

LTCC: Representation Theory of Finite Groups

Exercise Set 3

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. Let G be a finite group. Find a basis for $\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \mathbb{C}[G])$.

Solution: Let $G = \{g_1, \dots, g_n\}$. Following the proof of Prop 4 in Section 2.2, we have a basis for $\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \mathbb{C}[G])$ given by $\phi_i : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$, where $\phi_i(h) = hg_i$ for all $h \in G$.

2. Let $V_1, \dots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$ -modules, and let V, W be arbitrary $\mathbb{C}[G]$ -modules. Suppose that for $1 \leq i \leq k$,

$$\dim(\text{Hom}_{\mathbb{C}[G]}(V, V_i)) = d_i \text{ and } \dim(\text{Hom}_{\mathbb{C}[G]}(W, V_i)) = e_i.$$

Show that $\dim(\text{Hom}_{\mathbb{C}[G]}(V, W)) = \sum_{i=1}^k d_i e_i$.

Solution: Let $V = X_1 \oplus \dots \oplus X_r$ and $W = Y_1 \oplus \dots \oplus Y_s$, where each X_a and each Y_b is an irreducible $\mathbb{C}[G]$ -module. Then $\dim(\text{Hom}_{\mathbb{C}[G]}(V, W))$ is equal to the number of ordered pairs (a, b) such that $X_a \cong Y_b$. This is given by

$$\sum_{i=1}^k \{(a, b) \mid X_a \cong Y_b \cong V_i\}.$$

Now note that the number of integers a with $X_a \cong V_i$ is $\dim(\text{Hom}_{\mathbb{C}[G]}(V, V_i)) = d_i$, and similarly the number of integers b with $Y_b \cong V_i$ is $\dim(\text{Hom}_{\mathbb{C}[G]}(W, V_i)) = e_i$. Therefore, $\dim(\text{Hom}_{\mathbb{C}[G]}(V, W)) = \sum_{i=1}^k d_i e_i$.

3. Given a group G , show that character of the regular representation G is given by $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$ in G .

Solution: Let ρ be the regular representation, and observe that for all $g \in G$, $\rho(g)$ permutes the elements of G , and hence is represented by a permutation matrix (with a single 1 in each row and column and 0s elsewhere). Therefore, the trace of $\rho(g)$ is given by the number of elements which are fixed under left multiplication by g . We observe that if $gh = h$ for some $h \in G$, then $g = e$. Therefore, since e fixes every element in G , we have $\chi(e) = |G|$ and since $g \neq e$ cannot fix any element in G , we have $\chi(g) = 0$.

4. Let S_n act on an n -dimensional space V with basis $\{v_1, \dots, v_n\}$ by $\sigma v_i = v_{\sigma(i)}$. (V is called the *permutation module* of S_n .)

- (a) Let χ be the character of the permutation representation of S_7 . Find $\chi(x)$ for $x = (1\ 2)$ and $x = (1\ 6)(2\ 3\ 5)$.

Solution: $\chi((1\ 2)) = 5$

$\chi((1\ 6)(2\ 3\ 5)) = 2$

- (b) Show that in general the character χ of a permutation representation is given by $\chi(\sigma) = |\text{fix}(\sigma)|$ where $\text{fix}(\sigma) = \{i \mid \sigma(i) = i\}$.

Solution: Let ρ be the permutation representation. As in Q 1, for each $\sigma \in S_n$, the matrix $\rho(\sigma)$ will be a permutation matrix, and so $\chi(\sigma)$ equals the number of basis vectors fixed by the action of σ . This is precisely equal to $|\text{fix}(\sigma)|$.

5. Let $\rho : G \rightarrow GL(V)$ be a representation, and let χ be the character of ρ . Prove the following statements:

- (a) If $g \in G$ is an element of order 2, then $\chi(g)$ is an integer such that $\chi(g) \equiv \chi(e) \pmod{2}$.

Solution: Since $g^2 = e$, we know that $\rho(g)^2 = I_V$. Therefore, the eigenvalues of g are all 1 or -1, and hence $\chi(g) = \text{tr}(\rho(g)) = k - \ell$ where k is the multiplicity of 1, ℓ is the multiplicity of -1 and $k + \ell = \dim(V) =: n$. Then, mod 2, we have $\chi(g) = k - \ell = k + \ell = n = \chi(e)$.

- (b) $|\chi(g)| = \chi(e)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. (Note that the absolute value of a complex number is given by $|x + iy| = \sqrt{x^2 + y^2}$.)

Solution: If $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\chi(g) = \text{tr}(\rho(g)) = n\lambda$, and hence $|\chi(g)| = n|\lambda|$. But note that λ must be a root of unity since $\rho(g)$ has finite order, and so $|\lambda| = 1$, and hence $|\chi(g)| = n = \chi(e)$.

Now suppose $|\chi(g)| = n = \chi(e)$. Then $\chi(g)$, which is the sum of m^{th} roots of unity, must in fact be the sum of n copies of the same root of unity ω . We know there is a basis in which $\rho(g)$ is diagonal (for example, by the proof of Section 3.1, Prop 3), and so in this basis $\rho(g) = (\text{diag})(\omega, \omega, \dots, \omega) = \omega I_n$. But any change of basis (conjugation) would leave this matrix unchanged, and hence $\rho = \omega I_n$ in any basis.

- (c) $\ker(\rho) = \{g \in G \mid \chi(g) = \chi(e)\}$

Solution: Suppose $g \in \ker(\rho)$. Then $\rho(g) = I_n = \rho(e)$, and hence $\chi(g) = \chi(e)$.

Now suppose $\chi(g) = \chi(e)$ for some $g \in G$. Then by the previous part, we know that $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. Therefore $\chi(g) = n\lambda$, but this can only equal $\chi(e) = n$ if $\lambda = 1$. Thus, $\rho(g) = I_n$ and hence $g \in \ker(\rho)$.

6. Prove that if χ is a faithful irreducible character of G , then the center of G is given by $Z(G) = \{g \in G \mid |\chi(g)| = \chi(e)\}$.

Solution: Let ρ be the corresponding representation.

Suppose g satisfies $|\chi(g)| = \chi(e)$. If $g \in Z(G)$, then since ρ is irreducible, by Week 1 Q4a we have $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$. Conversely, if $g = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\rho(g)$ commutes with $\rho(h)$ for all $h \in G$, and hence $g \in Z(G)$ since ρ is faithful.

Finally, by Q2b, we know that $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$ iff $|\chi(g)| = \chi(e)$, and so the result follows.

7. Let χ be an irreducible character of G , and suppose $z \in Z(G)$ is an element of order m . Show that there exists an m^{th} root of unity $\lambda \in \mathbb{C}$ such that for all $g \in G$, $\chi(zg) = \lambda\chi(g)$.

Solution: Let ρ be the corresponding representation. By Q2b, we know that $\rho(z) = \lambda I_n$ for some $\lambda \in \mathbb{C}$, and since $\rho(z^m) = \rho(z)^m = I_n$, we know λ must be an m^{th} root of unity. Therefore $\rho(zg) = \rho(z)\rho(g) = \lambda\rho(g)$ for all $g \in G$, and hence $\chi(zg) = \lambda\chi(g)$.

8. Let χ be a character of G . Show that χ is a homomorphism from G to $\mathbb{C}^\times$ if and only if χ is the character of a degree one representation. (Such characters are called *linear characters*.)

Solution: If χ has degree $n > 1$, then $\chi(e) = n$, so χ does not take identity in G to identity in $\mathbb{C}^\times$, and hence cannot be a homomorphism.

Now suppose χ has degree 1, so that $\chi(e) = 1$. Then if ρ is the corresponding representation, we have $\chi(g) = \rho(g)$, and a representation of G is a group homomorphism by definition. Thus, χ is in fact a homomorphism in this case.