

LTCC: Representation Theory of Finite Groups

Exercise Set 3

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. Let G be a finite group. Find a basis for $\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \mathbb{C}[G])$.
2. Let $V_1, \dots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$ -modules, and let V, W be arbitrary $\mathbb{C}[G]$ -modules. Suppose that for $1 \leq i \leq k$,

$$\dim(\text{Hom}_{\mathbb{C}[G]}(V, V_i)) = d_i \text{ and } \dim(\text{Hom}_{\mathbb{C}[G]}(W, V_i)) = e_i.$$

Show that $\dim(\text{Hom}_{\mathbb{C}[G]}(V, W)) = \sum_{i=1}^k d_i e_i$.

3. Given a group G , show that character of the regular representation G is given by $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$ in G .
4. Let S_n act on an n -dimensional space V with basis $\{v_1, \dots, v_n\}$ by $\sigma v_i = v_{\sigma(i)}$. (V is called the *permutation module* of S_n .)
 - (a) Let χ be the character of the permutation representation of S_7 . Find $\chi(x)$ for $x = (1\ 2)$ and $x = (1\ 6)(2\ 3\ 5)$.
 - (b) Show that in general the character χ of a permutation representation is given by $\chi(\sigma) = |\text{fix}(\sigma)|$ where $\text{fix}(\sigma) = \{i \mid \sigma(i) = i\}$.
5. Let $\rho : G \rightarrow GL(V)$ be a representation, and let χ be the character of ρ . Prove the following statements:
 - (a) If $g \in G$ is an element of order 2, then $\chi(g)$ is an integer such that $\chi(g) \equiv \chi(e) \pmod{2}$.
 - (b) $|\chi(g)| = \chi(e)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. (Note that the absolute value of a complex number is given by $|x + iy| = \sqrt{x^2 + y^2}$.)
 - (c) $\ker(\rho) = \{g \in G \mid \chi(g) = \chi(e)\}$
6. Prove that if χ is a faithful irreducible character of G , then the center of G is given by $Z(G) = \{g \in G \mid |\chi(g)| = \chi(e)\}$.

7. Let χ be an irreducible character of G , and suppose $z \in Z(G)$ is an element of order m . Show that there exists an m^{th} root of unity $\lambda \in \mathbb{C}$ such that for all $g \in G$, $\chi(zg) = \lambda\chi(g)$.
8. Let χ be a character of G . Show that χ is a homomorphism from G to $\mathbb{C}^\times$ if and only if χ is the character of a degree one representation. (Such characters are called *linear characters*.)