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Censored α-stable processes

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Outline

Problem statement

Definition: α-stable

A Lévy process $(X, \mathrm{P}_x)_{x\in\mathbb{R}}$ is called α-stable if it satisfies the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \overset{d}{=} X|_{\mathrm{P}_{cx}}, \quad c > 0$$

The problem

Let

$$\tau_{-1}^{1} = \inf\{t > 0 : X_t \in (-1, 1)\}$$

be the first hitting time of $(-1, 1)$.

What is $\mathrm{P}_x(X_{\tau_{-1}^{1}} \in \mathrm{d}y,\ \tau_{-1}^{1} < \infty)$?

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Problem: history

- Blumenthal, Getoor, Ray (1961): symmetric, d-dimensional
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Theorem (B-G-R)

Let $x > 1$. Then, when $\alpha \in (0, 1]$,

$$\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y,\ \tau_{-1}^1 < \infty)/\mathrm{d}y = \frac{\sin(\pi\alpha/2)}{\pi}(x^2 - 1)^{\alpha/2}(1 - y^2)^{-\alpha/2}(x - y)^{-1}$$

for $y \in (-1, 1)$.

Problem: history

- Blumenthal, Getoor, Ray (1961): symmetric, d-dimensional
- Port (1967): one-sided jumps

Theorem (B-G-R)

Let $x > 1$. Then, when $\alpha \in (1,2)$,

$$\begin{aligned} &\mathrm{P}_x(X_{\tau^1_{-1}} \in \mathrm{d}y)/\mathrm{d}y \\ &\quad = \frac{\sin(\pi\alpha/2)}{\pi}(x^2-1)^{\alpha/2}(1-y^2)^{-\alpha/2}(x-y)^{-1} \\ &\quad\quad - (\alpha-1)\frac{\sin(\pi\alpha/2)}{\pi}(1-y^2)^{-\alpha/2}\int_1^x (t^2-1)^{\alpha/2-1}\,\mathrm{d}t, \end{aligned}$$

for $y \in (-1,1)$.

Outline

1. Problem
2. Tools
   - pssMps and the Lamperti transform
   - α-stable processes
3. The censored process
   - Construction
   - Lamperti transform
   - Wiener-Hopf factorisation
4. Results

Positive, self-similar Markov processes

α-pssMp

$[0, \infty)$-valued Markov process,
equipped with initial measures P_x, $x > 0$,
with 0 an absorbing state,
satisfying the scaling property

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Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ α-pssMp $\leftrightarrow$ $(\xi, \mathbb{P}_y)_{y \in \mathbb{R}}$ killed Lévy

$X_t = \exp(\xi_{S(t)})$, $\xi_s = \log(X_{T(s)})$,

S a random time-change T a random time-change

Stable processes

Definition

A Lévy process X is called α-stable if it satisfies the scaling property

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Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \rightarrow$ BM, exclude this.]
The quantity $\rho = \mathrm{P}_0(X_t \geq 0)$ is well defined.

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We make two assumptions:

- X does not have one-sided jumps,
- When $\alpha = 1$, X is symmetric.

Outline

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- Let $A_t = \int_0^t \mathbb{1}_{(X_t > 0)} \, \mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Y}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Y_t = \check{Y}_t \mathbb{1}_{(t < T_0)}$.
  This is the censored stable process.

The Lamperti transform and its structure

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Censoring preserves self-similarity: Y is a pssMp.
Let ξ be the Lamperti transform of Y.

Wiener-Hopf factorisation

Recall: Wiener-Hopf factorisation (probabilistic part)

Any Lévy process ξ is associated with two increasing, possibly killed Lévy processes, H and $\hat{H}$.

H is a time-change of $\bar{\xi}$, the running maximum.

$-\hat{H}$ is a time-change of $\underline{\xi}$, the running minimum.

H and $\hat{H}$ can be identified by looking at the characteristic function of ξ_1.

Wiener-Hopf factorisation for ξ: $\alpha \in (0, 1]$

WHF for $\alpha \in (0, 1]$

- H: Lamperti-stable subordinator with parameters $(\alpha\rho, 1)$,
- $\hat{H}$: Lamperti-stable subordinator with parameters $(\alpha\hat{\rho}, \alpha)$.

Lamperti-stable subordinators are nice! We can calculate:

- The Lévy measure of ξ,
- The Lévy measures of H and $\hat{H}$,
- The renewal measures, $\mathbb{E} \int_0^\infty \mathbb{1}_{(H_t \in \cdot)} \, \mathrm{d}t$ and $\mathbb{E} \int_0^\infty \mathbb{1}_{(\hat{H}_t \in \cdot)} \, \mathrm{d}t$.

Wiener-Hopf factorisation for ξ: $\alpha \in (1,2)$

WHF for $\alpha \in (1,2)$

- H: Conjugate of $\mathcal{T}_{\alpha-1}$-transform of LSS$(1-\alpha\rho, \alpha\hat{\rho})$.
- $\hat{H}$: Conjugate of LSS$(1-\alpha\hat{\rho}, \alpha\rho)$.

Not as nice, but we can still calculate Lévy measures and renewal measures.

Outline

Results

Recall: the problem

Let X be a stable process and $x > 1$.

$$\mathrm{P}_x\left(X_{\tau_{-1}^1} \in \mathrm{d}y,\ \tau_{-1}^1 < \infty\right) = \text{what?}$$

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By shifting and scaling, reduce this to finding

$$\mathrm{P}_1\left(X_{\tau_0^b} \in \mathrm{d}z, \tau_0^b < \infty\right), \qquad 0 < b < 1.$$

Results

We are looking for
$$P_1\left(X_{\tau_0^b} \in dz, \tau_0^b < \infty\right) = P_1\left(Y_{\tau_0^b} \in dz, \tau_0^b < \infty\right).$$

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Recall: Lamperti transform

$$Y_t = \exp(\xi_{S(t)}), \quad \text{and} \quad \xi_s = \log Y_{T(s)},$$

where S, T are random, mutually inverse time-changes.

Key fact: if $a = \log b$, then $(0, b)$ corresponds to $(-\infty, a)$ and τ_0^b corresponds to $S_a^- = \inf\{s > 0 : \xi_s < a\}$. Then,

$$Y_{\tau_0^b} = \exp\big(\xi_{S_a^-}\big).$$

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$$Y_{\tau_0^b} = \exp\big(\xi_{S_a^-}\big).$$

So now we are looking for $\mathbb{P}\big(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty\big)$, for $a < 0$.

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Method for $\alpha \in (0, 1]$

Use the ladder process:

$$\begin{aligned}\mathbb{P}(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty) &= \mathbb{P}(\underline{\xi}_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty)\\ &= \mathbb{P}(-\hat{H}_{S_{-a}^+} \in \mathrm{d}w)\\ &= \int_{[0,-a]} \hat{U}(\mathrm{d}z)\Pi_{\hat{H}}(-\mathrm{d}w - z),\end{aligned}$$

recalling that $-\hat{H}$ is a time-change of the running minimum $\underline{\xi}$.

Results

Now we are looking for $\mathbb{P}(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty)$, for $a < 0$.

Method for $\alpha \in (1,2)$

Use the Pecherskii-Rogozin identity:

$$\int_0^\infty \int \exp(qa - \beta(a - \xi_{S_a^-}))\,\mathrm{d}\mathbb{P}\,\mathrm{d}a = \frac{\hat{\kappa}(q) - \hat{\kappa}(\beta)}{(q-\beta)\hat{\kappa}(q)},$$

for $a < 0$, $q, \beta > 0$, where $e^{-\hat{\kappa}(\lambda)} = \mathbb{E}\left(e^{-\lambda \hat{H}_1}\right)$.

The theorem

Theorem

Let $x > 1$. Then, when $\alpha \in (0,1]$,

$$\mathrm{P}_x(X_{\tau_{-1}^{1}} \in \mathrm{d}y,\ \tau_{-1}^{1} < \infty)/\mathrm{d}y$$
$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1},$$

for $y \in (-1,1)$.

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Let $x > 1$. Then, when $\alpha \in (1, 2)$,

$$\begin{aligned} &\mathrm{P}_x(X_{\tau^1_{-1}} \in \mathrm{d}y)/\mathrm{d}y \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1} \\ &\qquad - (\alpha-1)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \\ &\qquad\qquad \times \int_1^x (t-1)^{\alpha\hat{\rho}-1}(t+1)^{\alpha\rho-1}\,\mathrm{d}t, \end{aligned}$$

for $y \in (-1, 1)$.

Thank you!