

LTCC: Representation Theory of Finite Groups

Exercise Set 4

Exercises marked with $\star$ are especially useful for subsequent lectures.

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. Suppose χ is a nonzero, nontrivial character of G , and that $\chi(g)$ is a nonnegative real number for all $g \in G$. Show that χ must be reducible.
2. $\star$ (*From lecture*) Suppose χ is a character of G and λ is a linear character of G .
 - (a) Show that the product $\lambda\chi$ (given by $\lambda\chi(g) = \lambda(g)\chi(g)$) is also a character of G .
 - (b) Show that if χ is irreducible, then so is $\lambda\chi$.
3. $\star$ Let N be a normal subgroup of G and let $\tilde{\chi}$ be a character of G/N . Let $\chi : G \rightarrow \mathbb{C}$ be given by $\chi(g) = \tilde{\chi}(gN)$. Then χ is a character of G , and χ and $\tilde{\chi}$ have the same degree.
4. $\star$ Let χ be the character of a permutation representation of S_n (as discussed in Problem Set 3, Q 4). Show that $\tilde{\chi}(g) = |\text{fix}(g)| - 1$ is also a character of S_n . [*Hint*: Look for a trivial submodule of this permutation module.]
5. $\star$ Write out the character table of S_4 . [*Hint*: Consider the character $\tilde{\chi}$ defined in the previous exercise.]
6. $\star$ Let V and W be vector spaces. If $\{v_1, \dots, v_n\}$ is a basis for V and $\{w_1, \dots, w_m\}$ is a basis for W , then the *tensor product* $V \otimes W$ is the vector space with basis $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. [Note that for $v \in V$ and $w \in W$, we have $v \otimes w = (\sum_i \lambda_i v_i) \otimes (\sum_j \mu_j w_j) = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$.] If V and W are in fact $\mathbb{C}[G]$ -modules, we can define an action of G on $V \otimes W$ by $g \cdot (v \otimes w) = gv \otimes gw$ and extending linearly.
 - (a) Show that if the characters of V and W are χ and ψ , respectively, then the character of $V \otimes W$ is $\chi\psi$. [This shows that the product of any two characters of G is again a character of G . Note that this gives us an alternative proof of Exercise 1a, but Exercise 1a can also be solved more directly.]

- (b) Let V be a $\mathbb{C}[G]$ -module with basis $\{v_1, \dots, v_n\}$, and let $\varphi : V \otimes V \rightarrow V \otimes V$ be the map given by $\varphi(v_i \otimes v_j) = v_j \otimes v_i$. Show that $\text{Sym}(V) = \{x \in V \otimes V \mid \varphi(x) = x\}$ and $\text{Alt}(V) = \{x \in V \otimes V \mid \varphi(x) = -x\}$ are complementary submodules of $V \otimes V$.
- (c) Find the characters χ_S and χ_A of $\text{Sym}(V)$ and $\text{Alt}(V)$ in terms of the character χ of V , and verify that $\chi^2 = \chi_S + \chi_A$.
- (d) Consider the character $\tilde{\chi}$ of S_4 defined in Q 4. Find a decomposition of $\tilde{\chi}^2$ as a sum of irreducible characters. [This give us a way of decomposing the corresponding tensor product module as a direct sum of irreducible modules.]
7. $\star$ Let G' denote the *commutator* subgroup of G , i.e. $G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$. A standard fact in group theory is that the quotient group G/N is abelian if and only if $G' \subseteq N$. Show that the linear characters of G are precisely the lifts to G of the irreducible characters of G/G' . [This implies that there are exactly $|G/G'|$ linear characters of G .]
8. There exists a group G of order 10 with precisely four conjugacy classes with representatives g_1, g_2, g_3, g_4 , and has an irreducible character χ given by

$g_i :$	g_1	g_2	g_3	g_4
χ	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0

- (a) Find the sizes of the conjugacy classes of G . (*Hint:* It would be helpful to also have one other irreducible character for this.)
- (b) Complete the character table of G .
9. Find the character tables for
- (a) $D_4 = \langle r, f \mid r^4 = f^2 = e, fr = r^{-1}f \rangle$
- (b) $G = \langle a, b \mid a^6 = b^3 = 1, ba = ab^{-1} \rangle$.