Probabilistic Approaches to Neutron Transport Problems

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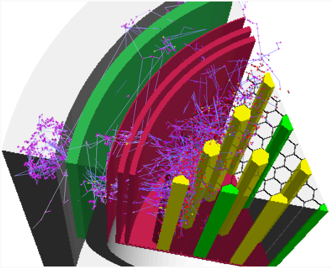
Growth & Division in Mathematics & Medicine, 4-6 November, UCL

What is this talk about

- Some probability tools for **spatial branching processes**.
- Applications to neutron transport problems.

  Based on joint work with Alexander M.G. Cox, Simon C. Harris, Emma Horton and Andreas E. Kyprianou.

Neutron flux



ⓒWood.

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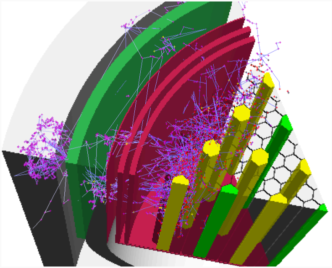
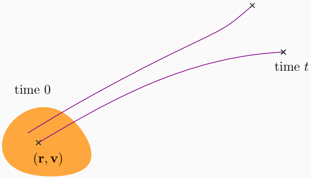


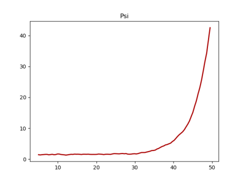
ⓒ**Wood.**

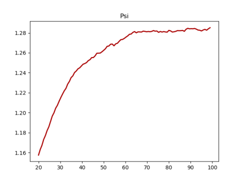
$\psi_t(\mathbf{r},\mathbf{v})$: **neutron flux** inside the nuclear reactor core at time t emitted from the configuration $(\mathbf{r},\mathbf{v})$.

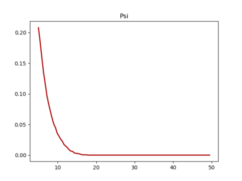


Criticality

- **Growth rate of $\psi_t(\mathbf{r}, \mathbf{v})$:**







Supercritical **Critical** **Subcritical**

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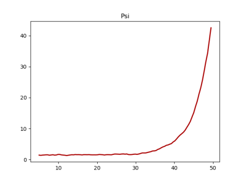

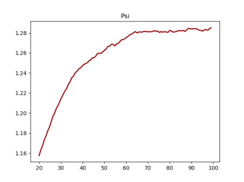

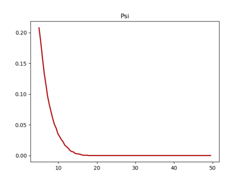

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Put another way, $\psi_t(\mathbf{r}, \mathbf{v}) \asymp e^{\lambda_0 t}$, where $\lambda_0 =$ lead eigenvalue.

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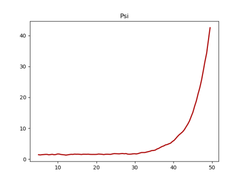

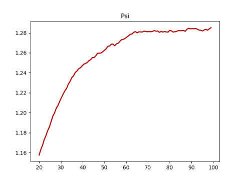

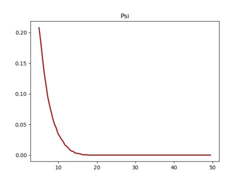

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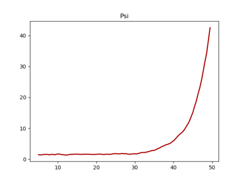

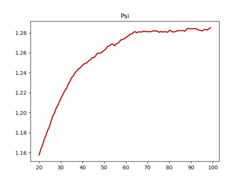

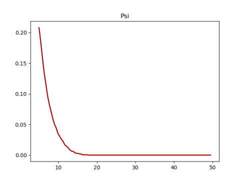

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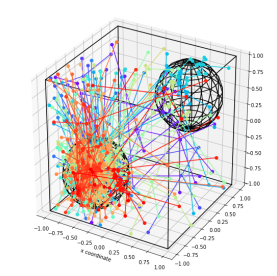
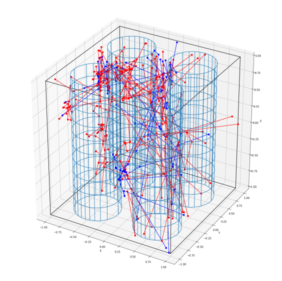
$\rightsquigarrow$ Monte-Carlo methods

Monte-Carlo methods

- A variety of Monte-Carlo methods:
  - Neutron branching process (basic)
  - Neutron random walk (many-to-one)
  - h-neutron random walk (h-transform)

- Modelling of the neutron process as a spatial branching process.





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Motion of a neutron is governed by

- **Fission**
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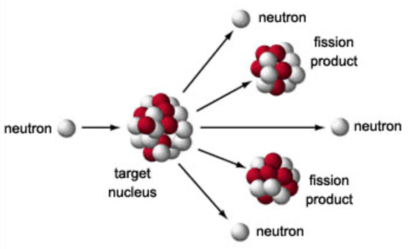
- Absorption



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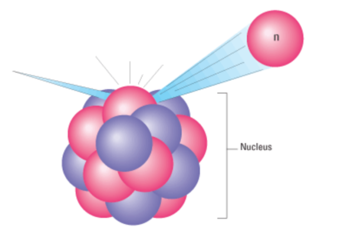

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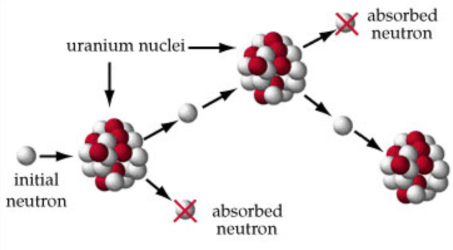

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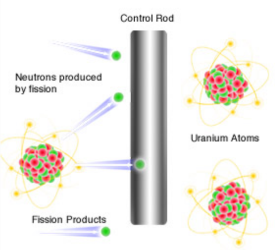

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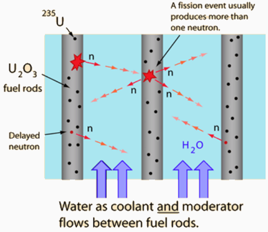

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- **branching** $\rightsquigarrow$ fission & absorption
- **spatial motion** $\rightsquigarrow$ transport, scattering & absorption

Remark. No neutron-neutron interactions.

Inhomogeneous branching and scattering rates.

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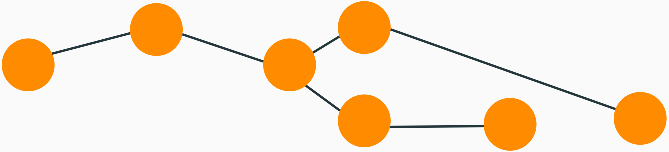

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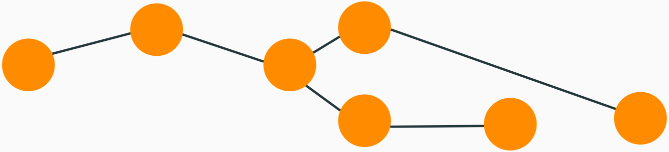

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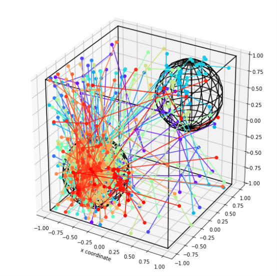
1^{st} MC method: Neutron branching process

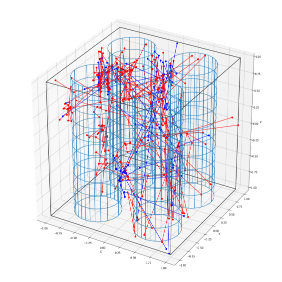
- Run a simulation of the neutron branching process starting from the configuration $(\mathbf{r}, \mathbf{v})$.

- Repeat for k times.
  N_t^i = number of the surviving neutrons at time t from the i-th simulation.

- $\psi_t(\mathbf{r}, \mathbf{v}) = \mathbb{E}[N_t^i] \approx \frac{1}{k} \sum_{i=1}^{k} N_t^i$. Then

$$\lambda_0 \approx \frac{1}{t} \log \psi_t(\mathbf{r}, \mathbf{v}) \approx \frac{1}{t} \log \left(\frac{1}{k} \sum_{i=1}^{k} N_t^i \right).$$

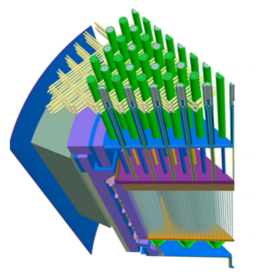


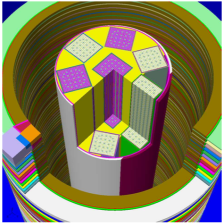


Neutron Branching Process: Analysis

- Relatively easy to implement and geometry insensitive.

- Slow convergence and costly to run.





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Towards other MC methods

- We only employ the first moment:

$$\psi_t(\mathbf{r}, \mathbf{v}) = \mathbb{E}[N_t^i].$$

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- Can we find another random variable $\tilde{N}_t$ also satisfying

$$\psi_t(\mathbf{r}, \mathbf{v}) = \mathbb{E}[\tilde{N}_t]$$

  but at the same time

  - either easier to simulate
  - or having a smaller variance

Many-to-one & neutron random walk

We can suppress branching by

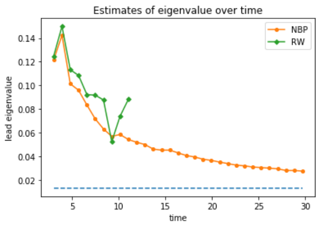
- simulating a single neutron path $(\mathbf{r}_s, \mathbf{v}_s)$
- then applying the many-to-one formula:

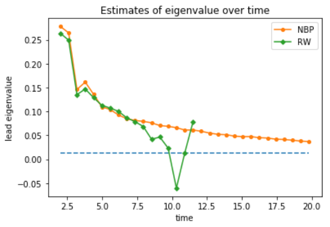
$$\psi_t(\mathbf{r}, \mathbf{v}) = \mathbb{E}\left[e^{\int_0^t \beta(\mathbf{r}_s, \mathbf{v}_s) ds}\, \mathbf{1}_{\{\text{survival at } t\}}\right]$$

where β depends only on the branching parameters.

Neutron random walk: Analysis

- Quick/cheap to run.
- The exponential weight $e^{\int_0^t \beta(\mathbf{r}_s, \mathbf{v}_s) ds}$ could increase variance.



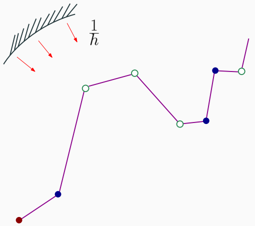


h-transform & h-neutron random walk

- Applying an h-transform to the law of the neutron path yields

$$\psi_t(\mathbf{r}, \mathbf{v}) = \mathbb{E}\left[\exp\left(\int_0^t \frac{\mathrm{L}h(\tilde{\mathbf{r}}_s, \tilde{\mathbf{v}}_s)}{h(\tilde{\mathbf{r}}_s, \tilde{\mathbf{v}}_s)} + \beta(\tilde{\mathbf{r}}_s, \tilde{\mathbf{v}}_s) ds \right) \frac{h(\mathbf{r}, \mathbf{v})}{h(\tilde{\mathbf{r}}_t, \tilde{\mathbf{v}}_t)} \mathbf{1}_{\{\text{survival at } t\}} \right]$$

where $(\tilde{\mathbf{r}}_s, \tilde{\mathbf{v}}_s)$ is a neutron path which scatters at rate $\propto \frac{1}{h}$ and where L is some differential-integral operator.



h-transform & h-neutron random walk (Cont')

- Optimal choice of h:

$$(L + \beta)h = \lambda_0 h$$

h-transform & h-neutron random walk (Cont')

- Optimal choice of h:
$$(L + \beta)h = \lambda_0 h$$
In general, such a solution is not known explicitly.

- Instead, we substitute with a guess.
    - Since the h-transform formula is valid for a general h, there is no loss in accuracy.
    - As soon as $h = 0$ at boundary, the scattering will force the neutron to stay inside.

Some questions

- What would be a practically good choice of h?
  - It seems more efficient to have $h = 0$ at boundary.
    On the other hand, this causes a divergence in the scattering rate.
    Will this be a hurdle?
  - How to update our knowledge on the eigenfunction and use it to improve the convergence?

Some questions (Cont')

- Other Monte-Carlo methods?

  Consider the following method:

  - Start with k particles.
  - Let the system evolve for some time T_0. Denote by K_0 the number of neutrons in the system. If $K_0 > k$, discard $K_0 - k$ particles. If $K_0 < k$, sample new particles from the distribution given by the current states.
  - Iterate.

  Long-time behaviour? Genealogy of the particles?

Some questions (Cont')

- Ergodicity of the neutron branching process?

  Denote by $(\mathbf{r}_s^i, \mathbf{v}_s^i), 1 \leq i \leq N_s$, the configurations of the neutrons at time s. Let $X_s = \frac{1}{N_s} \sum_i \delta_{(\mathbf{r}_s^i, \mathbf{v}_s^i)}$.

  - In the supercritical regime, Harris, Horton & Kyprianou recently showed that X_s converges to some deterministic measure. What is the speed of this convergence?
  - What about critical/subcritical regimes (conditioned upon survival)?

- Other applications for the many-to-one or h-transform methods?

  Note that there is an analogous theory for discrete-time/generation-based branching processes (ask Emma Horton!)