

$$\begin{aligned}
\frac{q*}{\Psi^*(\theta)} &= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat\rho-1}\int_0^\infty \mathbf{1}_{(y\le v)} v^{\mathrm{i}\theta-\alpha\hat\rho-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1}\mathrm{d}v\mathrm{d}y \\
&= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat\rho-1}y^{\mathrm{i}\theta-\alpha\hat\rho}\mathrm{d}y\frac{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat\rho+\mathrm{i}\theta)\Gamma(\alpha\hat\rho)\Gamma(\alpha+1)}{\alpha\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)\Gamma(\alpha\hat\rho)\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)},
\end{aligned}$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

Stable process killed on entry to $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t > 0)}\mathbf{1}_{(A)}\right] = E\left[\mathrm{e}^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau < \mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

- Noting that $\Psi^*(-\mathrm{i}\alpha\hat{\rho}) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$,has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1 + \alpha\hat{\rho} + \mathrm{i}z)}{\Gamma(1 + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = E[\xi_1^{\uparrow}] > 0$ so that $\lim_{t\to\infty}\xi_t^{\uparrow} = \infty$.

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Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = 0,$$

namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.
- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^{\downarrow}(z) = \Psi^*(z - \mathrm{i}(1 - \alpha\hat{\rho})) = \Psi^{\downarrow}(z) = \frac{\Gamma(1 + \alpha\rho - \mathrm{i}z)}{\Gamma(1 - \mathrm{i}z)} \frac{\Gamma(\mathrm{i}z + \alpha\hat{\rho})}{\Gamma(\mathrm{i}z)}.$$

- The choice of notation is pre-emptive since we can also check that $\Psi^{\downarrow}(0) = 0$ and $\Psi^{\downarrow\prime}(0) < 0$ so that if $\xi^{\downarrow}$ is a Lévy process with characteristic exponent $\Psi^{\downarrow}$, then $\lim_{t\to\infty} \xi^{\downarrow}_t = -\infty$.

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Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \leq t)$,

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  where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.
- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.
- In an appropriate sense, it turns out that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is the law of a stable process conditioned to continuously approach the origin from above.

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ξ^*, $\xi^\uparrow$ and $\xi^\downarrow$

- The three examples of pssMp offer quite striking underlying Lévy processes
- Is this exceptional?

Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t>0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbb{1}_{(t<T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1,2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\leadsto}{\xi}$. *Then* $\overset{\leadsto}{\xi}$ *is equal in law to* $\xi^{**} \oplus \xi^{\mathsf{C}}$, *with*

- ξ^{**} *equal in law to* ξ^{*} *with the killing removed,*
- ξ^{C} *a compound Poisson process with jump rate* $q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.

Moreover, the characteristic exponent of $\overset{\leadsto}{\xi}$ *is given by*

$$\overset{\leadsto}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written $\xi^{\odot}$, then it characteristic exponent is given by

$$\Psi^{\odot}(z) = 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+1))}{\Gamma(\frac{1}{2}(\mathrm{i}z+1-\alpha))}, \qquad z \in \mathbb{R}.$$

Hypergeometric Lévy processes

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2, \ \gamma, \hat{\gamma} \in (0,1) \ \hat{\beta} \geq -1, \text{ and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1 - \beta + \gamma - \mathrm{i}z)}{\Gamma(1 - \beta - \mathrm{i}z)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + \mathrm{i}z)}{\Gamma(\hat{\beta} + \mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1 + \gamma, \eta; \eta - \hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } \ x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1 + \hat{\gamma}, \eta; \eta - \gamma; \mathrm{e}^{x}\right), & \text{if } \ x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, for $|z| < 1$, ${}_2F_1(a, b; c; z) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

Starting from zero

- We have carefully avoided the issue of talking about pssMp issued from the origin.
- This should ring alarm bells when we look at the Lamperti transform

$$Z_t^{(x)}\mathbf{1}_{(t<\zeta^{(x)})} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\} = \exp\{\xi_{\varphi(x^{-\alpha}t)} + \log x\}, \qquad t \geq 0,$$

- On the one hand $\log x \downarrow -\infty$, which is the point of issue of ξ, but

$$\varphi(x^{-\alpha}t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha(\xi_u+\log x)}\mathrm{d}u > t\},$$

  meaning that we are sampling the Lévy process over a longer and longer time horizon.
- We know that 0 is an **absorbing point**, but it might also be an **entrance point** (can it be both?).
- We know that some of our new friends have no problem using the origin as an entrance point, e.g. $|X|$, where X is an α-stable process (or Brownian motion).
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- Could look at behaviour of the transition semigroup of Z as its initial value tends to zero. That is to say, to consider whether the weak limit below is well defined:

$$\mathbb{P}_0(Z_t \in \mathrm{d}y) := \lim_{x\downarrow 0} \mathbb{P}_x(Z_t \in \mathrm{d}y), \qquad t, y > 0.$$

- In that case, for any sequence of times $0 < t_1 \le t_2 \le \cdots \le t_n < \infty$ and $y_1, \cdots, y_n \in (0,\infty)$, $n \in \mathbb{N}$, the Markov property gives us

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- We would like a stronger sense of convergence e.g. we would like
$$\mathbb{E}_0[f(Z_s : s \leq t)] := \lim_{x \to 0} \mathbb{E}_x[f(Z_s : s \leq t)]$$
for an appropriate measurable function on cadlag paths of length t.
- The right setting to discuss *distributional convergence* is with respect to so-called *Skorokhod topology*.
- ROUGHLY: There is a metric on cadlag path space which does a better job of measuring how "close" two paths are than e.g. the uniform functional metric.
- This metric induces a topology (the Skorokhod topology). From this topology, we build a measurable space around the space of cadlag paths.
- Think of $\mathbb{P}_x$, $x > 0$ as a family of measures on this space and we want weak convergence "$\mathbb{P}_0 := \lim_{x\to 0} \mathbb{P}_x$" on this space.

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Starting from zero

Assume that Z is a pssMp with $\zeta = \infty$ a.s. Moreover, suppose that the Lévy process ξ, associated with Z through the Lamperti transform, is not a compound Poisson process.

Theorem

Under the assumption that $\mathbb{E}(\xi_1) > 0$, for any positive measurable function f and $t > 0$,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{\alpha \mathbb{E}(\xi_1)} E\left(\frac{1}{I_\infty^-} f\left(\left(\frac{t}{I_\infty^-} \right)^{1/\alpha} \right) \right),$$

where $I_\infty^- = \int_0^\infty \exp\{-\alpha \xi_s\} ds$.

Theorem

The limit $\mathbb{P}_0 := \lim_{x \to 0} \mathbb{P}_x$ exists in the sense of convergence with respect to the Skorokhod topology if and only if $\mathbb{E}(H_1^+) < \infty$ (H^+ is the ascending ladder process of ξ).

Sketch proof of the second theorem

- The basic idea is to give a pathwise construction of a candidate for "$(Z, \mathbb{P}_0)$" then check that there is weak convergence to it.
- Suppose we can identify ξ° which is a version of the underlying Lévy process ξ of $(Z, \mathbb{P}_x)$, $x > 0$ but now indexed by $\mathbb{R}$ rather than indexed by $[0, \infty)$, then we can identify the pathwise candidate for "$(Z, \mathbb{P}_0)$" by

$$Z_t^{(0)} = \exp\{\xi^\circ_{\varphi^\circ(t)}\}, \qquad t \geq 0,$$

where

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- If the above makes sense, then ξ° must "enter" from the space-time point $(-\infty, -\infty)$.
- It is the existence of an ξ° and "convergence" to it of $\xi + \log x$ on $[-s, t]$ as $x \to 0, s \to \infty$ which produces the necessary and sufficient condition that $E[H_1^+] < \infty$.

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Construction of ξ°

- If ξ° enters from $(-\infty, \infty)$, then it must make first passage over any level x in a "stationary" way.
- Specifically, we would need that $(\xi^\circ_{\sigma_a^+} - a, a - \xi^\circ_{\sigma_a^+ -})$ is independent of $a \in \mathbb{R}$, where $\sigma_a^+ = \inf\{t > -\infty : \xi^\circ_t > a\}$.This motivates the following construction:
- Take the stationary overshoot/undershoot law of ξ (which requires the necessary and sufficient condition $E[H_1^+] < \infty$)

$$\chi(\mathrm{d}y, \mathrm{d}z) = \frac{1}{\mathbb{E}[H_1^+]}\left(\widehat{U}_\xi(z)\Pi_\xi(z+\mathrm{d}y)\mathrm{d}z + \gamma\delta_0(\mathrm{d}y)\delta_0(\mathrm{d}z)\right), \qquad y, z \geq 0.$$

- Build the two-dimensional random variable $(\Delta, \Delta^\uparrow)$ has distribution χ. Then

$$\xi^\circ_t := \begin{cases} \xi_t & \text{under } P_\Delta \text{ if } t \geq 0, \\ -\xi^\uparrow_{|t|-} & \text{under } P^\uparrow_{\Delta^\uparrow} \text{ if } t < 0, \end{cases}$$

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- Build the two-dimensional random variable $(\Delta, \Delta^\uparrow)$ has distribution χ. Then

$$\xi^\circ_t := \begin{cases} \xi_t & \text{under } P_\Delta \text{ if } t \geq 0, \\ -\xi^\uparrow_{|t|-} & \text{under } P^\uparrow_{\Delta^\uparrow} \text{ if } t < 0, \end{cases}$$

where (ξ, P_x), $x > 0$ is an independent copy of the underlying Lévy process for Z and $\xi^\uparrow = \{\xi^\uparrow_t : t \geq 0\}$ under $P^\uparrow_x$ is an independent copy of the process ξ conditioned to stay positive.
- Hidden catch: Before constructing the entrance of Z from 0, we need to construct the entrance of $\xi^\uparrow$ from 0.

Construction of ξ°

- If ξ° enters from $(-\infty,\infty)$, then it must make first passage over any level x in a "stationary" way.
- Specifically, we would need that $(\xi^\circ_{\sigma_a^+}-a, a-\xi^\circ_{\sigma_a^+-})$ is independent of $a\in\mathbb{R}$, where $\sigma_a^+=\inf\{t>-\infty:\xi^\circ_t>a\}$.This motivates the following construction:
- Take the stationary overshoot/undershoot law of ξ (which requires the necessary and sufficient condition $E[H_1^+]<\infty$)

$$\chi(\mathrm{d}y,\mathrm{d}z)=\frac{1}{\mathbb{E}[H_1^+]}\left(\widehat{U}_\xi(z)\Pi_\xi(z+\mathrm{d}y)\mathrm{d}z+\gamma\delta_0(\mathrm{d}y)\delta_0(\mathrm{d}z)\right),\qquad y,z\geq 0.$$

- Build the two-dimensional random variable $(\Delta,\Delta^\uparrow)$ has distribution χ. Then

$$\xi^\circ_t:=\begin{cases}\xi_t & \text{under } P_\Delta \text{ if } t\geq 0,\\ -\xi^\uparrow_{|t|-} & \text{under } P^\uparrow_{\Delta^\uparrow} \text{ if } t<0,\end{cases}$$

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Recurrent extension

- The previous construction has insisted that Z is a *pssMp* with $\zeta = \infty$ a.s. But what about the case that $\zeta < \infty$ a.s.
- We can say something about the case that $\zeta < \infty$ a.s. and $X_{\zeta-} = 0$.
- A cadlag strong Markov process, $\vec{Z} := \{\vec{Z}_t \colon t \geq 0\}$ with probabilities $\{\vec{P}_x, x \geq 0\}$, is a *recurrent extension* of Z if, for each $x > 0$, the origin is not an absorbing state $\vec{P}_x$-almost surely and $\{\vec{Z}_{t \wedge \vec{\zeta}} \colon t \geq 0\}$ under $\vec{P}_x$ has the same law as (Z, P_x), where

$$\vec{\zeta} = \inf\{t > 0 : \vec{Z}_t = 0\}.$$

Theorem

Assume that Z is a non-conservative positive self-similar Markov process. Suppose that (ξ, P) is the (killed) Lévy process associated with Z through the second Lamperti transform. Then there exists a unique recurrent extension of Z which leaves 0 continuously if and only if there exists a $\beta \in (0, \alpha)$ such

$$\mathbb{E}(e^{\beta \xi_1}) = 1.$$

Here, as usual, α is the index of self-similarity.

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Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R}\times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbb{P}_{x,i}$, $x\in\mathbb{R}$, $i\in E$, if, for any $i\in E$, $s,t\geq 0$: Given $\{J(t)=i\}$,
    - $(\xi(t+s)-\xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u\leq t\}$,
    - $(\xi(t+s)-\xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0,i)$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n \geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n \geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n \geq 1}$ are the jump times of J,

the process ξ has the representation

$$\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

Lamperti-Kiu transform

- Take J to be irreducible on $E = \{1, -1\}$.
- Let

$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \leq t < T_0,$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t\right\}$$

and

$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$

- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.

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Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$\mathbf{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) - \mathbf{Q} \circ G(z),$$

(when it exists), where $\circ$ indicates elementwise multiplication.

- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbb{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{-\mathbf{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

(when it exists).

Entrance at zero

- Given the Lamperti-Kiu representation

$$Z_t = e^{\xi(\tau(|x|^{-\alpha}t))+\log|x|}J(\tau(|x|^{-\alpha}t)) \qquad 0 \le t < T_0,$$

  it is clear that we can think of a similar construction from zero to the case of pssMp.

- We need to construct a stationary version of the pair (ξ, J) which is indexed by $\mathbb{R}$ and pinned at space-time point $(-\infty, \infty)$.
- Just like the theory of Lévy processes, by observing the range of the process (ξ_t, J_t) $t \ge 0$, **only** at the points of its new suprema, we see a process (H_t^+, J_t^+), $t \ge 0$, which is also a MAP, where H^+ is has increasing paths.

Theorem

Suppose that $\pi\mathbf{Q} = 0$. Then the limit $\mathbb{P}_0 := \lim_{|x|\to 0}\mathbb{P}_x$ exists in the sense of convergence with respect to the Skorokhod topology if and only if $\pi_1\mathbb{E}_1(H_1^+) + \pi_{-1}\mathbb{E}_{-1}(H_1^+) < \infty$, and otherwise limit does not exist.

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An α-stable process is a rssMp

- An α-stable process up to absorption in the origin is a rssMp.
- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1-\alpha\hat{\rho}+z)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

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Esscher transform for MAPs

- If $\mathbf{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, *and*

$$M_t := \mathrm{e}^{\gamma(\xi(t)-\xi(0))-\chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ *such that* $\chi(\gamma)$ *is defined. Then,* M_t, $t \geq 0$, *is a unit-mean martingale. Moreover, under the change of measure*

$$\mathrm{d}\mathbf{P}^{\gamma}_{x,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{x,i}|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) *remains in the class of MAPs with new exponent given by*

$$\mathbf{\Psi}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1}\mathbf{\Psi}(z - \mathrm{i}\gamma)\mathbf{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ *is the identity matrix and* $\mathbf{\Delta}_v(\gamma) = \mathrm{diag}(v(\gamma))$.

Esscher and drift

- Suppose that χ is defined in some open interval D of $\mathbb{R}$, then, it is smooth and convex on D.
- Since $\mathbf{\Psi}(0) = -\mathbf{Q}$, we always have $\chi(0) = 0$ and $\mathbf{v}(0) = (1, \cdots, 1)$. So $0 \in D$ and $\chi'(0)$ is well defined and finite.
- With all of the above

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Esscher and the stable-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det\Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, which makes

$$\boldsymbol{\Psi}^{\bullet}(z) = \boldsymbol{\Delta}^{-1}\boldsymbol{\Psi}(z+\alpha-1)\boldsymbol{\Delta} = \begin{bmatrix} \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

  where $\boldsymbol{\Delta} = \mathrm{diag}(\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$.
- When $\alpha \in (0,1)$, $\chi'(0) > 0$ (because the stable process never touches the origin a.s.) and $\boldsymbol{\Psi}^{\bullet}(z)$-MAP drifts to $-\infty$
- When $\alpha \in (1,2)$, $\chi'(0) < 0$ (because the stable process touches the origin a.s.) and $\boldsymbol{\Psi}^{\bullet}(z)$-MAP drifts to $+\infty$.

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Riesz-Bogdan-Zak transform

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\bullet_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\bullet_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\bullet_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\boldsymbol{\Psi}^\bullet(z)$.

What is the $\mathbf{\Psi}^\bullet$-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}^\bullet_x)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin.
- When $\alpha \in (1,2)$, $(X, \mathbb{P}^\bullet_x)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin.

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