Representation Theory of Finite Groups

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Throughout, $\mathbb{F}$ is a field.

Sec 1.1) Fundamentals

Defn 1: An *associative algebra* over $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a bilinear map $A \times A \to A$, $(a,b) \mapsto ab$, such that $(ab)c = a(bc)$.

Defn 2: A *unit* in an assoc. alg. A is an elt. $1_A \in A$ s.t. $a1_A = 1_A a = a \quad \forall a \in A$.

Prop 1: If A has a unit, it must be unique.
Pf: Let $1, 1'$ be units in A. Then $1 \cdot 1' = 1 = 1'$. □

Henceforth, by an *algebra*, we will mean an associative algebra with a unit.

Examples:

1) $A = \mathbb{F}$.
2) $A = \mathbb{F}[x_1, \ldots, x_n]$ (polynomials over $\mathbb{F}$)
3) Let V be a vect. sp. over $\mathbb{F}$.
   $A = \mathrm{End}(V)$ (algebra of linear maps $V \to V$)
4) The *free algebra* $A = \mathbb{F}\langle x_1, \ldots, x_n \rangle$, with basis consisting of *words* in $x_1, \ldots, x_n$ and multiplication given by concatenation of words.

Defn 3: An algebra is *commutative* if $ab = ba \quad \forall a, b \in A$.

One more imp. example:

5) G group. The *group algebra* $A = \mathbb{F}[G]$ with basis indexed by elts of G.

Defn 4: A *homomorphism of algebras* $T: A \to B$ is a linear map s.t. $T(xy) = T(x)T(y) \quad \forall x, y \in A$ and $T(1_A) = 1_B$.

Sec 1.2) Representations

Defn 1: A *representation* of an algebra A is a hom. of algebras $\rho : A \to \mathrm{End}(V)$ for some vector space V. We call V a (left) *A-module*.

rho →

$$A \times V \longrightarrow V$$
$$(a, v) \longmapsto (\rho(a))(v).$$

Notation: We will often write $\rho(a)(v)$ as $a \cdot v$ or simply av. Then the hom. property is given
$$(ab)v = a(bv).$$

Examples:

1) $V = 0$, ρ zero map.
2) $V = A$, $\rho : A \to \mathrm{End}(A)$ given by $\rho(a)b = ab$
   This is called the *regular repn of A*
3) $A = \mathbb{F}\langle x_1, \dots, x_n \rangle$. Then any left A-mod. is a vect sp. V over $\mathbb{F}$ with a collection of arbitrary lin. ops. $\rho(x_1), \rho(x_2), \dots, \rho(x_n)$.

Defn 2: A *submodule* of a left A-module V is a subspace $W \leq V$ which is invariant under $\rho(a) \in \mathrm{End}(V)$ $\forall\, a \in A$.

$$A \times W \longrightarrow W.$$

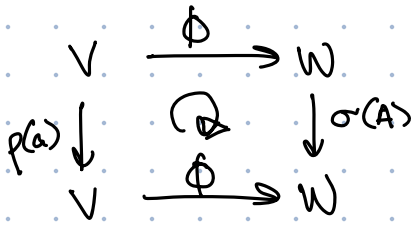
The hom. $\rho_W : A \to \mathrm{End}(W)$ is a *subrepresentation* of ρ.

Defn 3: A module $V \neq 0$ (and its associated repn) of A is *irreducible*, or *simple*, if its only submodules are 0 and V.

This property depends on the action $\rho : A \to \mathrm{End}(V)$.

Defn 4: Let $\rho : A \to \mathrm{End}(V)$ and $\sigma : A \to \mathrm{End}(W)$ be two repns of A. Then a *homomorphism of representations* (or *intertwining operator*) $\Phi : V \to W$ is a lin. map which commutes with the action of A, ie
$$\Phi \circ \rho(a) = \sigma(a) \circ \Phi \quad \forall\, a \in A.$$

$$\Phi(a \cdot v) = a \cdot (\Phi(v))$$



This is said to be an isomorphism or equivalence of repns if it is an isom. of vect. spaces.

Defn 5: The set of all homs. of repns $V \to W$ is denoted $\mathrm{Hom}_A(V, W)$.

↳ This A is important to denote that these are not just lin maps, but also compatible with the action of A.

Defn 6: Let V, W be A-modules. Then $V \oplus W$ is also an A-module under the map:

$$a(v, w) = (av, aw) \qquad \forall v \in V, w \in W, a \in A$$

This is called the direct sum of V and W.

Defn 7: A nonzero A-mod V is said to be indecomposable if it is not isom. to a direct sum of two nonzero modules.

Note: Irred $\Rightarrow$ Indecomp

$\not\Leftarrow$

Example: Let $A = \mathbb{F}[x]$, let $V = \mathbb{F}^2$ and $\rho : A \to \mathrm{End}(V)$ be given by:

$$\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $\rho(x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $W = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \cong \mathbb{F}$ is a submodule,

and hence V is not irreducible.

Suppose $V = W \oplus U$. Then since $\dim(V) = 2$, $\dim(W) = 1$, we must have $\dim U = 1$.

Suppose $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \notin W$., then $u_2 \neq 0$, so

$$\rho(x)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 \\ u_2 \end{bmatrix} \neq \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ for any } \lambda \in \mathbb{F}.$$

Hence V is not decomposable.

Prop 1: (Schur's Lemma:) Let V, W be modules of an algebra A over field $\mathbb{F}$ (not necessarily algebraically closed). Let $\phi : V \to W$ be a nonzero hom. of reps. Then:
i) If V is irred, ϕ is injective
ii) If W is irred, ϕ is surjective.
Thus, if both V and W are irred, ϕ is an isomorphism.

Exercise: A alg, V, W A-mods
$\phi : V \to W$ hom.
i) $\ker \phi$ is a submod of V
ii) $\operatorname{im} \phi$ is a submod of W.

Pf: i) Since $K = \ker \phi$ is a submodule of V, and $\phi \neq 0$, and V is irred, we have $K = 0$. So ϕ is injective.
ii) Since $I = \operatorname{im} \phi$ is a submod of W, and $\phi \neq 0$, and W is irred, we have $I = W$. So ϕ is surj. □

Cor 1: (Schur's Lemma for algebraically closed fields:) Let V be a fin. dim. irreducible A-module over an alg. closed field $\mathbb{F}$. Let $\phi : V \to V$ be an intertwining op. Then $\phi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{F}$.

Pf: Let λ be an eigenvalue of ϕ, which exists because $\mathbb{F}$ is alg. closed. Then $\phi - \lambda \cdot \mathrm{Id}$ is also an intertwining op which is not an isom. (since its det. is 0). Then by Prop 1, this op. is the 0 map, and so $\phi = \lambda \cdot \mathrm{Id}$. □

Cor 2: Let A be a commutative algebra over an alg. closed field $\mathbb{F}$. Then every irred. fin. dim. A-module V is 1-dim.

Pf: Let V be irred. and fin. dim. Then $\forall a \in A$, $\rho(a) : V \to V$ is an intertwining operator, because A is commutative. Then, by

Schur's Lemma, $\rho(a)$ is a scalar op $\forall a \in A$. Hence, every subspace of V is a submodule of V. So if V is irred, it must have dimension 1. □

Defn 8: Let G be a gp and $A = \mathbb{F}[G]$. A <u>repn of G</u> (over $\mathbb{F}$) is a repn of A, ie a map $\rho : A \to \mathrm{End}(V)$ s.t. $\rho(ab) = \rho(a)\rho(b)$ and $\rho(1_A) = \mathrm{Id}_V$.

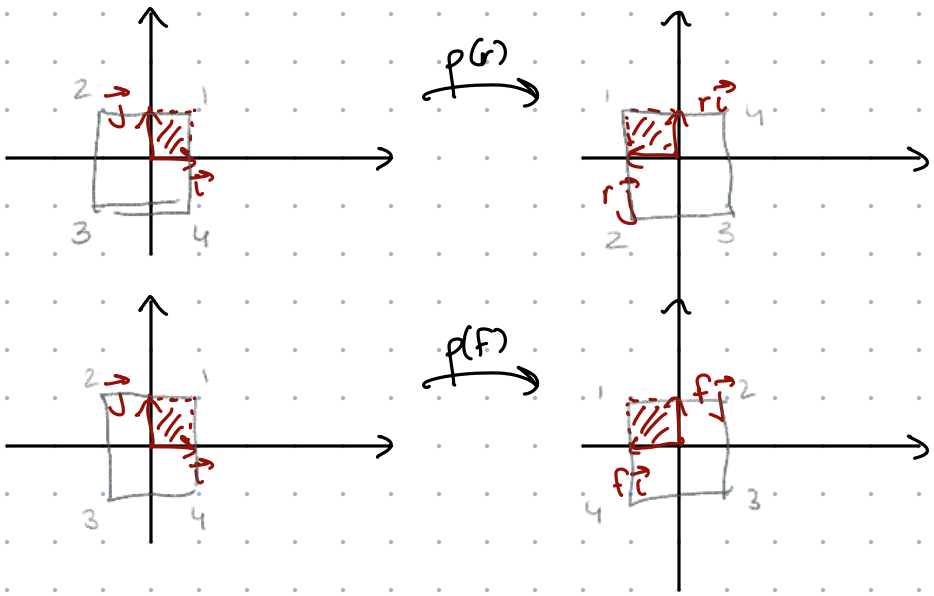
This is equivalent to a group homomorphism $\rho : G \to GL(V)$.

the set of invertible linear maps $V \to V$.

Example: $\mathbb{F} = \mathbb{C}$, $G = D_8 = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$
$= \langle r, f \mid r^4 = e = f^2,\ fr = r^{-1}f \rangle$

1) $\rho : G \longrightarrow GL(2,\mathbb{C}) = GL(\mathbb{C}^2)$

$$\rho(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho(f) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Q: Is this irred?

Note: This amounts to finding an invariant subsp. of dim 1, ie a common eigenvect. for $\rho(r)$ and $\rho(f)$.

Suppose $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is such a vect. Then

$$f \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$ so either $w_1 = 0$ or $w_2 = 0$.

Then:

$$r \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} = \mu \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$ so this implies $w_1 = w_2 = 0$.



So this is irred.