

LTCC: Representation Theory of Finite Groups

Exercise Set 2

Exercises marked with $\star$ are especially useful for subsequent lectures.

1. (*From lecture notes:*) Let A be the group algebra over $\mathbb{C}$ of $G = C_3 = \langle x | x^3 = e \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be an A -module with representation $x \cdot v_i = v_{i+1 \bmod 3}$. This module has a submodule $U = \langle v_1 + v_2 + v_3 \rangle$. Find a submodule W such that $V = U \oplus W$.

Solution: Let $W_0 = \langle v_1, v_2 \rangle$, so that $V = U \oplus W_0$ as vector spaces (note that W_0 is not a submodule). Let $\varphi : V \rightarrow U$ be the projection map from $V = U \oplus W_0$ to U . We can verify that for all basis vectors v_i , we have

$$\psi(v_i) = \frac{1}{|G|} \sum_{g \in G} g \varphi g^{-1}(v_i) = \frac{1}{3}(v_1 + v_2 + v_3).$$

Thus, $W = \ker(\psi) = \{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \}$.

2. Is the submodule W from the previous exercise irreducible?

Solution: The submodule $W = \{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \}$ is two dimensional, with a possible basis given by $v_1 - v_2, v_2 - v_3$. We have seen (as a corollary to Schur's Lemma) that all the irreducible submodules of an abelian group must be 1-dimensional, so this implies that W is not irreducible.

3. $\star$ (*From class discussion:*) Let V be an A -module and let U and W be submodules. Show that $U \cap W$ is also a submodule.

Solution: We recall that the intersection of two vector subspaces is again a subspace. We now observe that if for all $a \in A, u \in U, w \in W$, we have $au \in U$ and $aw \in W$, then if $v \in U \cap W$, we have $v \in U$, so $av \in U$ and similarly, $v \in W$, so $av \in W$. Hence, $av \in U \cap W$ in this case, and so $U \cap W$ is a submodule.

4. $\star$ (*From class discussion:*) Here, we show that a decomposition of a module into irreducible submodules is essentially unique. Suppose V is an A -module and $V = \bigoplus_{i=1}^n U_i$, where each U_i is an irreducible submodule. Suppose W is an irreducible submodule of V . By considering the possibilities for the intersections $W \cap U_i$, show that we must in fact have $W = U_i$ for some i .

Solution: From the previous exercise, we know that for each i , the intersection $W \cap U_i$ is a submodule of W , and since W is irreducible, we know that this intersection must be either 0 or all of W . Since

$$W \cap V = W \cap \left(\bigoplus_{i=1}^n U_i \right) = \bigoplus_{i=1}^n (W \cap U_i) = W,$$

we must have $W \cap U_i = W$ for exactly one i , say $i = k$. Then, since $W \cap U_k$ is also a submodule of U_k and U_k is irreducible, we have $W \cap U_k = U_k$, and hence $W = U_k$.

5. ★ Let V, W_1, W_2 be A -modules. Show that

$$\operatorname{Hom}_A(V, W_1 \oplus W_2) \cong \operatorname{Hom}_A(V, W_1) \oplus \operatorname{Hom}_A(V, W_2)$$

as vector spaces (and thus their dimensions are the same).

Solution: Let $W = W_1 \oplus W_2$ and let π_1, π_2 be the projection maps from W to W_1, W_2 , respectively, and note that the π are A -homomorphisms. Define a map $\varphi : \operatorname{Hom}_A(V, W_1 \oplus W_2) \rightarrow \operatorname{Hom}_A(V, W_1) \oplus \operatorname{Hom}_A(V, W_2)$ by $\varphi(f) = (\pi_1 \circ f, \pi_2 \circ f)$. We verify that φ is linear and bijective, and therefore an isomorphism of vector spaces.

6. Let G be a group. Suppose that ρ is a representation of G on a vector space V of dimension 1. Prove that $G/\ker(\rho)$ is abelian.

Solution: We note that $G/\ker(\rho) \cong \operatorname{im}(\rho)$ (as groups), and since $\operatorname{im}(\rho) \subset GL(V) \cong \mathbb{F}^\times$ (since V is 1-dimensional), we see that $\operatorname{im}(\rho)$ is a subgroup of an abelian group, and hence also abelian.

7. Let $\phi : G \rightarrow GL(n, \mathbb{F})$ be a representation of G . Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

is also a representation of G .

Solution: We observe that

$$\rho(gh) = \det(\phi(gh)) = \det(\phi(g)\phi(h)) = \det(\phi(g))\det(\phi(h)) = \rho(g)\rho(h),$$

and

$$\rho(e) = \det(\phi(e)) = \det(I_n) = 1.$$

Therefore, ρ is also a representation.

8. ★ Let $\rho : G \rightarrow GL(n, \mathbb{F})$ be a representation of G . Show that $\sigma : G/\ker(\rho) \rightarrow GL(n, \mathbb{F})$ given by $\sigma(g\ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Recall from Exercise Sheet 1, Question 6, that a faithful representation is one that is injective. Note that here you must show that it is a well-defined representation, as well as showing that it is faithful.)

Solution: Suppose $g\ker(\rho) = h\ker(\rho)$. This equivalent to the condition that $g^{-1}h \in \ker(\rho)$ and hence $\rho(g^{-1}h) = I_n$. Since ρ is a representation, we have $\rho(g)^{-1}\rho(h) = I_n$, and hence $\rho(g) = \rho(h)$. Thus, $\sigma(g\ker(\rho)) = \sigma(h\ker(\rho))$, so σ is well-defined.

We also observe that

$$\sigma(g \ker(\rho) h \ker(\rho)) = \sigma(gh \ker(\rho)) = \rho(gh) = \rho(g)\rho(h) = \sigma(g \ker(\rho))\sigma(h \ker(\rho)),$$

and

$$\sigma(e \ker(\rho)) = \rho(e) = I_n.$$

Therefore, σ is also a representation.

Finally, we suppose $\sigma(g \ker(\rho)) = \sigma(h \ker(\rho))$. Then $\rho(g) = \rho(h)$, and so $\rho(g)^{-1}\rho(h) = I_n$, and therefore $\rho(g^{-1}h) = I_n$. Thus, $g^{-1}h \in \ker(\rho)$, and so $g \ker(\rho) = h \ker(\rho)$. Therefore, σ is a faithful representation.

9. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G , there exists a faithful irreducible $\mathbb{C}[G]$ -module.

Solution: By Maschke's Theorem, the regular $\mathbb{C}[G]$ -module decomposes into irreducible submodules. Let ρ be the representation of G on one of the nontrivial submodules. (Note that nontrivial submodules must exist whenever G is a nontrivial group. When G is trivial, the trivial module is faithful.) We then have that action of G on this nontrivial irreducible module is faithful, because $\ker(\rho)$ is normal in G , and therefore must be trivial.

10. When constructing the regular representation of G over $\mathbb{F}$, we used the action of G on itself by *left multiplication*, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.

- (a) Define the *right multiplication* action $\cdot : G \times G \rightarrow G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.

Solution: We observe:

$$gh \cdot x = x(gh)^{-1} = xh^{-1}g^{-1} = g \cdot (h \cdot x)$$

and $e \cdot x = xe^{-1} = x$. Thus, this map is an action of G on $\mathbb{F}[G]$. This fails for the operation $g \cdot x = xg$, since in this case:

$$gh \cdot x = x(gh) = h \cdot (g \cdot x).$$

- (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.

Solution: Let $T : \mathbb{F}[G] \rightarrow \mathbb{F}[G]$ be the map given by $x \mapsto x^{-1}$ for all $x \in G$. (This is a permutation of basis vectors of $\mathbb{F}[G]$ and hence an invertible linear map.) We observe that for all $g \in G$, $g \cdot T(x) = g \cdot x^{-1} = x^{-1}g^{-1} = (gx)^{-1} = T(gx)$. Thus, T is an equivalence of representations.

- (c) Define the *conjugation* action $\cdot : G \times G \rightarrow G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.

Solution: We observe:

$$gh \cdot x = (gh)x(gh)^{-1} = ghxh^{-1}g^{-1} = g \cdot (h \cdot x)$$

and $e \cdot x = exe^{-1} = x$. Thus, this map is an action of G on $\mathbb{F}[G]$.

- (d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)

Solution: Let $G = C_3$. We observe that every element of G is fixed by the conjugation action of G , and hence so is every element of $\mathbb{F}[G]$, so $\mathbb{F}[G]$ decomposes into the direct sum of three copies of the trivial module. On the other hand, the G acts nontrivially on the regular $\mathbb{F}[G]$ -module.

Alternative solution: Let $|G| > 1$. Suppose there exists a bijective linear map $T : \mathbb{F}[G] \rightarrow \mathbb{F}[G]$ such that $T(gx) = gT(x)g^{-1}$ for all $g \in G, x \in \mathbb{F}[G]$. By setting $x = e$, we get $T(g) = gT(e)g^{-1} = e$ for all $g \in G$, which contradicts the fact that G is bijective.

11. Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$ -module has dimension 1. Show that G is abelian.

Solution: By Maschke's Theorem, the regular $\mathbb{C}[G]$ -module decomposes into irreducible submodules. Thus, we get that $\mathbb{C}[G] = \bigoplus_{i=1}^n U_i$ where each U_i has dimension 1 in this case. For each i let u_i be a nonzero vector in U_i . Then $\{u_1, \dots, u_n\}$ is a basis for $\mathbb{C}[G]$, and for each $g \in G, i \in I$, we have $gu_i = \lambda_i u_i$ for some $\lambda_i \in \mathbb{C}$. Thus, g is a diagonal matrix in this basis for all $g \in G$. Since any two diagonal matrices commute, we get that $gh = hg$ for all $g, h \in G$, and so the group is abelian.

12. Let G be a finite group and let $\rho : G \rightarrow GL(2, \mathbb{C})$ be a representation of G . Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

Solution: Suppose ρ is reducible. Then there exists a proper nonzero $\mathbb{C}[G]$ -submodule V of $\mathbb{C}^2$, which means that V is 1-dimensional. By Maschke's Theorem, there also exists a submodule W such that $\mathbb{C}^2 = V \oplus W$. Let v, w be nonzero vectors of V and W , respectively. Note that $\{v, w\}$ is a basis of $\mathbb{C}^2$ in this case, and so $\rho(g)$ and $\rho(h)$ act diagonally with respect to this basis and hence commute, which is a contradiction. Thus, ρ must be irreducible.