

LTCC: Representation Theory of Finite Groups

Exercise Set 2

Exercises marked with $\star$ are especially useful for subsequent lectures.

1. (*From lecture notes:*) Let A be the group algebra over $\mathbb{C}$ of $G = C_3 = \langle x | x^3 = e \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be an A -module with representation $x \cdot v_i = v_{i+1 \bmod 3}$. This module has a submodule $U = \langle v_1 + v_2 + v_3 \rangle$. Find a submodule W such that $V = U \oplus W$.
2. Is the submodule W from the previous exercise irreducible?
3. $\star$ (*From class discussion:*) Let V be an A -module and let U and W be submodules. Show that $U \cap W$ is also a submodule.
4. $\star$ (*From class discussion:*) Here, we show that a decomposition of a module into irreducible submodules is essentially unique. Suppose V is an A -module and $V = \bigoplus_{i=1}^n U_i$, where each U_i is an irreducible submodule. Suppose W is an irreducible submodule of V . By considering the possibilities for the intersections $W \cap U_i$, show that we must in fact have $W = U_i$ for some i .
5. $\star$ Let V, W_1, W_2 be A -modules. Show that

$$\operatorname{Hom}_A(V, W_1 \oplus W_2) \cong \operatorname{Hom}_A(V, W_1) \oplus \operatorname{Hom}_A(V, W_2)$$

as vector spaces (and thus their dimensions are the same).

6. Let G be a group. Suppose that ρ is a representation of G on a vector space V of dimension 1. Prove that $G/\ker(\rho)$ is abelian.
7. Let $\phi : G \rightarrow GL(n, \mathbb{F})$ be a representation of G . Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

is also a representation of G .

8. $\star$ Let $\rho : G \rightarrow GL(n, \mathbb{F})$ be a representation of G . Show that $\sigma : G/\ker(\rho) \rightarrow GL(n, \mathbb{F})$ given by $\sigma(g\ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Recall from Exercise Sheet 1, Question 6, that a faithful representation is one that is injective. Note that here you must show that it is a well-defined representation, as well as showing that it is faithful.)

9. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G , there exists a faithful irreducible $\mathbb{C}[G]$ -module.
10. When constructing the regular representation of G over $\mathbb{F}$, we used the action of G on itself by *left multiplication*, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.
 - (a) Define the *right multiplication* action $\cdot : G \times G \rightarrow G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.
 - (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.
 - (c) Define the *conjugation* action $\cdot : G \times G \rightarrow G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.
 - (d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)
11. Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$ -module has dimension 1. Show that G is abelian.
12. Let G be a finite group and let $\rho : G \rightarrow GL(2, \mathbb{C})$ be a representation of G . Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.