Wk 1, Ex 7) Suppose V is an A-module and W a submod. Then V/W is an A-mod as well.

Well-defined: If $v+W = v'+W$, then $a\cdot(v+W) = a\cdot(v'+W)$

$$a\cdot(v+W) = (a\cdot v) + W$$

Prop 1: Let A be an alg. over $\mathbb{F}$ and let $T: V \to W$ be a hom of A-mods. Then $V/\ker(T) \cong \operatorname{im}(T)$.

Sec 2.1) Semisimple Representations

Recall: "Simple" = "irreducible"

Defn 1: A semisimple (or completely reducible (completely decomposable)) A-mod is (isomorphic to) a direct sum of irreducible submodules.

Thm 1: Let A be fin. dim. Then every irreducible A-mod is fin. dim.

Pf: Let V be an irred. A-mod. Let $v_0 \in V$ be nonzero. Then $Av_0 = \{av_0 \mid a \in A\}$ is an A-submod, and fin-dim. But since V is irred, $Av_0 = V$. □

Now specialise: Let $A = \mathbb{F}[G]$ ($\mathbb{F}$ field, G group)

If G is finite, A is fin. dim.

Thm 2: (Maschke's Theorem:) Let G be a fin. gp and let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Then every fin. dim. $\mathbb{F}[G]$-module is semisimple.

Non-example: Let $G = C_3 = \langle x \mid x^3 = e\rangle$, $\mathbb{F} = \mathbb{F}_3 \cong \mathbb{Z}_3$, $V = \mathbb{F}^2 = \left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\}$

$\rho: G \to GL(V) = GL(2,\mathbb{F})$.

$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $x^2 \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\underset{e}{x^3} \mapsto \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $U = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is a submodule of V.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \checkmark$$

Suppose $b \neq 0$. Then:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix} \neq \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

PF: Let V be an $\mathbb{F}[G]$-mod, and U an $\mathbb{F}[G]$-submod.
[WTS: There is an $\mathbb{F}[G]$-submod W of V s.t. $V = U \oplus W$]

Let W_0 be a subsp. of V s.t. V is the direct sum $U \oplus W_0$ as a vector space. Define

$$\varphi : V \to V$$
$$\varphi(v) = \varphi(u+w) = u \quad (\text{where } u \in U, w \in W_0).$$

↓ projection onto U.

[We will modify φ to an $\mathbb{F}[G]$-hom.]

Let $\Psi(v) := \frac{1}{|G|} \sum_{g \in G} g \circ \varphi \circ g^{-1}(v)$

Observe: Ψ is a lin. map s.t. $\operatorname{im} \Psi = U$.
To verify that this is an $\mathbb{F}[G]$-hom, observe for any $h \in G$:

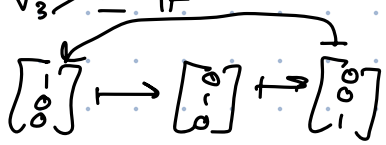
$$\begin{aligned} \Psi \circ h(v) &= \Psi(hv) \\ &= \frac{1}{|G|} \sum_{g \in G} g \varphi g^{-1} h (v) \\ &= \frac{1}{|G|} \sum_{k \in G} hk^{-1} \varphi k (v) \\ &= h \circ \Psi(v). \end{aligned}$$

$k = g^{-1}h$
$\Rightarrow k h^{-1} = g^{-1}$
$\Rightarrow g = hk^{-1}$

Thus, $\ker \Psi =: W$ is a submod, and $V \cong \operatorname{im}(\Psi) \oplus \ker(\Psi) = U \oplus W$. □

Example: $G = C_3 = \langle x \rangle$, $\mathbb{F} = \mathbb{C}$, $V = \langle v_1, v_2, v_3 \rangle \cong \mathbb{F}^3$

$x \cdot v_i = v_{i+1 \bmod 3}$



$U = \langle v_1 + v_2 + v_3 \rangle$, $W_0 = \langle v_2, v_3 \rangle$

Q: What is $\Psi(v_i)$?

$Q(v_1) = v_1 + v_2 + v_3 \Rightarrow \Psi(v_1) = \frac{1}{3}(v_1 + v_2 + v_3 + 0 + 0)$

$Q(v_2) = 0 \qquad \Psi(v_2)$

$Q(v_3) = 0 \qquad \Psi(v_3)$

$\Psi(v_1 + v_2 + v_3) = \frac{1}{3}(v_1 + v_2 + v_3)(3) = v_1 + v_2 + v_3$.

$\Psi(v_1 - v_2) = 0$

$\Psi(v_2 - v_3) = 0$

So $W = \langle v_1 - v_2, v_2 - v_3 \rangle$. Can check this is a submod.

Cor 1: Let G be a fin. gp, $\mathbb{F}$ a field s.t. $\mathrm{char}(\mathbb{F}) \nmid |G|$ (does not divide). Then any irred. $\mathbb{F}[G]$-module is a direct summand of the reg. $\mathbb{F}[G]$-module.

↓

The repn. $V = \mathbb{F}[G]$, and $\mathbb{F}[G]$ acts on itself via left-multiplication.

PF: Let U be an irred. $\mathbb{F}[G]$-mod. Let $u \in U$ be nonzero. Define an $\mathbb{F}[G]$-hom $T: \mathbb{F}[G] \to U$ by:

$$T\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \lambda_g \, g \cdot u$$

This is clearly a module hom. Since U is irred, and $\mathrm{im}(T) \neq 0$ (for example, $T(e) = u \neq 0$), we know $\mathrm{im}(T) = U$.

Then, by Maschke's Thm, $\mathbb{F}[G] = \ker(T) \oplus W$ for some W. Since $\mathbb{F}[G]/\ker(T) \cong W$ and $\mathbb{F}[G]/\ker(T) \cong \mathrm{im}(T) = U$, we have $U \cong W$.

□

Big Q: Which submodules appear in the reg. repn? And with what multiplicities?

Cor 2: If G is a fin. gp, then there are only finitely many non-isomorphic irred $F[G]$-modules (char$(F) \nmid |G|$)

Sec 2.2) The space of $\mathbb{C}[G]$-homs.

Throughout, let G be a fin. gp, let $A = \mathbb{C}[G]$.

Prop 1: Suppose V, W are irred A-mods. Then:

$$\dim(\mathrm{Hom}_A(V,W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Pf: If $V \not\cong W$, then $\mathrm{Hom}_A(V,W) = 0$ by Schur's Lemma. Suppose $V \cong W$. Let $\varphi: V \to W$ be an A-mod isom. [WTS: every other A-hom is a scalar mult. of φ.] Let $\psi \in \mathrm{Hom}_A(V,W)$. Then $\varphi^{-1} \circ \psi$ is an A-hom $V \to V$, and hence $\varphi^{-1}\psi = \lambda \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$. Therefore, $\psi = \lambda\varphi$, so $\mathrm{Hom}_A(V,W) = \langle \varphi \rangle$. ∎

Defn 1: If V is an A-mod and U is an irred A-mod, we say U is a composition factor of V if V has an A-submod isom to U.

Prop 2: Let V, W be A-mods, and suppose $\mathrm{Hom}_A(V,W) \neq 0$. Then V, W have a common composition factor.

Pf: Let $\varphi \in \mathrm{Hom}_A(V,W)$ be nonzero. Then $V = \ker\varphi \oplus U$ for some submod. U of V (by Maschke's Theorem). Let X be an irred. submod. of U. Then $\varphi(X) \neq \{0\}$, and by Schur's lemma, $\mathrm{im}(\varphi) \cong X$ and $\mathrm{im}(\varphi) \subseteq W$. □

Prop 3: Let V_i, W_j be irred A-mods. Then $\dim(\mathrm{Hom}_A(\bigoplus_{i=1}^r V_i, \bigoplus_{j=1}^s W_j)$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \dim(\mathrm{Hom}_A(V_i, W_j))$$

Pf: Exercise

Cor 1: Let V be an A-mod with $V = U_1 \oplus \cdots \oplus U_s$, where each U_i is an irred. submod. Let W be any irred. A-mod. Then $\dim(\mathrm{Hom}_A(V,W))$ and $\dim(\mathrm{Hom}_A(W,V))$ both equal the number of submodules U_i s.t. $U_i \cong W$.

Prop 4: If U is an A-mod, then $\dim(\mathrm{Hom}_A(\underset{\text{reg. repn.}}{A}, U)) = \dim U$.

Pf: Let $d = \dim U$. Let $u_1, u_2, \ldots, u_d$ be a basis for U. Define $\varphi_i : A \to U$ by $\varphi_i(a) = au_i$

Then $\forall a, b \in A$, $\varphi_i(ab) = (ab)u_i = a(bu_i) = a\varphi_i(b)$, so $\varphi_i \in \mathrm{Hom}_A(A,U)$. [We wish to show $\{\varphi_i\}$ form a basis for $\mathrm{Hom}_A(A,U)$.]

Let $\psi \in \mathrm{Hom}_A(A,U)$. Then

$$\psi(1) = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_d u_d$$

for some $\lambda_i \in \mathbb{C}$. Since ψ is an A-hom:

$$\psi(a) = \psi(a1) = a\psi(1) = \lambda_1(au_1) + \cdots + \lambda_d(au_d)$$

$$= \lambda_1\varphi_1(a) + \cdots + \lambda_d\varphi_d(a).$$

Thus, the φ_i span $\mathrm{Hom}_A(A,U)$.

Now suppose $\sum \lambda_i \varphi_i = 0$. Then, evaluating both sides at 1 gives us:

$$\sum \lambda_i \varphi_i(1) = 0$$

$$\Rightarrow \quad \sum \lambda_i u_i = 0.$$

So $\lambda_i = 0$ $\forall i$ (since the u_i are lin. indep), and thus the φ_i are lin. indep.

Hence $\dim(\mathrm{Hom}_A(A,U)) = d = \dim U$. ∎

Thm 1: Suppose $A = U_1 \oplus \cdots \oplus U_r$ (as A-modules), where each U_i is irreducible. If U is any irred A-mod, then the number of U_i s.t. $U_i \cong U$ is equal to $\dim U$.

Pf: By Prop 4, $\dim U = \dim(\mathrm{Hom}_A(A, U))$, and by Cor 1, $\dim(\mathrm{Hom}_A(A, U))$ equals the number of submods U_i isom. to U. □

Thm 2: Let $V_1, \dots, V_k$ form a complete set of non-isom. irred. A-modules. Then:

$\mathbb{C}[G]$

$$\sum_{i=1}^{k} (\dim V_i)^2 = |G|.$$

Example: Let G be a gp of order 8. Then the possibilities for the dims d_i of irred. modules are:

$8 = 2^2 + 2^2$ ⟶ not possible, no trivial repn.

$8 = 2^2 + 1^2 + 1^2 + 1^2 + 1^2$ (non-abelian)

$8 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$ (abelian)

Note: Let V be a 1-dim vect. sp $\langle v \rangle$ s.t.

$$g \cdot v = v \quad \forall\, g \in G.$$

This is called the trivial repn of G.

This is always an irred repn of G (for any gp G).