LTCC: Representation Theory of Finite Groups
Exercise Set 2

Exercises marked with $\star$ are especially useful for subsequent lectures.

1. In class we considered the following: Let A be the group algebra over $\mathbb{C}$ of $G = C_3 = \langle x | x^3 = e \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be an A-module with representation $x \cdot v_i = v_{i+1 \mod 3}$. This module has a submodule $U = \langle v_1 + v_2 + v_3 \rangle$. We then found a submodule W such that $V = U \oplus W$.

   Is this submodule W irreducible?

2. $\star$ Let V be an A-module and let U and W be submodules. Show that $U \cap W$ is also a submodule.

3. *Here, we show that a decomposition of a module into irreducible submodules is essentially unique.* Suppose V is an A-module and $V = \bigoplus_{i=1}^{n} U_i$, where each U_i is an irreducible submodule. Suppose W is an irreducible submodule of V. By considering the the possibilities for the intersections $W \cap U_i$, show that we must in fact have $W = U_i$ for some i.

4. $\star$ Let V, W_1, W_2 be A-modules. Show that

$$\mathrm{Hom}_A(V, W_1 \oplus W_2) \cong \mathrm{Hom}_A(V, W_1) \oplus \mathrm{Hom}_A(V, W_2)$$

   as vector spaces (and thus their dimensions are the same).

5. Let G be a group. Suppose that ρ is a representation of G on a vector space V of dimension 1. Prove that $G/\ker(\rho)$ is abelian.

6. Let $\phi : G \to GL(n, \mathbb{F})$ be a representation of G. Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

   is also a representation of G.

7. $\star$ Let $\rho : G \to GL(n, \mathbb{F})$ be a representation of G. Show that $\sigma : G/\ker(\rho) \to GL(n, \mathbb{F})$ given by $\sigma(g \ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Recall from Exercise Sheet 1, Question 6, that a faithful representation is one that is injective. Note that here you must show that it is a well-defined representation, as well as showing that it is faithful.)

8. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}[G]$-module.

9. When constucting the regular representation of G over $\mathbb{F}$, we used the action of G on itself by *left multiplication*, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.

   (a) Define the *right multiplication* action $\cdot : G \times G \to G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.

   (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.

   (c) Define the *conjugation* action $\cdot : G \times G \to G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.

   (d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)

10. $\star$ Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$-module has dimension 1. Show that G is abelian.

11. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.