Variance–Gamma distributions and (other) functionals of Normals and their approximation

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The class of Variance-Gamma distributions

The Variance-Gamma distributions are a quite general class of distributions that were introduced into the financial literature by Madan and Seneta (1990). Their semi–heavy tails make them appropriate for applications in finance. They form a four parameter family with parameters $r > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\mu \in \mathbb{R}$ and p.d.f.

$$\begin{aligned} p(x; r, \theta, \sigma, \mu) &= \frac{1}{\sigma\sqrt{\pi}\Gamma(r/2)}\left(\frac{|x-\mu|}{2\sqrt{\theta^2+\sigma^2}}\right)^{\frac{r-1}{2}} e^{\frac{\theta}{\sigma^2}(x-\mu)} \\ &\quad K_{\frac{r-1}{2}}\left(\frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2}|x-\mu|\right), \quad x \in \mathbb{R}, \qquad (1) \end{aligned}$$

where $K_\nu(\cdot)$ is the modified Bessel function of the second kind of order $\nu \in \mathbb{R}$. If a random variable X has density (1) then we write $X \sim VG(r, \theta, \sigma, \mu)$.

Basic Properties of Variance-Gamma distributions

Variance-Gamma random variables have the following characterisation in terms of Normal and Gamma random variables:

Suppose that $S_1, S_2, \ldots, S_r$ and $T_1, T_2, \ldots, T_r$ be a sequence of independent $N(0,1)$ distributed random variables, then

$$\mu + \theta \sum_{i=1}^{r} S_i^2 + \sigma \sum_{i=1}^{r} S_i T_i \sim VG(r/2, \theta, \sigma, \mu). \tag{2}$$

Suppose now that V_1, V_2 are independent random variables, distributed as $V_1 \sim \Gamma(r, \lambda_1)$ and $V_2 \sim \Gamma(r, \lambda_2)$, then

$$V_1 - V_2 \sim VG(r/2, \lambda_1^{-1} - \lambda_2^{-1}, 2(\lambda_1 \lambda_2)^{-1/2}, 0). \tag{3}$$

Basic Properties of Variance-Gamma distributions

The class of Variance-Gamma distributions includes many well known distributions:

$$\begin{aligned} VG(r/2, \lambda^{-1}, 0, 0) &= \Gamma(r, \lambda); \\ VG(1, 0, \sigma, \mu) &= Laplace(\mu, \sigma); \\ \lim_{r \to \infty} VG(r/2, 0, \sigma r^{-1/2}, \mu) &= N(\mu, \sigma^2). \end{aligned}$$

These results are easily obtained by using either the characterisations given in (2) and (3) or the asymptotic properties of modified Bessel functions of the second kind.

Suppose $X_{1,1},...,X_{n,r}$, $Y_{1,1},...,Y_{m,r}$ is a sequence of i.i.d. random variables with zero mean and unit variance. Then the statistics

$$U_r = \frac{1}{\sqrt{mn}} \sum_{i,j,k=1}^{n,m,r} X_{ik} Y_{jk} \quad (4)$$

$$V_r = \frac{1}{n} \sum_{k=1}^{r} \left(\sum_{i=1}^{n} X_{ik} \right)^2 \quad (5)$$

are asymptotically $VG(r,0,1,0)$ and $VG(r,2,0,0)$ distributed respectively, by a simple application of the central limit theorem and by the characterisation (2). We now see how Stein's method can be used to produce a bound on the error in approximating U_r and V_r by their asymptotic distributions.

What is Stein's Method?

Stein's Method is a powerful technique which allows the distance between two probability distributions to be bounded with respect to a probability metric. This allows us to obtain bounds on the rate of convergence in important situations.

Stein's Method has been used to obtain approximation results for many standard probability distributions, including the Normal, Poisson and Gamma distributions.

A major strength of the method is that it has a relatively simple application under various dependence structures.

A measure of distance

Let X and Y be random variables. A distance between the distributions of X and Y can be given by

$$d(X,Y) = \sup_{h \in \mathcal{H}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|$$

For some class of functions $\mathcal{H}$.

If $d(X_n, X) \to 0$, as $n \to \infty$, for all h from a sufficiently large class of functions $\mathcal{H}$ then $X_n \xrightarrow{\mathcal{D}} X$, as $n \to \infty$.

An example

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with zero mean, unit variance and $\mathbb{E}|X^3| < \infty$. Let $Z \sim N(0,1)$.
By the central limit theorem, we have

$$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} Z, \quad as \quad n \to \infty.$$

By Stein's method, for any smooth h,

$$|\mathbb{E}h(W_n) - Nh| \leq \frac{1}{\sqrt{n}} ||h'||_\infty \left(2 + \mathbb{E}|X^3|\right),$$

where Nh denotes $\mathbb{E}h(X)$ for $X \sim N(0,1)$.

Stein's Method for Variance-Gamma distributions

The study of Variance-Gamma approximations via Stein's method rests on the following lemma which characterises Variance-Gamma distributions.

Lemma 1. $Z \sim VG(r, \theta, \sigma, \mu)$, if and only if

$$\mathbb{E}\{\sigma^2(Z-\mu)f''(Z)+(r\sigma^2+2\theta(Z-\mu))f'(Z)+(r\theta-(Z-\mu))f(Z)\} = 0 \quad (6)$$

for all piecewise twice continuously differentiable f such that $\mathbb{E}|Zf''(Z)|$, $\mathbb{E}|f'(Z)|$, $\mathbb{E}|Zf'(Z)|$, $\mathbb{E}|f(Z)|$ and $\mathbb{E}|Zf(Z)|$ are finite.

This gives rise to the following ODE, known as the Stein equation:

$$\sigma^2(x-\mu)f''(x)+(\sigma^2r+2\theta(x-\mu))f'(x)+(r\theta-(x-\mu))f(x) \\ = h(x)-VG_{\sigma,\mu}^{r,\theta}h, \quad (7)$$

where $h:\mathbb{R}\rightarrow\mathbb{R}$ is a test function, and $VG_{\sigma,\mu}^{r,\theta}h$ denotes $\mathbb{E}h(X)$ for $X\sim VG(r,\theta,\sigma,\mu)$. Taking expectations and evaluating both sides at any random variable W gives

$$\begin{aligned} & \mathbb{E}h(W)-VG_{\sigma,\mu}^{r,\theta}h \\ = \; & \mathbb{E}\{\sigma^2(W-\mu)f''(W)+(r\sigma^2+2\theta(W-\mu))f'(W) \\ & +(r\theta-(W-\mu))f(W)\}. \end{aligned} \quad (8)$$

Note that (7) reduces, in the appropriate limits, to the Stein equations for Normal and Gamma distributions that were given in Stein (1972) and Luk (1994).

Our aim is the following: for a given h, bound

$$\mathbb{E}h(W) - VG^{r,\theta}_{\sigma,\mu}h.$$

We have now reduced this to: for a given h, solve the Stein equation 7 for f and bound

$$\mathbb{E}\{\sigma^2(W-\mu)f''(W)+(r\sigma^2+2\theta(W-\mu))f'(W)+(r\theta-(W-\mu))f(W)\} \tag{9}$$

We will bound (2) using a Taylor expansion about a random variable coupled with W. Therefore the bound will involve derivatives of f. These derivatives can be bounded as follows.

Lemma 2. The Stein equation (7) has a unique bounded solution and it's derivatives up to fourth order can be bounded in terms of the derivatives of the test function h as follows

$$\|f^{(n)}\|_\infty \leq C_{0,n}\|h - VG^{r,\theta}_{\sigma,\mu}h\|_\infty + \sum_{i=1}^{n-1} C_{i,n}\|h^{(i)}\|_\infty,$$

where $n \in \{0, 1, \ldots, 4\}$ and the $C_{i,n}$ are $O(r^{-1/2})$ constants.

Having solved the Stein equation for f and bounded its derivatives in terms of the derivatives of h we are now in a position to obtain approximation results for Variance Gamma distributions.

We can bound the distance between the distribution of a random variable W and a $VG(r, \theta, \sigma, \mu)$ distribution by bounding (2). This is done by a Taylor expansion about a random variable coupled with W.

Here is an example:

Theorem 3. Let $U_r = \frac{1}{\sqrt{mn}} \sum_{i,j,k=1}^{n,m,r} X_{ik} Y_{jk}$. Then for $h \in C_b^3(\mathbb{R})$ we have

$$|\mathbb{E}h(U_r) - VG_{1,0}^{r,0}h| \leq A_0 \|h - VG_{1,0}^{r,0}h\|_\infty + \sum_{i=1}^{3} A_i \|h^{(i)}\|_\infty,$$

where the A_i are expressions involving the moments, up to eight order, of the X_{ik} and Y_{jk}, and are of order $O(r^{1/2}(m^{-1} + n^{-1} + (mn)^{-1/2}))$.

Normal Approximation

Taking $\mu = 0$, $\sigma = r^{-1/2}$, $\theta = 0$ and letting $r \to \infty$ in Lemma 1 we obtain the following result of Stein (1972):

(Stein's Lemma) $Z \sim N(0,1)$, if and only if

$$\mathbb{E}\{f'(Z) - Zf(Z)\} = 0$$

for all piecewise differentiable f such that $\mathbb{E}f'(Z) < \infty$.

This gives rise to the Stein equation:

$$f'(x) - xf(x) = h(x) - Nh. \qquad (10)$$

Given a test function h, we can obtain the unique bounded solution to (10), which is given by

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} (h(t) - Nh) e^{-t^2/2} dt.$$

It is straightforward to bound f and its derivatives in terms of the test function and its derivatives, e.g. $\|f''\|_\infty \leq 2\|h'\|_\infty$.

Normal Approximation

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Put $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ and $W_i = W - \frac{X_i}{\sqrt{n}}$. Then

$$\begin{aligned}
\mathbb{E}Wf(W) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}X_i f(W) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}X_i f(W_i) + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i^2 f'(W_i) + R_1 \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}f'(W_i) + R_1 \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}f'(W) + R_1 + R_2 \\
&= \mathbb{E}f'(W) + R_1 + R_2.
\end{aligned}$$

Therefore

$$
\begin{aligned}
|\mathbb{E}h(W) - Nh| &= |\mathbb{E}f'(W) - Wf(W)| \\
&\leq |R_1| + |R_2| \\
&\leq \frac{1}{\sqrt{n}} ||h'||_\infty \left(2 + \mathbb{E}|X^3|\right).
\end{aligned}
$$

A more general problem

Let $X_k, X_{1k}, X_{2k}, \ldots, X_{nk}$, for $1 \leq k \leq d$ be a sequence of i.i.d. random variables with $\mathbb{E}X_k = 0$ and $\mathrm{Var}X_k = 1$. Define $W_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ik}$, and let $Z_k \sim N(0,1)$, for $1 \leq k \leq d$.

Then, we would like to obtain explicit bounds on the error in approximating $g(W_1, \ldots, W_d)$ by $g(Z_1, \ldots, Z_d)$, where $g : \mathbb{R}^d \to \mathbb{R}$ is continuous.

In particular, we are interested in determining when the convergence will be of order $n^{-1/2}$ and when it will be of the improved n^{-1} rate.

A general $n^{-1/2}$ type limit theorem

Theorem 4. Suppose that $g : \mathbb{R} \to \mathbb{R}$ and that its first derivative has polynomial growth rate, with $|g'(w)| \leq A + B|w|^k$, where A, B and k are non-negative constants. Let $X, X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$, $\mathrm{Var}X = 1$ and $\mathbb{E}|X|^{k+4} < \infty$. Suppose that $h \in C_b^1(\mathbb{R})$, then there exists an absolute constant C, independent of n, such that

$$|\mathbb{E}h(g(W)) - \mathbb{E}h(g(Z))| \leq \frac{C}{\sqrt{n}}\mathbb{E}|X|^{k+4}\|h'\|.$$

Theorem 5. Suppose that $g:\mathbb{R}\rightarrow\mathbb{R}$ is a symmetric function, and that $|g'(w)|\leq A_1+B_1|w|^{k_1}$ and $|g''(w)|\leq A_2+B_2|w|^{k_2}$, where A_j, B_j and k_j, for $j=1,2$, are non-negative constants. Let $X, X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}X=0$, $\mathrm{Var}X=1$, $\mathbb{E}|X|^{2k_1+5}<\infty$ and $\mathbb{E}|X|^{k_2+5}<\infty$. Suppose that $h\in C_b^2(\mathbb{R})$, then there exists an absolute constant C, independent of n, such that

$$|\mathbb{E}h(g(W))-\mathbb{E}h(g(Z))|\leq\frac{C}{n}(\mathbb{E}|X|^{2k_1+5}+\mathbb{E}|X|^{k_2+5})(\|h'\|_\infty+\|h''\|_\infty).$$

Corollary 6. Let the the X_i and $g : \mathbb{R} \to \mathbb{R}$ be defined as in Theorem 5, but suppose we drop the assumption that g is symmetric and replace it with the condition that $\mathbb{E}X^3 = 0$. Then, for $h \in C_b^2(\mathbb{R})$, there exists an absolute constant C such that

$$|\mathbb{E}h(g(W)) - \mathbb{E}h(g(Z))| \leq \frac{C}{n}(\mathbb{E}|X|^{2k_1+5} + \mathbb{E}|X|^{k_2+5})(\|h'\|_\infty + \|h''\|_\infty).$$

Sketch of the proof of Theorem 5

Provided that various expectations exist, we can apply a Taylor expansion to obtain

$$\begin{aligned}\mathbb{E}h(g(W)) - \mathbb{E}h(g(Z)) &= \mathbb{E}\{f'(W) - Wf(W)\} \\ &= \frac{1}{2\sqrt{n}}\mathbb{E}X^3\mathbb{E}f''(W) + O(n^{-1}),\end{aligned}$$

where f is the solution to the Stein equation:

$$f'(x) - xf(x) = h(g(x)) - \mathbb{E}h(g(Z)). \tag{11}$$

It therefore follows that the convergence will be of order n^{-1} if $\mathbb{E}X^3 = 0$ or $\mathbb{E}f''(W) = O(n^{-1/2})$.

Sketch of the proof Theorem 5

If it was true that $\mathbb{E}f''(Z) = 0$ then it would follow that $\mathbb{E}f''(W) = O(n^{-1/2})$. This is because $\mathbb{E}f''(Z) = 0$ would mean that

$$\mathbb{E}f''(W) = \mathbb{E}\psi'(W) - \mathbb{E}W\psi(W), \tag{12}$$

where ψ is the solution to the ODE $\psi'(x) - x\psi(x) = f''(x)$. We could then use Stein's method for Normal approximation to bound the RHS of (12) to order $n^{-1/2}$, which would mean that $\mathbb{E}f''(W) = O(n^{-1/2})$, as required.

Sketch of the proof of Theorem 5

The following key lemma, together with the above argument, shows when g is a symmetric function we are able to obtain the desired $O(n^{-1})$ convergence rate.

Lemma 7. Suppose that the second derivative of the solution to (11) exists and that $g : \mathbb{R} \to \mathbb{R}$ is a symmetric function, then $\mathbb{E}f''(Z) = 0$.

Combining Lemma 7 with the above argument, and some rather lengthy calculations, gives the required bound.

Extensions to more general theorems

The above results can be generalised in a number of ways, such as the following:

- Weakening the assumptions on the growth rate of g' and g''. This requires stronger conditions on the existence of expectations involving the random variable X.
- Generalising to multivariate limit theorems, with $g : \mathbb{R}^d \to \mathbb{R}$. This requires higher order derivatives of g and the test function, h, to exist.
- Generalising to locally dependent random variables.

Application: a χ^2 limit theorem

Taking $g(x) = x^2$ in Theorem 5 gives:

Theorem 8. Let $X, X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variable with $\mathbb{E}X = 0$, $\mathrm{Var}X = 1$ and $\mathbb{E}|X|^7 < \infty$. Suppose that $h \in C_b^2(\mathbb{R})$, then there exists an absolute constant C such that

$$|\mathbb{E}h(W^2) - \chi^2_{(1)}h| \leq \frac{C\mathbb{E}|X|^7}{n}(\|h'\|_\infty + \|h''\|_\infty),$$

where $\chi^2_{(1)}h$ denotes the expectation of $h(S)$, for $S \sim \chi^2_{(1)}$.

Application: a Variance-Gamma limit theorem

We can use a multivariate generalisation of Theorem 5, taking $g(x, y) = xy$, to obtain:

Theorem 9. Let U_1 be defined as per equation (4). Suppose, that $\mathbb{E}|X|^5 < \infty$, $\mathbb{E}|Y|^5 < \infty$, and that $h \in C_b^2(\mathbb{R})$, then there exists an absolute constant C such that

$$\begin{aligned} & |\mathbb{E}h(U_1) - VG^{1,0}_{1,0}h| \\ \leq \quad & C\left(\frac{1}{n} + \frac{1}{m}\right)(\mathbb{E}|X|^5 + \mathbb{E}|Y|^5)^2(\|h''\|_\infty + \|h^{(3)}\|_\infty + \|h^{(4)}\|_\infty). \end{aligned}$$

Can we do better than $O(n^{-1})$?

Yes!

Theorem 10. Let $X, X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}X^k = \mathbb{E}Z^k$, where $Z \sim N(0,1)$, for all positive integers $k \leq p$, and that $\mathbb{E}|X|^{p+1} < \infty$. Then, for all $h \in C_b^{p+1}(\mathbb{R})$, we have

$$|\mathbb{E}h(W) - Nh| \leq \frac{M_p}{p!n^{(p-1)/2}}(p(2^{p-1}-1)\mathbb{E}|X|^{p-1} + \mathbb{E}|X|^{p+1}),$$

where

$$M_p = \min\left\{\frac{\sqrt{\pi}\Gamma(\frac{p+1}{2})}{2\Gamma(\frac{p}{2}+1)}\|h^{(p)}\|_\infty, \frac{\|h^{(p+1)}\|_\infty}{p+1}\right\}.$$

Further Work

- Obtain Variance-Gamma approximation results for non-smooth test functions.
- Extend the characterizing Lemma 1 for Variance-Gamma distributions to one for the class of Generalised Hyperbolic distributions.
- Apply these approximation results to examples, such as Freidman's χ^2 statistic and alignment–free sequence comparison.
- Determine necessary conditions for $O(n^{-1})$ convergence rates. For example, it is conjectured that it is necessary for g to be symmetric in the case of non matching moments.

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