Branching Brownian motion in a strip: Survival near criticality

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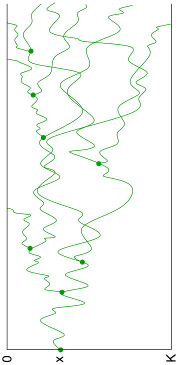

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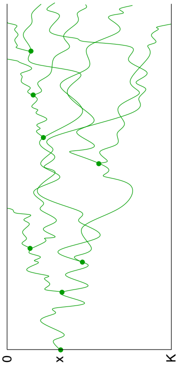

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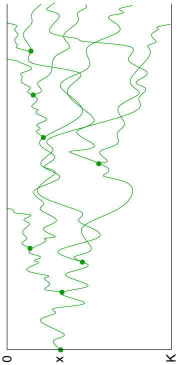
- particles perform Brownian motion with drift $-\mu$ and are killed on hitting 0 or K, i.e. according to diffusion operator $L = \frac{1}{2}\frac{d}{dx^2} - \mu\frac{d}{dx}$ for $\{u \in C(0, K) : u(0+) = u(K-) = 0\}$



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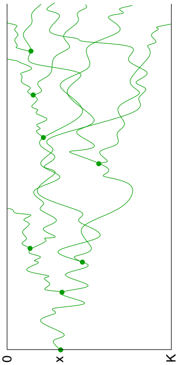

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- they undergo dyadic branching at rate β,
- offspring particles move off independently from their birth position and repeat their parent's stochastic behaviour.

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Let $x \in (0, K)$ and $\lambda_c = \beta - \mu^2/2 - \pi^2/2K^2$.
(i) If $\lambda_c > 0$, then $P_x^K(\zeta^K = \infty) > 0$.
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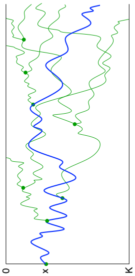
So we have positive survival probability if and only if $K > K_0 = \frac{\pi}{\sqrt{2\beta - \mu^2}}$.

Critical width

Conditioning on survival using the martingale

$$Z(t) = \sum_{u \in N_t} e^{\mu X_u(t) - (\beta - \mu^2/2 - \pi^2/2K^2)t} \sin(\pi X_u(t)/K), \quad t \geq 0,$$



induces a spine decomposition

- run a Brownian motion conditioned to stay in (0,K) - the spine
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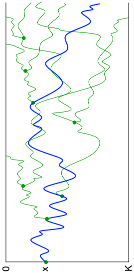

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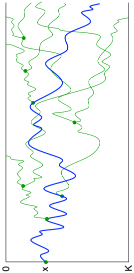

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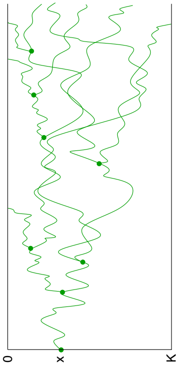
$\rightarrow$ If $K = K_0$ then $\{\text{extinction}\} = \{Z(\infty) = 0\}$ and we can't use the conditioning to get a spine decomposition !

Blue and red tree



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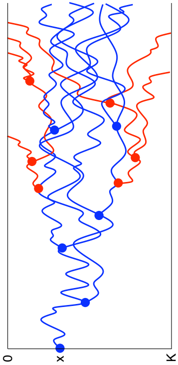

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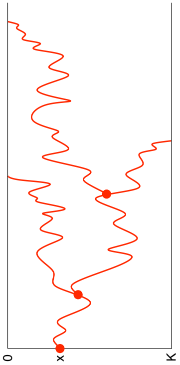

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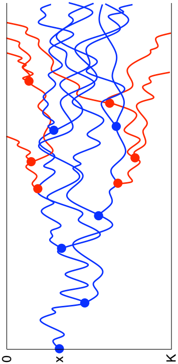

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- motion $L^B = \frac{1}{2}\frac{d}{dx^2} - (\mu + \frac{w'}{1-w})\frac{d}{dx}$ (h-transform $h = 1 - w$)

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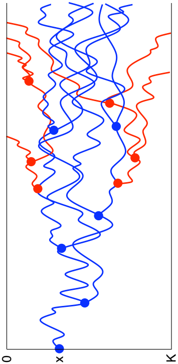

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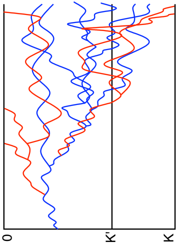
$\rightarrow P_x^K(\cdot|\zeta^K = \infty)$ has the same law as observing a dressed blue tree starting from x.

What happens to dressed blue tree as $K \downarrow K_0$?



Recall: Blue tree

- branching rate $\beta(1-w)$
- diffusion operator $L^B = \frac{1}{2}\frac{d}{dx^2} - (\mu + \frac{w'}{1-w})\frac{d}{dx}$

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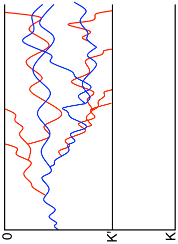

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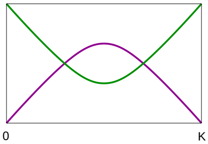

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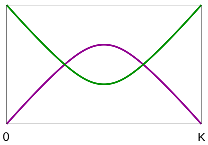

Theorem

Let $x \in (0, K_0)$. As $K \downarrow K_0$,

$$1 - w_K(x) \sim c_K \sin(\pi x / K_0) e^{\mu x},$$

where $c_K \sim \lambda_c(K) \frac{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}{12\beta\pi^3(e^{\mu K_0}+1)}$.

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Theorem

Let $x \in (0, K_0)$. Then $\lim_{K \downarrow K_0} P_x^K(\cdot | \zeta^K = \infty) = \mathbb{Q}_x^{K_0}(\cdot)$, where $\mathbb{Q}_x^{K_0}$ is the law of a particle system which consist of

- *a spine performing BM conditioned to stay in $(0, K_0)$,*
- *immigration of P^{K_0}-BBM at rate 2β.*