Scaling limits of large random trees

using Lévy processes

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Part I: Introduction

1. Motivations
2. Real trees & Gromov-Hausdorff topology
3. First scaling limits: Galton-Watson trees

Trees

In this lecture, a tree with n nodes is implicitly a connected graph with n (unlabelled) vertices and no cycle, which is moreover rooted (a node is distinguished and called the root)

A leaf is a node of degree 1, different from the root

If exceptionally the tree is ordered (plane) or with labelled nodes/leaves, it will be specified

Goal: Study the scaling limits of large random trees

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Ex. 1: Combinatorial trees

Take a tree T_n uniformly at random amongst a set of trees with "size" n, e.g.:

- T_n is a uniform rooted tree with n nodes
- T_n is a uniform rooted ordered tree with n nodes
- T_n is a uniform labelled tree with n nodes
- T_n is a uniform rooted (ordered) binary tree with n nodes, etc.

What happens when n is large?

Goal: Study the scaling limits of large random trees

Ex. 2: Conditioned Galton-Watson trees

η: proba. on $\mathbb{Z}_+ = \{0, 1, 2, ...\}$ (offspring distribution), such that $\eta(1) < 1$, with mean m

Extinction probability $= 1$ in subcritical ($m < 1$) and critical ($m = 1$) cases

$\in [0, 1)$ in supercritical cases ($m > 1$)

T_n: critical Galton-Watson tree conditioned to have n nodes (for integers n for which it is possible)

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Connections with combinatorial trees:

- If $\eta \sim$ Geo(1/2), T_n is uniform amongst the set of rooted ordered trees with n nodes
- If $\eta \sim$Poisson(1), T_n is uniform amongst the set of rooted trees with n labelled nodes
- If $\eta \sim \frac{1}{2}(\delta_0 + \delta_2)$, T_n is uniform amongst the set of rooted ordered binary trees with n nodes

However, not all combinatorial trees are conditioned GW trees, ex: the uniform rooted trees with n nodes

Goal: Study the scaling limits of large random trees

Ex. 3: Dynamical models of tree growth

Ex.: Rémy's algorithm (85): Generates trees uniformly distributed amongst the set of rooted binary trees with n labelled leaves

$T_{\mathrm{R}}(n)$ rooted binary tree with n labelled leaves $n \geq 1$

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let's start with $T_{\mathrm{R}}(1)$:

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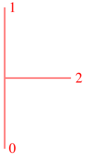

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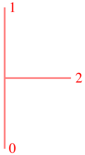

at each step :

- an edge is selected uniformly at random
- a new edge-leaf is branched on its "middle"

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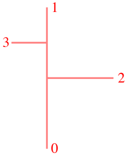

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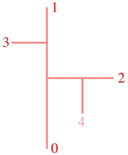

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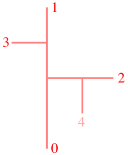

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This tree is distributed as a (planted) Galton-Watson tree with offspring distribution $\frac{1}{2}(\delta_0 + \delta_2)$ conditioned to have $2n-1$ nodes

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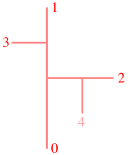

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Generalizations: At each step (1) we add k edges instead of 1 or more generally (1bis) a rooted tree (possibly random) or (2) we choose edges non-uniformly, and put possibly some weights on the nodes, etc.

A parenthesis on the notion of scaling limits: A basic example

I.i.d sequence of *centered* random variables $X_i \in \{-1, 1\}$:

$$-1,\ 1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ 1,\ -1,\ -1,\ 1,\ 1,\ 1,\ 1, ...$$

Centered random walk: $S_n = X_1 + ... + X_n$

How does S_n behave when n is large ?

(1) what is the growth rate ?

(2) what is the limit after rescaling ?

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Central limit theorem:

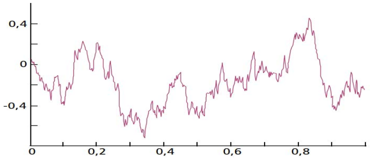
$$\frac{S_n}{\sqrt{n}} \xrightarrow{\text{law}} \mathcal{N}(0,1)$$

Functional version:

DONSKER'S theorem (51):

$$\left(\frac{S_{[nt]}}{\sqrt{n}}, t \in [0,1] \right) \xrightarrow{\text{law}} (B(t), t \in [0,1])$$

where B is a standard Brownian motion



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More generally:

- Invariance principle: Random walks with i.i.d. centered increments $X_i, i \geq 1$ with finite variance $\sigma^2 > 0$ converge in $\sigma\sqrt{n}$ towards a standard Brownian motion

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More generally:

- Invariance principle: Random walks with i.i.d. centered increments $X_i, i \geq 1$ with finite variance $\sigma^2 > 0$ converge in $\sigma\sqrt{n}$ towards a standard Brownian motion

- However: Random walks with i.i.d. increments $X_i, i \geq 1$ such that

$$\mathbb{P}(X_1 > x) \sim C_1 x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X_1 < -x) \sim C_2 x^{-\alpha} \quad \text{for some } \alpha \in (0,2)$$

  with $C_1 + C_2 > 0$ (and centered if $\alpha \in [1,2)$) converge in $n^{1/\alpha}$ towards an $\alpha-$stable Lévy process.

Scaling limits of large random trees

Goal: Describe similarly the scaling limits of large random trees using self-similar “continuous" trees

Beyond the topic of random trees, there are many applications to the study of other large random graphs. E.g.:

- Random planar maps: Bijection between different classes of planar maps and certain labelled trees (see e.g. Le Gall-Miermont 12 lecture for an Introduction)
- Erdös-Rényi random graph inside the critical window, description of the scaling limit by Addario Berry-Broutin-Goldschmidt 12

2. Real trees

A real tree is a metric space "connected and with no loop". Formally:

A metric space $(\mathcal{T}, d)$ is a real tree (or $\mathbb{R}$ tree) if for all $(x, y) \in \mathcal{T}^2$:

- $\exists!$ isometry $\varphi : [0, d(x, y)] \to \mathcal{T}$ s.t. $\varphi(0) = x$ and $\varphi(d(x, y)) = y$
- For every continuous, injective function $c : [0, 1] \to \mathcal{T}$ with $c(0) = x$ and $c(1) = y$, one has $c([0, 1]) = \varphi([0, d(x, y)])$

Let $[[x, y]] := \varphi([0, d(x, y)])$

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The degree of a point x of $\mathcal{T}$ is the number of connected components of $\mathcal{T} \backslash \{x\}$

A leaf is a point of degree 1

A branch point is a point of degree at least 3

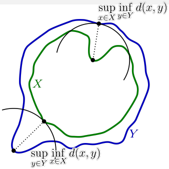
Notation (rescaling): $a\mathcal{T} := (\mathcal{T}, ad)$

A discrete tree can be seen as a real tree by replacing its edges by segments (usually of length 1)

Most of our trees will be rooted (a point is distinguished and called the root)

Gromov-Hausdorff distance

For X, Y compact subsets of a metric space (Z, d_Z)



$$d_{\text{Hausdorff}}(X, Y) = \max\left(\sup_{x\in X} d_Z(x, Y), \sup_{y\in Y} d_Z(y, X)\right)$$

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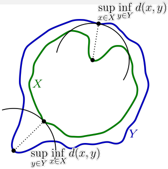
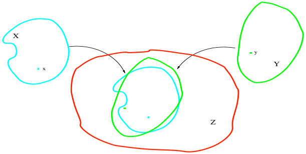


<u>Gromov-Hausdorff distance</u>: Let (X, x), (Y, y) be compact, pointed, metric spaces

$$d_{\mathrm{GH}}\left((X, x), (Y, y)\right) := \inf\left(d_{\mathrm{Hausdorff}}(\varphi_1(X), \varphi_2(Y)) \vee d_Z(\varphi_1(x), \varphi_2(y))\right)$$

the infimum being on all isometric embeddings $\varphi_1 : X \hookrightarrow Z$ and $\varphi_2 : Y \hookrightarrow Z$ into a same metric space (Z, d_Z).



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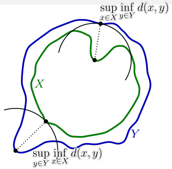

<u>Gromov-Hausdorff distance</u>: Let (X,x), (Y,y) be compact, pointed, metric spaces

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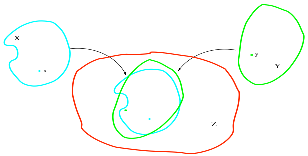

(X,x) and (Y,y) are *equivalent* if $\exists\ \varphi$ isometry: $Y = \varphi(X)$, $y = \varphi(x)$ (then, $d_{\mathrm{GH}}\left((X,x),(Y,y)\right) = 0$)

d_{GH}: distance on the set of equivalence classes

Gromov-Hausdorff-Prokhorov distance

Measured metric spaces: Equipped with a probability measure (on their Borel sigma-field)

Prokhorov distance: μ, μ' probability measures on (Z, d_Z),

$$d_{\mathrm{Prokhorov}}(\mu, \mu') = \inf\{\varepsilon > 0 : \mu(A) \leq \mu'(A_\varepsilon) + \varepsilon \text{ and } \mu(A') \leq \mu(A_\varepsilon) + \varepsilon, \forall A \subset Z \text{ closed}\}$$

where $A_\epsilon = \{x \in Z : d_Z(A, x) \leq \varepsilon\}$. If (Z, d_Z) is separable, it is a metrization of the topology of weak convergence.

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Gromov-Hausdorff-Prokhorov distance: (X, x, μ_X), (Y, y, μ_Y) compact, pointed, measured metric spaces

$$\begin{aligned} & d_{\text{GHP}}\left((X, x, \mu_X), (Y, y, \mu_Y)\right) \\ := \quad & \inf\left(d_{\text{Hausdorff}}(\varphi_1(X), \varphi_2(Y)) \vee d_Z(\varphi_1(x), \varphi_2(y)) \vee d_{\text{Prokhorov}}(\varphi_{1*}\mu_X, \varphi_{2*}\mu_Y)\right) \end{aligned}$$

the infimum being on all isometric embeddings $\varphi_1 : X \hookrightarrow Z$ and $\varphi_2 : Y \hookrightarrow Z$ into a same metric space (Z, d_Z). Here $\varphi_{1*}\mu_X, \varphi_{2*}\mu_Y$ denote the push-forwards of μ_X, μ_Y by φ_1, φ_2.

Gromov-Hausdorff-Prokhorov distance

$\mathfrak{T}$: set of (root preserving) isometry classes of compact rooted real trees

$\mathfrak{T}_m$: set of (root and measure preserving) isometry classes of compact rooted measured real trees

Theorem

$(\mathfrak{T}, d_{\mathrm{GH}})$ and $(\mathfrak{T}_m, d_{\mathrm{GHP}})$ are Polish spaces.

Ref. : Burago-Burago-Ivanov 01, Evans-Pitman-Winter 06, Evans-Winter 06, Miermont 09, Abraham-Delmas-Hoscheit 13

Lecture notes: Le Gall 05, Evans 08

3. First scaling limits: Galton-Watson trees

Context: GW tree with critical offspring distribution η with variance $0 < \sigma^2 < \infty$

T_n: version conditioned to have n nodes, equipped with the uniform measure μ_n on its nodes

Universal limit: The Brownian tree

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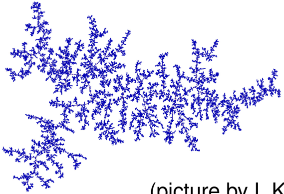

(picture by I. Kortchemski)

Theorem (Aldous 93)

$\exists$ a compact measured real tree $(\mathcal{T}_{\mathrm{Br}}, \mu_{\mathrm{Br}})$ s.t.

$$\left(\frac{T_n}{\sqrt{n}}, \mu_n\right) \xrightarrow[\mathrm{GHP}]{\mathrm{law}} \left(\frac{2}{\sigma}\mathcal{T}_{\mathrm{Br}}, \mu_{\mathrm{Br}}\right)$$

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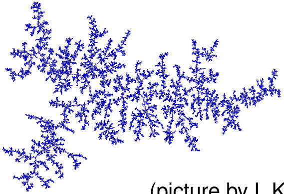

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Combinatorial applications:

(1) If T_n is uniform amongst the set of rooted ordered trees with n nodes, $n^{-1/2}T_n \to \mathcal{T}_{B_r}$

(2) If T_n is uniform amongst the set of rooted trees with n labelled nodes, $n^{-1/2}T_n \to 2\mathcal{T}_{B_r}$

This global perspective provides the behavior of several statistics of the trees (maximal height, height of a typical node, diameter, etc.) that first interested combinatorists.

Galton-Watson trees: Stable cases

Now, assume $\eta(k) \underset{k\to\infty}{\sim} \kappa k^{-1-\alpha}$, $\alpha \in (1,2)$ $(\Rightarrow \sigma^2 = \infty)$

Theorem (Duquesne 03)

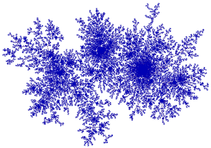
$\exists$ a compact measured real tree $(\mathcal{T}_\alpha, \mu_\alpha)$, called α-stable tree, s.t.

$$\left(\frac{T_n}{n^{1-1/\alpha}}, \mu_n\right) \xrightarrow[\text{GHP}]{\text{law}} \left(\left(\frac{\alpha(\alpha-1)}{\kappa\Gamma(2-\alpha)}\right)^{1/\alpha} \cdot \mathcal{T}_\alpha, \mu_\alpha\right)$$

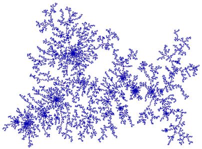
Stable trees: $\{\mathcal{T}_\alpha, \alpha \in (1,2]\}$ with the convention $\mathcal{T}_2 := \sqrt{2}\mathcal{T}_{\mathrm{Br}}$.

They belong to the class of Lévy trees, introduced by Le Gall-Le Jan 98 (see also Duquesne-Le Gall 02, Duquesne-Le Gall 05)

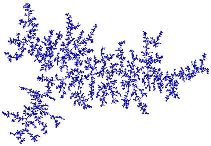
Strong connections with CSBP, fragmentation, coalescence processes, random planar maps



$\alpha = 1.2$



$\alpha = 1.5$



$\alpha = 2$ (Brownian)

(picture by I. Kortchemski)

Where are the Lévy processes?

Lukasiewicz path (see Igor's lecture): a random walk in the Galton-Watson tree $\rightarrow$ spectrally positive stable Lévy process.

Contour function (see Igor's lecture): the contour function of the Brownian tree is a positive Brownian excursion conditioned to have length 1.

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Height of a tagged leaf (central idea in the following):

Let L be a leaf distributed $\sim \mu_\alpha$ given $(\mathcal{T}_\alpha, \mu_\alpha)$

$\Lambda_*(t) := \mu_\alpha$-mass of the connected component of $\{v \in \mathcal{T}_\alpha : \mathrm{dist}(\rho, v) > t\}$ containing L.

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The process $(\Lambda_*(t), t \geq 0)$ is a positive self-similar Markov process, more precisely,

$$\Lambda_*(t) = \exp(-\xi_{\tau(t)}), \forall t \geq 0,$$

where ξ is a subordinator with Lévy measure

$$\Pi(\mathrm{d}x) = \frac{(1-\alpha^{-1})e^x \mathrm{d}x}{\Gamma(1+\alpha^{-1})(e^x-1)^{2-\alpha^{-1}}}$$

and τ the time-change $\tau(t) = \inf\{u : \int_0^u \exp(-\gamma\xi_t)\mathrm{d}r > t\}$ (Bertoin 02, Miermont 03).

Some properties of the stable trees

Vertices degrees (Duquesne-Le Gall 05): with probability one,

- For $\alpha = 2$: the vertices of $\mathcal{T}_\alpha$ are of degree 1,2 or 3 (binary tree)
- For $\alpha \in (1,2)$: the vertices of $\mathcal{T}_\alpha$ are of degree 1,2 or ∞ (branch points of infinite multiplicity)

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Set of leaves (Duquesne-Le Gall 05): with probability one,

- The set of leaves of $\mathcal{T}_\alpha$ is dense in $\mathcal{T}_\alpha$
- The measure μ_α is fully supported by the set of leaves

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Vertices degrees (Duquesne-Le Gall 05): with probability one,

- For $\alpha = 2$: the vertices of $\mathcal{T}_\alpha$ are of degree 1,2 or 3 (binary tree)
- For $\alpha \in (1,2)$: the vertices of $\mathcal{T}_\alpha$ are of degree 1,2 or ∞ (branch points of infinite multiplicity)

Set of leaves (Duquesne-Le Gall 05): with probability one,

- The set of leaves of $\mathcal{T}_\alpha$ is dense in $\mathcal{T}_\alpha$
- The measure μ_α is fully supported by the set of leaves

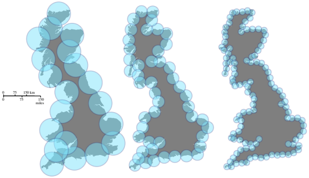
Fractal properties:

- *Self-similarity* (Miermont 03) – see Part 2 for a precise definition
- *Hausdorff dimension* (Duquesne-Le Gall 05, H.-Miermont 04)

$$\dim_{\mathrm{Haus}}(\mathcal{T}_\alpha) = \frac{\alpha}{\alpha - 1} \text{ a.s.}$$

Background on Hausdorff dimension

Ref. : Falconer 03



For all $r > 0$, r-dimensional Hausdorff measure of a metric space (Z, d_Z):

$$\mathcal{M}^r(Z) = \lim_{\varepsilon \to 0} \inf_{\{(C_i)_{i \in \mathbb{N}} : \mathrm{diam}\, C_i \leq \varepsilon\}} \left\{ \sum_{i \geq 1} \mathrm{diam}(C_i)^r : Z \subset \cup_{i \geq 1} C_i \right\}.$$

Then, the Hausdorff dimension of Z is given by

$$\dim_{\mathrm{H}}(Z) = \inf\{r : \mathcal{M}^r(Z) = 0\} = \sup\{r : \mathcal{M}^r(Z) = \infty\}.$$

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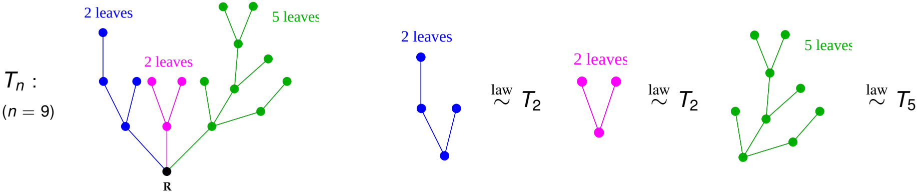
Part 2: Markov branching trees

1. Definitions and main result
2. Self-similar fragmentation trees
3. Proof of the main result
4. Applications: First examples

1. Markov branching trees

$(T_n, n \geq 1)$: T_n random rooted tree with n leaves

Markov branching property:



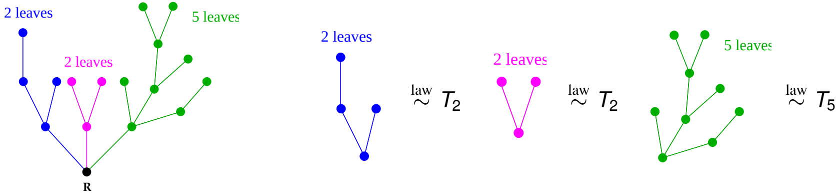
Conditional on "the root of T_n has p children-trees with $\lambda_1 \geq ... \geq \lambda_p$ leaves", T_n is distributed as the tree obtained by gluing on a common root p independent trees with respective distributions those of $T_{\lambda_1}, \ldots, T_{\lambda_p}$ and then forgetting the order.

1. Markov branching trees

$(T_n, n \geq 1)$: T_n random rooted tree with n leaves

Markov branching property:

T_n :
$(n = 9)$



Conditional on "the root of T_n has p children-trees with $\lambda_1 \geq ... \geq \lambda_p$ leaves", T_n is distributed as the tree obtained by gluing on a common root p independent trees with respective distributions those of $T_{\lambda_1}, \ldots, T_{\lambda_p}$ and then forgetting the order.

This property only depends on the distributions of each T_n, $n \geq 1$.

Similar definitions for sequences of trees indexed by the number of nodes, or more general notions of "size"

First ex.: Galton-Watson trees conditioned to have n leaves (respectively n nodes).

Markov branching trees

$(T_n, n \geq 1)$ Markov branching sequence indexed by leaves (T_n has n leaves)

Markov branching property $\Rightarrow$ the distributions of the $T_n, n \geq 1$ are entirely characterized by the splitting probabilities:

$$q_n(\lambda_1, \ldots, \lambda_p) := \mathbb{P}\,(\text{the root of } T_n \text{ has } p \text{ children-trees with } \lambda_1 \geq \ldots \geq \lambda_p \text{ leaves}), \ \forall n \geq 1$$

q_n: probability on the set $\mathcal{P}_n$ of partitions of n, i.e., for $n \geq 2$, $\mathcal{P}_n$ is the set of finite sequences of integers

$$\lambda = (\lambda_1, \ldots, \lambda_p) \text{ s.t. } \lambda_1 \geq \ldots \geq \lambda_p \geq 1 \text{ and } \sum_{i=1}^{p} \lambda_i = n$$

Note that necessarily $q_n((n)) < 1$ (but we may have $q_n((n)) > 0$).

For $n = 1$, $\mathcal{P}_1 := \{(1), \emptyset\}$.

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Note that necessarily $q_n((n)) < 1$ (but we may have $q_n((n)) > 0$).

For $n = 1$, $\mathcal{P}_1 := \{(1), \emptyset\}$.

Ex.: η: offspring distribution on $\mathbb{Z}_+$, GW_η: distribution of the total number of leaves in a η-GW tree. Assume $\mathrm{GW}_\eta(n) > 0$. Then,

$$q_n(\lambda_1, \ldots, \lambda_p) = \frac{p!}{\prod_{j \geq 1} m_j(\lambda)!} \eta(p) \frac{\prod_{i=1}^{p} \mathrm{GW}_\eta(\lambda_i)}{\mathrm{GW}_\eta(n)}$$

where $m_j(\lambda) = \#\{i : \lambda_i = j\}$.

Reciprocally, from each sequence (q_n) – where q_n proba. on $\mathcal{P}_n$ such that $q_n((n)) < 1$ – one can build a Markov branching sequence (T_n).

Goal

Goal: Find conditions on (q_n) to determine the scaling limit of (T_n)

Exercise: What happens when $q_n(n) = 1 - n^{-\alpha}$ and $q_n(\lceil n/2 \rceil, \lfloor n/2 \rfloor) = n^{-\alpha}$, $\alpha \geq 0$?

Main hypothesis

Partitions, a continuous counterpart:

$$\mathcal{S}^{\downarrow} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0 \text{ and } \sum_{i \geq 1} s_i = 1 \right\}$$

endowed with the distance $d_{\mathcal{S}^{\downarrow}}(\mathbf{s}, \mathbf{s}') = \sup_{i \geq 1} |s_i - s'_i|$.

Hypothesis (H)

$\exists\ \gamma > 0$ and ν a non-trivial $\sigma-$finite measure on $\mathcal{S}^{\downarrow}$ satisfying $\int_{\mathcal{S}^{\downarrow}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$ and $\nu(1, 0, \ldots) = 0$, such that

$$n^{\gamma} \sum_{\lambda \in \mathcal{P}_n} q_n(\lambda_1, \ldots, \lambda_p) \left(1 - \frac{\lambda_1}{n}\right) f\left(\frac{\lambda_1}{n}, \ldots, \frac{\lambda_p}{n}, 0, \ldots\right) \underset{n \to \infty}{\longrightarrow} \int_{\mathcal{S}^{\downarrow}} (1 - s_1) f(\mathbf{s}) \nu(\mathrm{d}\mathbf{s}).$$

$\forall$ continuous $f : \mathcal{S}^{\downarrow} \to \mathbb{R}$.

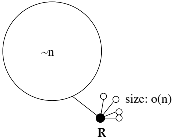
Ex.: If $q_n(n) = 1 - cn^{-\alpha}$ and $q_n(\lceil n/2 \rceil, \lfloor n/2 \rfloor) = cn^{-\alpha}$, $\alpha > 0$, then (H) is satisfied with

$$\gamma = \alpha \quad \text{and} \quad \nu(\mathrm{d}\mathbf{s}) = c\delta_{\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots\right)}$$

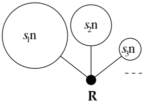
Main hypothesis

Informally: macroscopic branchings are rare:

with proba.~ 1 :



with proba. $\sim \frac{\nu(\mathrm{d}\mathbf{s})}{n^{\gamma}}$:



More precisely, when ν is infinite: $n \mapsto n\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$ s.t. $s_1 < 1 - \varepsilon$ for some $\varepsilon \in (0, 1)$ occurs asymptotically with proba. $\sim n^{-\gamma} \mathbf{1}_{\{s_1 < 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s})$.

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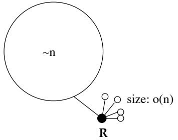
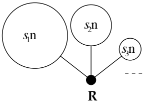
with proba.~ 1 : 

with proba. $\sim \frac{\nu(\mathrm{d}\mathbf{s})}{n^\gamma}$: 

More precisely, when ν is infinite: $n \mapsto n\mathbf{s}$, $\mathbf{s} \in \mathcal{S}^{\downarrow}$ s.t. $s_1 < 1 - \varepsilon$ for some $\varepsilon \in (0,1)$ occurs asymptotically with proba. $\sim n^{-\gamma} \mathbf{1}_{\{s_1 < 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s})$.

Remark: in case of macroscopic branchings at each step, there is at most a logarithmic growth

Ex.: Full binary tree with $n = 2^k$ leaves. Then maximal height $= \ln(n)/\ln(2)$.

More generally: Broutin-Devroye-McLeish-De La Salle 08:

$$\text{macroscopic branchings} \Rightarrow \text{maximal height of } T_n \sim \mathrm{cste}\ln(n),$$

but in general no scaling limit of the whole tree for the GH topology.

Main result

$(T_n, n \geq 1)$ Markov branching indexed by leaves

Theorem (H.-Miermont 12)

Assume (H). Then $\exists$ compact measured real tree $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$ s.t.

$$\left(\frac{T_n}{n^\gamma}, \mu_n\right) \xrightarrow[\text{GHP}]{\text{law}} (\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu}),$$

where μ_n is the uniform probability on the leaves of T_n.

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where μ_n is the uniform probability on the leaves of T_n.

$(T_n, n \geq 1)$ Markov branching indexed by nodes, p_n splitting probabilities on $\mathcal{P}_{n-1}$ (it describes how the $n-1$ non-root nodes are split into subtrees above the root, necessarily $p_2((1)) = 1$).

Theorem (H.-Miermont 12)

Assume (H) for (p_n) for some $0 < \gamma < 1$. Then $\exists$ random compact measured real tree $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$ s.t.

$$\left(\frac{T_n}{n^\gamma}, \mu_n\right) \xrightarrow[\text{GHP}]{\text{law}} (\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu}),$$

where μ_n is the uniform probability on the nodes of T_n.

Further results

In (H), the n^γ can be replaced by a regularly varying sequence.

See also Rizzolo 11+ for a generalization to scaling limits of Markov branching trees whose size is specified by the number of nodes whose out-degree (=number of children) lies in a given set.

See also H.-Miermont-Pitman-Winkel 08 for similar (and previous) results on scaling limits of Markov branching trees satisfying a consistent property (namely, T_n is distributed as the tree with n leaves obtained by removing an edge-leaf uniformly at random in T_{n+1}).

2. The limiting trees: self-similar fragmentation trees

The measured tree $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$ is γ-self-similar.

- In general, for a random compact measured real tree $(\mathcal{T}, \mu)$, $\forall t \geq 0$ let,

$$\mathcal{T}_t^{(i)}, i \geq 1 \text{ be the connected components of } \{v \in \mathcal{T} : \mathrm{dist}(\rho, v) > t\}$$
$$\text{and } \mu_t^{(i)} \text{ be the probability corresponding to the restriction of } \mu \text{ to } \mathcal{T}_t^{(i)}, i \geq 1$$

(ranked in decreasing order of μ-masses).

We say that $(\mathcal{T}, \mu)$ is γ-self-similar, $\gamma > 0$, if for all $t \geq 0$:

given $\mu_s^{(i)}, \forall i \geq 1, \forall s \leq t$, the measured trees $(\mathcal{T}_t^{(i)}, \mu_t^{(i)}), i \geq 1$ are independent, and

$$\left(\mathcal{T}_t^{(i)}, \mu_t^{(i)}\right) \overset{\text{law}}{=} \left(\left(\mu\left(\mathcal{T}_t^{(i)}\right)\right)^{\gamma} \cdot \mathcal{T}, \mu\right)$$

γ is called the index of self-similarity.

Such trees are studied in H.- Miermont 04, Stephenson 13.

Self-similar fragmentation trees

- Letting t increases and be a time-parameter, the process of masses

$$\left((\mu(\mathcal{T}_t^{(i)}), i \geq 1)\ t \geq 0\right)$$

is a self-similar fragmentation as introduced by Bertoin 02

$\Rightarrow$ its distribution is characterized by 3 parameters.

Very roughly, Bertoin 02 shows that

(1) After appropriate "Lamperti type" time changes that depend on the past of each fragment, the self-similar fragmentation is transformed in a homogeneous fragmentation (in which each fragment splits at the same rate)

(2) There is a "Lévy-Itô type decomposition" of each homogeneous fragmentation, that can be constructed from:

   - a Poisson point process driven by a σ-finite measure κ (called the dislocation measure) on the set $\left\{\mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq ... \geq 0 \text{ and } \sum_{i \geq 1} s_i \leq 1\right\}$, such that $\int (1 - s_1)\kappa(\mathrm{d}\mathbf{s}) < \infty$ & $\kappa(1, 0, \ldots) = 0$, that describes the distribution of the relative masses when a fragment splits
   - a continuous erosion of the fragments a at rate $c \geq 0$.

Self-similar fragmentation trees

As a consequence, the distribution of a self-similar tree is characterized by 3 parameters:

- the index of self-similarity (positive)
- a dislocation measure
- an erosion coefficient (non-negative)

The limiting tree $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$ arising in our setting has γ for index of self-similarity, ν for dislocation measure and 0 for erosion coefficient (= no erosion).

Remark: the Lévy-Itô decomposition of a homogeneous fragmentation can be extended to construct homogeneous compensated fragmentations (Bertoin 14+).

First examples: the stable trees

- Bertoin 02 notices that the Brownian tree $(\mathcal{T}_{\mathrm{Br}}, \mu_{\mathrm{Br}})$ is self-similar and calculates its characteristics: $\gamma = 1/2$ and $\nu(s_1 + s_2 < 1) = 0$,

$$\nu_{\mathrm{Br}}(s_1 \in \mathrm{d}x) = \frac{\sqrt{2}}{\sqrt{\pi} x^{3/2}(1-x)^{3/2}}, \quad 1/2 < x < 1$$

- Miermont 03 proves that each stable tree is self-similar and calculates its characteristics when $\alpha \in (1,2)$: $\gamma = 1 - 1/\alpha$ and

$$\int_{\mathcal{S}} f(\mathbf{s}) \nu_\alpha(\mathrm{d}\mathbf{s}) = C_\alpha \mathbb{E}\left[\sigma_1 f\left(\frac{\Xi_i}{\sigma_1}, i \geq 1\right)\right],$$

where

$$C_\alpha = \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)}$$

and $(\sigma_t, t \geq 0)$ is a stable subordinator of Laplace exponent $\lambda^{1/\alpha}$ and $(\Xi_i, i \geq 1)$ the sequence of its jumps before time 1, ranked in the decreasing order.

Some properties of the limiting trees

Support of $\mu_{\gamma,\nu}$: it is a.s. the set of leaves

(this is not necessarily the case for a general fragmentation tree – see Stephenson 13)

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Support of $\mu_{\gamma,\nu}$: it is a.s. the set of leaves

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Height of a typical leaf: let L be a leaf distributed $\sim \mu_{\gamma,\nu}$ given $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$

$\Lambda_*(t) := \mu_{\gamma,\nu}$-mass of the connected component of $\{v \in \mathcal{T}_{\gamma,\nu} : \mathrm{dist}(\rho, v) > t\}$ containing L

and $\Lambda_*(t) := 0$ if : $\mathrm{dist}(\rho, L) \leq t$.

The process $(\Lambda_*(t), t \geq 0)$ is a positive self-similar Markov process, more precisely,

$$\Lambda_*(t) = \exp(-\xi_{\tau(t)}), \forall t \geq 0,$$

where ξ is a subordinator with Laplace exponent $\phi(\lambda) = \int_{\mathcal{S}^{\downarrow}} \sum_i (1 - s_i^{\lambda}) s_i \nu(\mathrm{d}\mathbf{s})$ and ρ the time-change $\tau(t) = \inf\{u : \int_0^u \exp(-\gamma \xi_t)\mathrm{d}r > t\}$ (Bertoin 02).

In particular: Height of L = $\inf\{t \geq 0 : \Lambda_*(t) = 0\} = \int_0^{\infty} \exp(-\gamma \xi_r)\mathrm{d}r$.

Some properties of the limiting trees

Hausdorff dimension: If $\int_{\mathcal{S}^\downarrow}(s_1^{-1}-1)\nu(\mathrm{d}\mathbf{s})<\infty$, then

$$\dim_{\mathrm{H}}(\mathcal{T}_{\gamma,\nu}) = \max(1, 1/\gamma) \quad \text{a.s.}$$

(H.-Miermont 04). More generally Stephenson 13 computes the Hausdorff dimension of a general fragmentation tree.

3. Proof of the main result

Reminder: $(T_n, n \geq 1)$ Markov branching indexed by leaves, with splitting proba. $(q_n, n \geq 1)$

Theorem (H.-Miermont 12)

Assume that for all suitable functions f

$$n^{\gamma} \sum_{\lambda \in \mathcal{P}_n} q_n(\lambda_1, \ldots, \lambda_p)\left(1 - \frac{\lambda_1}{n}\right) f\left(\frac{\lambda_1}{n}, ..., \frac{\lambda_p}{n}, 0, ..\right) \underset{n \to \infty}{\to} \int_{\mathcal{S}^{\downarrow}} (1 - s_1) f(\mathbf{s}) \nu(\mathrm{d}\mathbf{s}).$$

Then, if μ_n denotes the uniform probability on the leaves of T_n

$$\left(\frac{T_n}{n^{\gamma}}, \mu_n\right) \xrightarrow[\mathrm{GHP}]{\mathrm{law}} (\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu}),$$

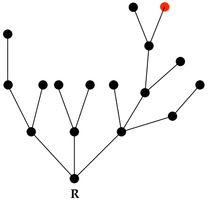
where $(\mathcal{T}_{\gamma,\nu}, \mu_{\gamma,\nu})$ is a fragmentation tree with parameters (γ, ν).

Outline of proof:

1. Height of a random leaf
2. Scaling limit of the tree spanned by k random leaves (finite dimensional cv)
3. Tightness criterion

Proof of the main result: A Markov chain in the Markov branching sequence of trees

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves



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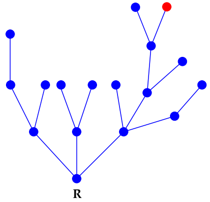
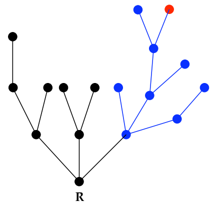


$X_n(0) = 9$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

Proof of the main result: A Markov chain in the Markov branching sequence of trees

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves

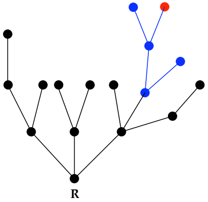


$X_n(0) = 9, X_n(1) = 5$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

Proof of the main result: A Markov chain in the Markov branching sequence of trees

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves

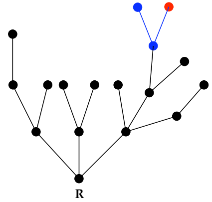


$X_n(0) = 9, X_n(1) = 5, X_n(2) = 3$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

Proof of the main result: A Markov chain in the Markov branching sequence of trees

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves

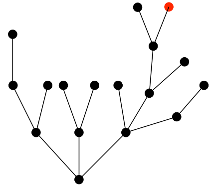


$X_n(0) = 9, X_n(1) = 5, X_n(2) = 3, X_n(3) = 2$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

Proof of the main result: A Markov chain in the Markov branching sequence of trees

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves



$X_n(0) = 9, X_n(1) = 5, X_n(2) = 3, X_n(3) = 2, X_n(4) = 1$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

It is a Markov chain!

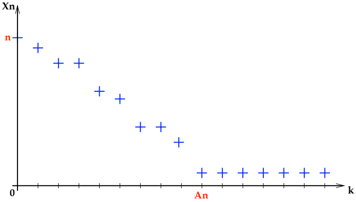
A_n = absorption time at 1= height of the marked leaf, up to geometric distribution

Scaling limit of non-increasing Markov chain

More generally: consider $(X(k), k \geq 0)$ a non-increasing $\mathbb{Z}_+ = \{0, 1, \ldots\}$-valued Markov chain

$(X_n(k), k \geq 0)$: chain starting from $X_n(0) = n$

Absorption time: $A_n = \inf\{i : X_n(i) = X_n(j), \forall j \geq i\}$



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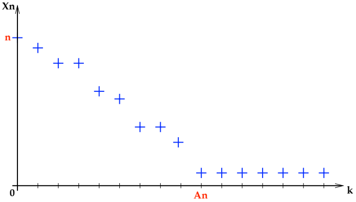

What is the behavior of $\frac{X_n(\cdot)}{n}$ and A_n when $n \to \infty$?

Scaling limit of non-increasing Markov chain

More generally: consider $(X(k), k \geq 0)$ a non-increasing $\mathbb{Z}_+ = \{0, 1, \ldots\}$-valued Markov chain

$(X_n(k), k \geq 0)$: chain starting from $X_n(0) = n$

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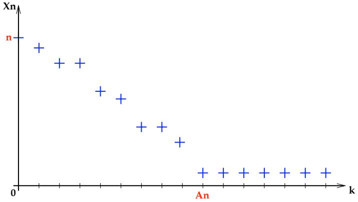

What is the behavior of $\dfrac{X_n(\cdot)}{n}$ and A_n when $n \to \infty$?

Hypothesis (H')

$\exists \gamma > 0$ and μ a finite measure on $[0, 1]$ ($\mu([0, 1]) > 0$) such that

$$n^{\gamma} \mathbb{E}\left[f\left(\frac{X_n(1)}{n}\right)\left(1 - \frac{X_n(1)}{n}\right)\right] \longrightarrow \int_{[0,1]} f(x)\mu(\mathrm{d}x)$$

for all continuous functions $f : [0, 1] \to \mathbb{R}$.

Scaling limit of non-increasing Markov chain

I.e., starting from n, “macroscopic" (with size proportional to n) jumps are rare

$$\mathbb{E}\left[\frac{n - X_n(1)}{n}\right] \sim \frac{\mu([0,1])}{n^\gamma}$$

and

$$\mathbb{P}\left(n - X_n(1) \geq n\varepsilon\right) \sim \frac{1}{n^\gamma}\int_{[0,1-\varepsilon]} \frac{\mu(\mathrm{d}x)}{1-x} \qquad \text{for a.e.} \quad 0 < \varepsilon \leq 1.$$

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Theorem (H.-Miermont 11)

Under (H'), $\exists$ positive $1/\gamma$-self-similar Markov process X_∞ such that

$$\left(\frac{X_n\left(\lfloor n^\gamma t \rfloor\right)}{n}, t \geq 0\right) \xrightarrow[n\to\infty]{\text{law}} \left(X_\infty(t), t \geq 0\right),$$

for the Skorokhod topology on the set $\mathbb{D}([0,\infty),[0,\infty))$.

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Lamperti: $X_\infty = \exp(-\xi_\rho)$, $\rho(t) = \inf\left\{u \geq 0 : \int_0^u \exp(-\gamma\xi_r)\mathrm{d}r \geq t\right\}$

Here ξ is a subordinator s.t. $\mathbb{E}[\exp(-\lambda\xi_t)] = \exp(-t\phi(\lambda))$, with

$$\phi(\lambda) = \mu(\{1\})\lambda + \int_{(0,1)} (1 - x^\lambda)\frac{\mu(\mathrm{d}x)}{1-x} + \mu(\{0\}), \quad \lambda \geq 0.$$

Absorption time

Fact: $\inf\{t \geq 0 : X_\infty(t) = 0\} = \int_0^\infty \exp(-\gamma\xi_r)\mathrm{d}r < \infty$ a.s.

Theorem (H.-Miermont 11)

Assume (H'). Then, jointly with the previous convergence,

$$\frac{A_n}{n^\gamma} \xrightarrow[n\to\infty]{\text{law}} \int_0^\infty \exp(-\gamma\xi_r)\mathrm{d}r.$$

This is *not* a direct corollary of the previous theorem, since the function:

non-increasing càdlàg function $\to$ its absorption time

is not continuous.

Also, under (H'):

$$\mathbb{E}\left[\left(\frac{A_n}{n^\gamma}\right)^p\right] \underset{n\to\infty}{\to} \mathbb{E}\left[\left(\int_0^\infty \exp(-\gamma\xi_r)\mathrm{d}r\right)^p\right], \quad \forall p \geq 0$$

and when $p \in \mathbb{Z}_+$,

$$\mathbb{E}\left[\left(\int_0^\infty \exp(-\gamma\xi_r)\mathrm{d}r\right)^p\right] = \frac{p!}{\prod_{i=1}^p \phi(\gamma i)} \quad \text{(by Carmona-Petit-Yor 97)}$$

Remark: Extension of all of these results to regularly varying sequences (instead of n^γ)

Main steps of the proof of the cv to the pssM process

Assume (H')

- Let : $Y_n(t) := n^{-1} X_n(\lfloor n^\gamma t \rfloor)$, then $(Y_n, n \geq 1)$ is tight
- Let Y' be a possible limit: $\exists$ a subsequence $(n_k, k \geq 1)$ s.t. $Y_{n_k} \xrightarrow{\text{law}} Y'$

  let $\tau_{Y_n}(t) := \inf\{u : \int_0^u Y_n^{-\gamma}(r) \mathrm{d}r > t\}$, $\tau_{Y'}(t) := \inf\{u : \int_0^u (Y'(r))^{-\gamma} \mathrm{d}r > t\}$

$$Z_n(t) := Y_n\left(\tau_{Y_n}(t)\right) \quad \text{and} \quad Z'(t) = Y'\left(\tau_{Y'}(t)\right)$$

  Fact: $Y'(t) = Z'(\tau_{Y'}^{-1}(t)) = Z'(\inf\{u : \int_0^u Z'^\gamma(r) \mathrm{d}r > t\})$

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  Fact: $Y'(t) = Z'(\tau_{Y'}^{-1}(t)) = Z'(\inf\{u : \int_0^u Z'^\gamma(r)\mathrm{d}r > t\})$
- For all $\lambda \geq 0$, and $n \geq 1$, let $G_n(\lambda) := \mathbb{E}\left[(X_n(1)/n)^\lambda\right]$. Then,

$$M_n^{(\lambda)}(t) := Z_n^\lambda(t)\left(\prod_{i=0}^{\lfloor n^\gamma \tau_{Y_n}(t)\rfloor - 1} G_{X_n(i)}(\lambda)\right)^{-1}, \quad t \geq 0$$

  is a martingale (consequence of the Markov property of X_n)
- $M_{n_k}^{(\lambda)} \overset{\text{law}}{\to} (Z')^\lambda \exp(\phi(\lambda)\cdot)$, which is also a martingale
- $\Rightarrow$ $\ln Z'$ is a Lévy process.

Applications and extensions

Apart from applications to Markov branching trees, these results can be used to describe the asymptotic behavior of:

(1) random walks with a barrier (H.-Miermont 11)

(2) the number of collisions in Λ-coalescent processes (H.-Miermont 11)

(3) the number of cuts in a Cayley tree needed to isolate the root (Bertoin 12).

Extensions: Recently, Bertoin-Kortchemski 14+ set up similar results to non-monotone Markov chains and develop several applications (to random walks conditioned to stay positive, to the number of particles in some coagulation-fragmentations processes, they also mention connections with random planar triangulations).

Also in H.-Stephenson 15+ (in progress) we study similar convergences for typed Markov chains towards "Lamperti time changed" Markov additive processes. This will have applications to dynamical models of tree growth.

Back to the of proof of scaling limits of MB trees

(T_n) MB sequence of trees indexed by leaves, with transition proba. (q_n) satisfying (H) with limiting parameters (γ, ν)

We want to show that $n^{-\gamma} T_n$ cv. to a (γ, ν)-fragmentation tree for the GHP topology

First step: Height of a leaf chosen uniformly at random amongst the set of n leaves

X_n: Markov chain corresponding to the size the subtree containing the marked leaf, started at n

$A_n = \inf\{k : X_n(k) = 1\}$

The chain X is non-increasing with transition probabilities

$$p_{k,i} = \sum_{\lambda \in \mathcal{P}_k} q_k(\lambda) \frac{i}{k} \#\{r : \lambda_r = i\}, \quad i \leq k$$

Fact: (q_n) satisfies (H) with limiting parameters $(\gamma, \nu) \Rightarrow$ the transition probabilities $(p_{n,\cdot})$ satisfies (H) with limiting parameters (γ, μ) where $(1-x)^{-1}\mu(\mathrm{d}x) = \sum_i s_i \nu(s_i \in \mathrm{d}x)$.

Back to the of proof of scaling limits of MB trees

Hence (q_n) satisfies (H) with limiting parameters (γ, ν)

$$\frac{H_n}{n^\gamma} \xrightarrow{\text{law}} \int_0^\infty \exp(-\gamma \xi_r) \mathrm{d}r$$

where ξ is a subordinator with Laplace exponent $\phi(\lambda) = \int_{\mathcal{S}^\downarrow} \sum_i (1 - s_i^\lambda) s_i \nu(\mathrm{d}\mathbf{s})$

This is the height of a typical leaf in a (γ, ν) fragmentation tree!

4. Applications: First examples

We discuss here several applications of the convergence of rescaled Markov branching trees to self-similar fragmentation trees.

Application 1: Conditioned Galton-Watson trees

- **GW trees conditioned by their number of nodes**: We recover Aldous 93 and Duquesne03

T_n: GW tree with offspring distribution η with mean 1, $\eta(1) \neq 1$, conditioned to have n nodes

Theorem (Aldous 93, Duquesne 03)

If η has finite non-zero variance or $\eta(k) \sim \kappa k^{-1-\alpha}$, $\alpha \in (1,2)$, then

$$\left(\frac{T_n}{n^{1-1/\alpha}}, \mu_n\right) \xrightarrow[n\to\infty]{\text{law}} (\mathrm{cst}_{\alpha,\kappa}\mathcal{T}_\alpha, \mu_\alpha)$$

(with the convention $\alpha = 2$ in case of finite variance).

Indeed: OK for the Markov-branching property.

Applications: Conditioned Galton-Watson trees

Moreover, the splitting proba. p_n (on $\mathcal{P}_{n-1}$) is given by

$$p_n(\lambda) = \frac{p!}{\prod_{j\geq 1} m_j(\lambda)!}\eta(p)\frac{\prod_{i=1}^{p}\overline{\mathrm{GW}}_\eta(\lambda_i)}{\overline{\mathrm{GW}}_\eta(n)}$$

where $\overline{\mathrm{GW}}_\eta(n)$ is the probability that a η-GW tree has n nodes.

Lemma (H.-Miermont 12)

If η has a finite variance σ^2, then (p_n) satisfies (H) with $\gamma = 1/2$ and $\nu = \frac{\sigma}{2}\nu_{\mathrm{Br}}$.

Together with the cv of MB trees $\rightarrow$ gives Aldous 93.

Applications: Conditioned Galton-Watson trees

Sketch of proof of the lemma: (1) Otter-Dwass formula (or cyclic lemma)

$$\overline{\mathrm{GW}}_\eta(n) = \frac{1}{n}\mathbb{P}(S_n = -1)$$

where S_n is a random walk with i.i.d. increments of law $(\eta_{i+1}, i \geq -1)$.

(2) Local limit theorem: $\mathbb{P}(S_n = -1) \underset{n\to\infty}{\sim} (2\pi\sigma^2 n)^{-1/2}$

(3) Riemann sums:

$$\sqrt{n} \sum_{\lambda \in \mathcal{P}_{n-1}} p_n(\lambda) \left(1 - \frac{\lambda_1}{n}\right) f\left(\frac{\lambda}{n}\right) \underset{n\to\infty}{\sim} \frac{\sigma}{\sqrt{2\pi}} \frac{1}{n} \sum_{\lambda_1 = \lceil (n-1)/2 \rceil}^{n-1} f\left(\frac{\lambda}{n}\right) \left(\frac{\lambda_1}{n}\right)^{-3/2} \left(\frac{n-\lambda_1}{n}\right)^{-3/2}$$

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We obtain in a similar (but more involved) way that when $\eta_k \sim ck^{-\alpha-1}$ for some $\alpha \in (1,2)$,

Lemma (H.-Miermont 12)

Then (p_n) satisfies (H) with $\gamma = 1 - 1/\alpha$ and $\nu = \left(c\Gamma(2-\alpha)\alpha^{-1}(\alpha-1)^{-1}\right)^{1/\alpha} \nu_\alpha$.

$\rightarrow$ gives Duquesne 03.

Applications: Conditioned Galton-Watson trees

▶ **GW trees conditioned by their number of nodes with out-degree in a given set**

Assume η has mean 1 and variance $0 < \sigma^2 < \infty$ and fix $A \subset \mathbb{Z}_+$.

T_n^A: version of the GW tree conditioned to have n nodes with out-degree (= number of children) in A (ex.: when $A = \{0\}$, the tree is conditioned to have n leaves)

Theorem (Rizzolo 11+)

$$\left(\frac{T_n^A}{\sqrt{n}}, \mu_n\right) \xrightarrow[\text{GHP}]{\text{law}} \left(\frac{2}{\sigma\sqrt{\eta(A)}}\mathcal{T}_{\mathrm{Br}}, \mu_{\mathrm{Br}}\right)$$

Main steps of the proof:

(1) Extension to any A of the previous results on cv of MB trees (by easy coupling argument)
(2) Evaluation of the splitting probabilities, by generalizing the Otter-Dwass formula (using couplings with others GW trees).

See also Kortchemski 12 for similar results proved via contour functions.

Applications: Pólya trees

Application 2: Pólya trees

Let $T_n^{(\mathrm{P})}$: uniform amongst the set of rooted trees with n nodes (non-ordered, non-labelled)

Aldous' conjecture 91: the scaling limit is the Brownian tree, up to a multiplicative constant

Broutin-Flajolet 08: study the maximal height's behavior

Drmota-Gittenberger 10: study the profile's behavior (profile: sequence of sizes of generations)

Marckert-Miermont 11: prove that the scaling limit of a tree picked uniformly amongst the set of rooted, binary trees with n nodes converges towards the Brownian tree.

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Theorem (H.-Miermont 12)

$$\left(\frac{T_n^{(\mathrm{P})}}{\sqrt{n}}, \mu_n\right) \xrightarrow[\mathrm{GHP}]{\mathrm{law}} (c_{\mathrm{P}} \mathcal{T}_{\mathrm{Br}}, \mu_{\mathrm{Br}}), \quad c_{\mathrm{P}} \sim 1.491$$

with μ_n the uniform measure on the nodes of $T_n^{(\mathrm{P})}$.

Applications: Pólya trees

Analog results for a uniform rooted tree with n (non-ordered, non-labelled) nodes and with at most m children per node (replacing c_{P} by a constant c_m).

Recently, Panagiotou and Stufler 15 give a more combinatorial proof of this result and extend it to Pólya trees with out-degrees in a given set.

Applications: Pólya trees

Sketch of proof of the theorem:

(1) The sequence $(T_n^{(\mathrm{P})})$ is not Markov branching, however it is "not far" from being so

(2) We can couple this sequence with a Markov branching sequence (T'_n) such that

$\mathbb{E}[d_{\mathrm{GHP}}(n^{-\varepsilon}T_n^{\mathrm{P}}, n^{-\varepsilon}T'_n)] \to 0, \forall \varepsilon > 0$

and $T_n^{(\mathrm{P})}$ and T'_n have the same splitting probability p_n

(3) For $\lambda \in \mathcal{P}_{n-1}$,

$$p_n(\lambda) = \frac{\prod_{j=1}^{n-1} \#F_j(m_j(\lambda))}{\#\mathbb{T}_n}$$

where $m_j(\lambda) = \{i : \lambda_i = j\}$, $\#\mathbb{T}_n$:nb. of rooted trees with n nodes

and $F_j(k)$: set of multisets with k elements in $\mathbb{T}_j$ (convention $F_j(0) := \{\emptyset\}$)

Ex.: $\#F_j(1) = \#\mathbb{T}_j$, $\#F_j(2) = \#\mathbb{T}_j(\#\mathbb{T}_j - 1)/2 + \#\mathbb{T}_j$, etc.

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(4) Otter 48:

$$\#\mathbb{T}_n \underset{n\to\infty}{\sim} \mathrm{c}\frac{\rho^n}{n^{3/2}}, \mathrm{c} > 0, \rho > 1.$$

(5) Riemann sums:

$$\sqrt{n} \sum_{\lambda \in \mathcal{P}_{n-1}} p_n(\lambda) \left(1 - \frac{\lambda_1}{n}\right) f\left(\frac{\lambda}{n}\right) \underset{n\to\infty}{\sim} \frac{c}{n} \sum_{\lambda_1 = \lceil (n-1)/2 \rceil}^{n-1} f\left(\frac{\lambda}{n}\right)\left(\frac{\lambda_1}{n}\right)^{-3/2}\left(\frac{n-\lambda_1}{n}\right)^{-3/2}$$

so that finally $c_{\mathrm{P}} = \sqrt{2}/(\mathrm{c}\sqrt{\pi})$.

To complete the picture on combinatorial trees asymptotics:

Theorem (Stufler 15)

Let $T_n^{(\mathrm{P},*)}$: uniform amongst the set of unrooted trees with n nodes (unordered, unlabelled). Then,

$$\frac{T_n^{(\mathrm{P})}}{\sqrt{n}} \xrightarrow[\mathrm{GH}]{\mathrm{law}} c_{\mathrm{P}}\mathcal{T}_{\mathrm{Br}}$$

(with the same c_{P}).

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Part 3: Dynamical models of tree growth

1. Rémy's algorithm (85)

$T_{\mathrm{R}}(n)$ rooted binary tree with n labelled leaves $n \geq 1$, built recursively:

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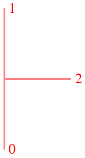
let's start with $T_{\mathrm{R}}(1)$:

1. Rémy's algorithm (85)

$T_{\mathrm{R}}(n)$ rooted binary tree with n labelled leaves $n \geq 1$, built recursively:

$T_{\mathrm{R}}(2)$:



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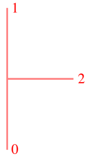


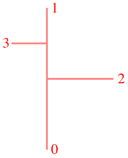
At each step :

- an edge is selected uniformly at random

- a new edge-leaf is branched on its "middle"

$T_{\mathrm{R}}(n)$ rooted binary tree with n labelled leaves $n \geq 1$, built recursively:

$T_{\mathrm{R}}(3)$:



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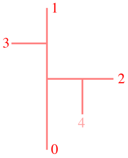
1. Rémy's algorithm (85)

$T_{\mathrm{R}}(n)$ rooted binary tree with n labelled leaves $n \geq 1$, built recursively:

$T_{\mathrm{R}}(4)$:

At each step :

- an edge is selected uniformly at random
- a new edge-leaf is branched on its "middle"



$T_{\mathrm{R}}(n)$ tree is distributed as a (planted) Galton-Watson tree with offspring distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ conditioned to have $2n - 1$ nodes (or n leaves).

As such, $T_{\mathrm{R}}(n)$ is distributed as the shape of the n-marginal of $\mathcal{T}_{\mathrm{Br}}$, i.e. as the subtree of $\mathcal{T}_{\mathrm{Br}}$ spanned by the root and n leaves taken independently according to μ_{Br} (Duquesne-Le Gall 02).

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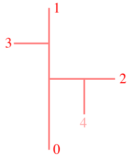
- an edge is selected uniformly at random
- a new edge-leaf is branched on its "middle"



$T_{\mathrm{R}}(n)$ tree is distributed as a (planted) Galton-Watson tree with offspring distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ conditioned to have $2n-1$ nodes (or n leaves).

As such, $T_{\mathrm{R}}(n)$ is distributed as the shape of the n-marginal of $\mathcal{T}_{\mathrm{Br}}$, i.e. as the subtree of $\mathcal{T}_{\mathrm{Br}}$ spanned by the root and n leaves taken independently according to μ_{Br} (Duquesne-Le Gall 02).

Aldous 93:

$$\frac{T_{\mathrm{R}}(n)}{\sqrt{n}} \xrightarrow[\mathrm{GHP}]{\mathrm{law}} 2\sqrt{2}\mathcal{T}_{\mathrm{Br}}.$$

Generalizations of Rémy's algorithm (85)

Several goals:

- Generate algorithmically the shapes of the n-marginals of the stable trees (Marchal 08), study the scaling limits and applications.
- Non-uniform choice of the selected edge. Ex.: Ford's models for phylogenetic trees (Ford 05). Study the scaling limits.
- Other models of tree growth where we add more than one edge on the selected edge. Study the scaling limits.

2. Growing k-ary trees

We are interested in the following modification of Rémy's algorithm. Let $k \geq 2$ be an integer.

Let $T_k(1)$ be the rooted tree composed by a single edge.

Then construct $(T_k(n), n \geq 1)$ recursively:

At each step,

- an edge is selected uniformly at random
- $k-1$ new edge-leaves are branched on its "middle".

What is the scaling limit?

Growing k-ary trees

Theorem (H.-Stephenson 14+)

Let $\mu_k(n)$ be the uniform measure on the leaves of $T_k(n)$. Then,

$$\left(\frac{T_k(n)}{n^{1/k}}, \mu_k(n)\right) \xrightarrow[\text{GHP}]{\mathbb{P}} (\mathcal{T}_k, \mu_k)$$

where $(\mathcal{T}_k, \mu_k)$ is a self-similar fragmentation tree, with index of self-similarity $1/k$ and dislocation measure

$$\nu_k(\mathrm{d}\mathbf{s}) = \frac{(k-1)!}{k(\Gamma(\frac{1}{k}))^{k-1}} \prod_{i=1}^{k} s_i^{-(1-1/k)} \left(\sum_{i=1}^{k} \frac{1}{1-s_i}\right) \mathbf{1}_{\{s_1 \geq s_2 \geq \ldots \geq s_k\}} \mathrm{d}\mathbf{s},$$

supported on the simplex of dimension $k-1$.

Note: the limiting tree $\mathcal{T}_k$ has Hausdorff dimension k a.s.

Growing k-ary trees

Proof: (1) Markov branching property.

(2) Let $T_n^1, \dots T_n^k$ be the k subtrees above the first node of $T_k(n)$, the label 1 refers to the subtree containing the very first leaf, the others are given arbitrarily.

Let $\tilde{q}_n$ be the distribution of their sizes, where size=nb. of internal nodes. It is a proba. on the set of compositions $\lambda = (\lambda_1, \dots, \lambda_k)$ of $n-1$ (i.e. of sequences of integers ≥ 1). Then,

$$\tilde{q}_n(\lambda) = \frac{1}{k(\Gamma(\frac{1}{k}))^{k-1}} \left(\prod_{i=1}^{k} \frac{\Gamma(\frac{1}{k} + \lambda_i)}{\lambda_i!} \right) \frac{n!}{\Gamma(\frac{1}{k} + n + 1)} \left(\sum_{j=1}^{\lambda_1+1} \frac{\lambda_1!}{(\lambda_1 - j + 1)!} \frac{(n-j+1)!}{n!} \right).$$

Next, evaluate the usual splitting probabilities by re-ordering in decreasing order.

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Next, evaluate the usual splitting probabilities by re-ordering in decreasing order.

(3) The convergence can be improved to a convergence in probability, by using urns models (in particular, triangular urns schemes – see Janson 05 – and the Chinese Restaurant Process of Pitman – see Pitman 06). Actually, it is possible to prove the convergence directly using urns models, but this does not permit to identify the limit as a self-similar fragmentation tree.

3. General models of “uniform" tree growth

(work in progress with Robin Stephenson 15+)

Now, we add a each step the same rooted discrete tree, say τ, having N nodes.

Then, in distribution for the GHP-topology

- The tree grows in $n^{1/N+1}$
- The limiting tree after normalization by $n^{1/N+1}$ is a multitype fragmentation tree, in which each branch point distributes its mass into its subtrees according to a dislocation measure that depends on its type. The number of types is the cardinal of

$$\mathcal{B}_\tau = \{\tau_v, v \in \tau \backslash \{\text{root}\}\},$$

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Generalizations to random τ possible (growth in $n^{1/\mathbb{E}[N+1]}$).

Technically: Convergence of typed non-increasing Markov chains to a “Lamperti-time changed Markov addive process".

4. The stable cases: Marchal's algorithm (08)

Goal: generate recursively the shape of the *n*-marginals of a α-stable tree $(\mathcal{T}_\alpha, \mu_\alpha)$ $(1 < \alpha \leq 2)$

$T_\alpha(n)$ rooted, n labelled leaves $n \geq 1$

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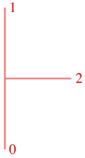

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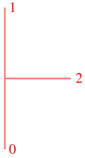

Weight : $\circ$ $\alpha - 1$ on each edge
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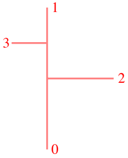

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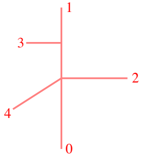

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Remark: $\alpha = 2 \Rightarrow 3 - 1 - \alpha = 0 \Rightarrow$ binary trees (Rémy's algorithm)

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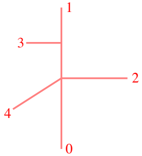

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Marchal 08: $T_\alpha(n) \stackrel{(d)}{=}$ shape of the subtree of $\mathcal{T}_\alpha$ spanned by the root and n leaves taken independently according to μ_α (including leaves labels)

Duquesne-Le Gall 02: this shape is distributed as a Galton-Watson tree whose offspring distribution has probability generating function $z + \alpha^{-1}(1-z)^\alpha$, conditioned to have n leaves.

Marchal's algorithm

Remarks: (1) There are sequences of GW trees conditioned by their number of nodes that cannot be constructed as growing trees, by adding vertices one by one (Janson 06).

(2) It is also possible to built the subtrees of $\mathcal{T}_\alpha$ spanned by the root and n independent leaves, $n \geq 1$, with lengths, algorithmically (Goldschmidt-H. 15).

Marchal's algorithm

$\mu_\alpha(n) :=$ uniform probability measure on the leaves of $T_\alpha(n)$

Theorem

Almost surely,

$$\left(\frac{T_\alpha(n)}{n^{1-1/\alpha}}, \mu_\alpha(n)\right) \xrightarrow[\mathrm{GHP}]{} (\alpha\mathcal{T}_\alpha, \mu_\alpha).$$

Marchal 08: proves almost sure finite dimensional convergence

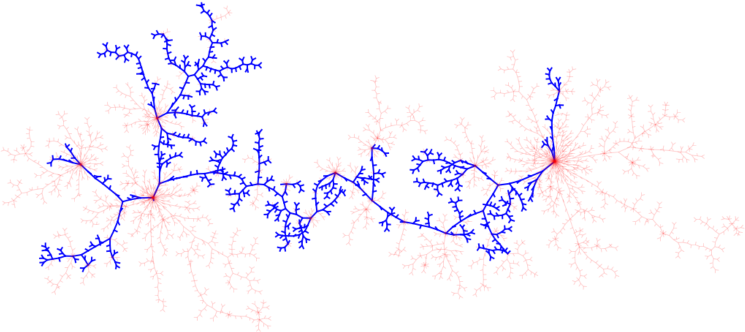
H.-Miermont-Pitman-Winkel 08: proves the convergence in probability for the topology GHP

(the convergence in distribution can be recovered from Part 2, since $(T_\alpha(n))$ is a Markov branching sequence, the splitting probabilities can be found in Miermont 03)

H.-Curien 13: cv a.s. for GHP.

5. The stable trees are nested!

A Brownian tree in a stable tree:



$\alpha = 1.8$ (red) $\qquad \alpha' = 2$ (blue)

(picture by N. Curien)

Hausdorff dimension

Recall that for all $\alpha \in (1,2]$

$$\dim_{\mathrm{Haus}}(\mathcal{T}_\alpha) = \frac{\alpha}{\alpha-1} = 1 + \frac{1}{\alpha-1} \text{ a.s.}$$

Extracting stable trees from stable trees

- Random scaling factor: for $\alpha \in (1,2]$

$$J_\alpha \overset{(d)}{=} \alpha(\Gamma_{2-1/\alpha})^{1-1/\alpha}$$

where $\Gamma_{2-1/\alpha}$ has a density on $(0,\infty)$ proportional to $x^{2-1/\alpha-1}\exp(-x)$

- Rescaled stable tree:

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Theorem (Curien-H. 13)

Let $1 < \alpha < \alpha' \leq 2$. There exists a closed subtree $\mathfrak{T}_{\alpha,\alpha'}$ of $J_\alpha \cdot \mathcal{T}_\alpha$ such that

$$\mathfrak{T}_{\alpha,\alpha'} \overset{(d)}{=} J_{\alpha'} \cdot \mathcal{T}_{\alpha'}$$

- $\mathfrak{T}_{\alpha,\alpha'}$ is not unique
- $\mu_\alpha(\mathfrak{T}_{\alpha,\alpha'}) = 0$ (with μ_α uniform probability measure on $J_\alpha \cdot \mathcal{T}_\alpha$)

Extracting a (rescaled) Brownian tree from $\mathcal{T}_\alpha$

- X_0: root of $\mathcal{T}_\alpha$ $\quad$ $(X_i, i \geq 1)$ i.i.d. sample $\sim \mu_\alpha$

  $\mathfrak{t}_n$: tree spanned by $X_0, X_1, ..., X_n$ $\quad$ Δ_{n+1}: projection of X_{n+1} on $\mathfrak{t}_n$

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- recursively, let $\tau_{n+1} := \begin{cases} \tau_n \cup [[\Delta_{n+1}, X_{n+1}]] & \text{if this union is a binary tree} \\ \tau_n & \text{otherwise} \end{cases}$

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- set: $\mathrm{Prun}_{\alpha,2}(\mathcal{T}_\alpha; (X_i, i \geq 0)) := \overline{\cup_{n\geq 1}\tau_n}$

Extracting a (rescaled) Brownian tree from $\mathcal{T}_\alpha$

Proposition (Brownian case, Curien-H. 13)

$$\mathrm{Prun}_{\alpha,2}(\mathcal{T}_\alpha; (X_i, i \geq 0)) \stackrel{(\mathrm{d})}{=} \frac{2}{\alpha}\sqrt{\mathrm{ML}_{2(1-1/\alpha),1-1/\alpha}} \cdot \mathcal{T}_2$$

where $\mathrm{ML}_{2(1-1/\alpha),1-1/\alpha}$ is a generalized Mittag-Leffler distribution.

Generalized Mittag-Leffler distribution $\mathrm{ML}_{\beta,\theta}$, with $\beta \in (0,1)$, $\theta > -\beta$:

$$\mathbb{E}\left[\mathrm{ML}_{\beta,\theta}^{p}\right] = \frac{\Gamma(\theta+1)\Gamma(\theta/\beta+p+1)}{\Gamma(\theta/\beta+1)\Gamma(\theta+p\beta+1)}, \quad p \geq 0.$$

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The proof of this crucial proposition is based on Marchal's algorithm (see later).

Proof of the theorem (Brownian extraction):

$$J_\alpha \cdot \frac{2}{\alpha}\sqrt{\mathrm{ML}_{2(1-1/\alpha),1-1/\alpha}} \overset{(\mathrm{d})}{=} J_2$$

Hence:

$$J_\alpha \cdot \mathcal{T}_\alpha \supset J_\alpha \cdot \mathrm{Prun}_{\alpha,2}(\mathcal{T}_\alpha; (X_i, i \geq 0)) \overset{(\mathrm{d})}{=} J_2 \cdot \mathcal{T}_2$$

Extracting a (rescaled) α'-stable tree from $\mathcal{T}_\alpha$

For $1 < \alpha < \alpha' \leq 2$ and $d \geq d'$ integers

$$p_{\alpha,\alpha',d,d'} = \begin{cases} \frac{(d'-1-\alpha')(\alpha-1)}{(d-1-\alpha)(\alpha'-1)} & \text{for } d \geq d' > 2 \\ 1 & \text{for } d \geq d' = 2 \\ 0 & \text{for } d \geq 2,\, d' = 0 \end{cases} \in [0,1]$$

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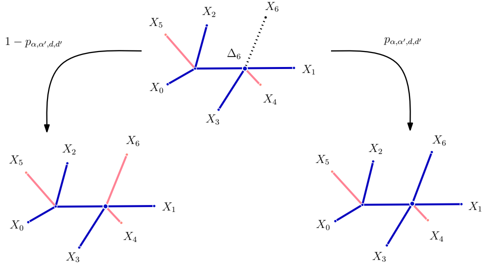
- recursively, $\tau_{n+1} = \tau_n \cup [[\Delta_{n+1}, X_{n+1}]]$ with proba. $p_{\alpha,\alpha',d,d'}$
  $\tau_{n+1} = \tau_n$ otherwise

  where: $\quad d$: degree of Δ_{n+1} in $\mathfrak{t}_n$ $\quad\quad d'$: degree of Δ_{n+1} in τ_n



with $\mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0)) := \overline{\cup_{n\geq 1}\tau_n}$

there is the following extension of the previous proposition:

Proposition (general case, Curien-H. 13)

$$\mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0)) \overset{(\mathrm{d})}{=} \frac{\alpha'}{\alpha}(\mathrm{ML}_{(1-1/\alpha)/(1-1/\alpha'),1-1/\alpha})^{1-1/\alpha'} \cdot \mathcal{T}'_\alpha$$

Proof of the theorem in the general case:

$$J_\alpha \cdot \frac{\alpha'}{\alpha}(\mathrm{ML}_{(1-1/\alpha)/(1-1/\alpha'),1-1/\alpha})^{1-1/\alpha'} \overset{(\mathrm{d})}{=} J_{\alpha'}$$

Hence:

$$J_\alpha \cdot \mathcal{T}_\alpha \supset J_\alpha \cdot \mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha, (X_i, i \geq 0)) \overset{(\mathrm{d})}{=} J_{\alpha'} \cdot \mathcal{T}_{\alpha'}.$$

□

Proof of the proposition: coloring Marchal's algorithm

It remains to prove the proposition. In that aim, we color the edges of $T_\alpha(n)$ recursively as follows:

Rules: blue/red coloring

- $T_\alpha(1)$ is blue
- an edge-leaf branched on a blue edge is blue
- an edge-leaf branched on a node is blue with proba. $p_{\alpha,\alpha',d,d'}$ and red otherwise
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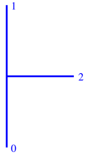

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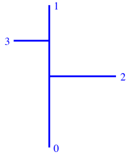

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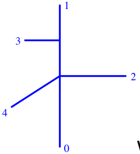

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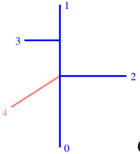

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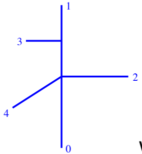
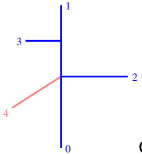


with proba. $p_{\alpha,\alpha',3,3}$



otherwise.

Ex.: when $\alpha' = 2$, the blue subtree is binary and follows Rémy's algorithm!

Proof of the proposition: coloring Marchal's algorithm

L_n: number of blue leaves in $T_\alpha(n)$

Lemma

$$\big(\text{blue subtree of } T_\alpha(n), n \geq 1\big) \stackrel{(\mathrm{d})}{=} \big(T_{\alpha'}(L_n), n \geq 1\big),$$

with $T_{\alpha'}$ independent of $(L_n, n \geq 1)$ in the right-hand side.

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Lemma

$$\frac{L_n}{n^{(1-1/\alpha)/(1-1/\alpha')}} \to \mathrm{ML}_{(1-1/\alpha)/(1-1/\alpha'),1-1/\alpha} \quad \text{almost surely.}$$

Proof. An urns argument: $(L_{n+1} - 1, n \geq 1)$ Markov chain distributed as the number of tables in a Chinese Restaurant Process with parameters $((1 - 1/\alpha)/(1 - 1/\alpha'), 1 - 1/\alpha)$ – see Pitman 06 (or Janson 05).

Remark: $\alpha < \alpha' \Rightarrow L_n \ll n$.

Proof of the proposition: coloring Marchal's algorithm

Reminder;

Proposition

$$\mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0)) \overset{(d)}{=} \frac{\alpha'}{\alpha} (\mathrm{ML}_{(1-1/\alpha)/(1-1/\alpha'),1-1/\alpha})^{1-1/\alpha'} \cdot \mathcal{T}'_\alpha$$

Proof:

- Th. Scaling limit $\Rightarrow \dfrac{T_\alpha(n)}{n^{1-1/\alpha}} \to \alpha \mathcal{T}_\alpha$ a.s.
- Th. Scaling limit + First lemma + Second lemma $\Rightarrow$ a.s.,

$$\begin{aligned} \frac{\text{blue subtree of } T_\alpha(n)}{n^{1-1/\alpha}} &= \frac{\text{blue subtree of } T_\alpha(n)}{L_n^{1-1/\alpha'}} \times \frac{L_n^{1-1/\alpha'}}{n^{1-1/\alpha}} \\ &\to \alpha' \mathcal{T}_{\alpha'} \cdot \left(\mathrm{ML}_{(1-1/\alpha)/(1-1/\alpha'),1-1/\alpha}\right)^{1-1/\alpha'} \end{aligned}$$

  for some version of $\mathcal{T}_{\alpha'}$ independent of the Mittag-Leffler r.v.

- Th. Scaling limit $\Rightarrow \dfrac{\text{blue subtree of } T_\alpha(n)}{n^{1-1/\alpha}} \to \mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0))$ a.s.

Mass of the blue subtree

- μ_n : uniform proba measure on the leaves of $T_\alpha(n)$

  since
  $$\mu_n(\text{blue subtree of } T_\alpha(n)) = \frac{L_n}{n} \to 0 \text{ a.s.} \quad \text{(by the second lemma)}$$

  it is intuitively clear that

  $$\mu_\alpha(\mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0))) = 0 \text{ a.s.}$$

  (but this is not a proof)

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  (but this is not a proof)

- $\mathrm{Prun}_{\alpha,\alpha'}(\mathcal{T}_\alpha; (X_i, i \geq 0))$: α'-stable tree (up to scaling)

  where is its natural uniform mass ?

  ► possible to recover it via fragmentation theory and martingales “Malthusian" techniques (see Section 4.3 in Curien-H. 13)

Embedded k-ary trees

As for the stable trees, the limiting trees $(\mathcal{T}_k)$ arising as limits of k-ary trees (see Paragraph 2) can be nested:

Theorem (H.-Stephenson 14+)

Let $2 \leq k' < k$. Then there is a subtree of $\mathcal{T}_k$ distributed as

$$\mathrm{ML}_{k'/k,1/k}^{1/k'} \cdot \mathcal{T}_{k'}$$

where $\mathrm{ML}_{k'/k,1/k}$ has a generalized Mittag-Leffler distribution with parameters $(k'/k, 1/k)$ and is independent of $\mathcal{T}_{k'}$.

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