Analytic and probabilistic fractional diffusion limit for the Boltzmann equation

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A model between two scales for collisional particles

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Microscopic

Newton's laws

Macroscopic

Hydrodynamical equations

A model between two scales for collisional particles

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Newton's laws	**Boltzmann equation**	Hydrodynamical equations

$$\underbrace{\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v)}_{\text{transport}} = \underbrace{Q(f,f)}_{\text{collisions}}$$

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Partial Differential Equations | Probability

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Partial Differential Equations

Probability

Density function $\quad f(t,x,v)dxdv$

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Baby model: **linear** Boltzmann equation $[0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \underbrace{F(v) \int f(t,x,v')\,dv'}_{\textit{part. changing from } v' \textit{ to } v} - \overbrace{f(t,x,v)}^{\textit{part. no longer at vel. } v}$$

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$$F \sim \frac{c_0}{\|v\|^{N+\alpha}} \quad \text{for } \|v\| \text{ large, } \alpha \in (0,2)$$

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- Fractional diffusion operator theory

$$\boxed{\partial_t \rho(t,x) + \kappa(-\Delta)^{\alpha/2}\rho(t,x) = 0}$$

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'Fractional diffusion limit for collisional kinetic equations"
Mellet, Mischler, Mouhot

'Limit theorems for additive functionals of a markov chain'
Jara, Komorowski, Olla

Tools and a probabilistic model

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Fractional Laplace operator:

$$(-\Delta)^{\alpha/2}\rho :=$$

$$\mathcal{F}^{-1}\left(|k|^{\alpha}(\mathcal{F}(\rho)(k)\right)$$

α-**stable process** Z_t: Lévy process with α-**scaling property**

$$k Z_{k^{-\alpha}t} \overset{f.d}{\sim} Z_t \quad k > 0$$

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Probabilistic model

Single particle dynamics

$$(X_t, V_t) \in \mathbb{R}^N \times \mathbb{R}^N$$

$(X_0, V_0) \sim f(0, x, v)$
$(T_i)_{i\geq 0} \sim E(1)$ indep. (jump times)
$(U_i)_{i\geq 0} \sim F$ indep. (velocity after i-jump)
$\Longrightarrow$ It is a Markov process.

Single particle dynamics

$$(X_t, V_t) \sim f(t, x, v)\,.$$

f **is solution to LBE !**

Y_t: position at the jumps in velocity.

Y_t is a Lévy process.

Rescale space and time
$(\epsilon x,\ \epsilon^{\alpha} t)$

$$\epsilon^{\alpha}\partial_t f^{\epsilon}+\epsilon v\cdot\nabla_x f^{\epsilon}=F\int f^{\epsilon}dv'-f^{\epsilon}$$

$$f(0,x,v)=f^{in}(x,v)$$

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$$f^\epsilon \underset{\epsilon \to 0}{\longrightarrow} \rho F = \rho(t,x) F(v)$$

satisfies the

fractional diffusion equation

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$$Z^\epsilon := \epsilon Y_{\epsilon^{-\alpha}t}$$

converges weakly to an

α-stable Lévy process.

Characteristic exponent:

$$\psi^\alpha(\theta) = C(\alpha)\int_S\int_0^\infty \frac{e^{i\theta^T\xi r}-1}{r^{1+\alpha}}drd\xi$$

THANK YOU FOR YOUR ATTENTION!!!!