

From last time:

1) $G = C_3 = \langle x \mid x^3 = e \rangle = \{e, x, x^2\}$

$\begin{matrix} g_i: \\ Z(g_i) \end{matrix}$	$\begin{matrix} e \\ 3 \end{matrix}$	$\begin{matrix} x \\ 3 \end{matrix}$	$\begin{matrix} x^2 \\ 3 \end{matrix}$
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

2) $G = D_3 = \langle r, f \mid r^3 = e, f^2 = e, fr = r^{-1}f \rangle$

$\begin{matrix} g_i \\ Cl_G(g_i) : \\ Z(g_i) : \end{matrix}$	$\begin{matrix} e \\ 1 \\ 6 \end{matrix}$	$\begin{matrix} r \\ 2 \\ 3 \end{matrix}$	$\begin{matrix} f \\ 3 \\ 2 \end{matrix}$
$u_1 \leftarrow \chi_1$	1	1	1
$u_2 \leftarrow \chi_2$	1	1	-1
$u_3 \leftarrow \chi_3$	2	-1	0

3) $G = A_4$

$\begin{matrix} g_i: \\ Cl_G(g_i) : \\ Z(g_i) : \end{matrix}$	$\begin{matrix} id \\ 1 \\ 12 \end{matrix}$	$\begin{matrix} \pm 1 \\ (12)(34) \\ 3 \\ 4 \end{matrix}$	$\begin{matrix} 1, \omega, \omega^2 \\ (123) \\ 3 \end{matrix}$	$\begin{matrix} (132) \\ 3 \end{matrix}$
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

From HW (Week 4, Q5):

4) $G = S_4$:

g_i $ Q(g_i) $ $ Z(g_i) $	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
	1	6	3	8	6
	24	4	8	3	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

$$\chi(g) = |\text{fix}(g)| - 1 = \chi_4 \text{ in this case.}$$

Ch 5) Representation Theory of Subgroups

5.1) Restriction:

Defn 1: Let H be a subgp of G . Then any $\mathbb{C}[G]$ -module V can be regarded as a $\mathbb{C}[H]$ -module, in which case, we denote it by $\text{Res}_H^G(V)$ or $\text{Res}_H(V)$ or $V \downarrow H$, and we call it the restriction of V to H . If V has char χ , then $V \downarrow H$ has char $\chi \downarrow H$.

Warning: χ irred $\not\Rightarrow \chi \downarrow H$ is irred.

Prop 1: Let $H \leq G$ and ψ a nonzero char of H . Then there exists an irred char χ of G s.t. $\langle \chi \downarrow H, \psi \rangle \neq 0$.

PF: Let $\chi_1, \dots, \chi_k$ be the irred. chars of G . Let Q be the reg. of G . Then $Q = \sum_i \chi_i(e) \chi_i$.

Now observe that the reg. $\mathbb{C}[G]$ -module contains $\mathbb{C}[H]$, and $\mathbb{C}[H]$ is closed under the action of H , so $\mathbb{C}[H]$ is a submod of $\mathbb{C}[G] \downarrow H$.

Suppose ψ is irred. Then the corr. $\mathbb{C}[H]$ -mod V is a submod of $\mathbb{C}[H]$. Therefore $\langle \psi \downarrow H, \psi \rangle \neq 0$, so

$$0 \neq \langle \sum \chi_i(e) \chi_i \downarrow H, \psi \rangle = \sum \chi_i(e) \langle \chi_i \downarrow H, \psi \rangle.$$

So $\langle \chi_i \downarrow H, \psi \rangle \neq 0$ for some i .

We can then extend this to reducible ψ by noting that for some constituent ψ_j of ψ , $\exists i$ s.t. $\langle \chi_i \downarrow H, \psi_j \rangle \neq 0$. $\square$

Thm 1: (Clifford's Thm:) Suppose $H \triangleleft G$, χ an irred char of G and $\psi_1, \dots, \psi_m$ the constituents of $\chi \downarrow H$. Then:

- 1) the ψ_i all have the same degree.
- 2) $\langle \chi \downarrow H, \psi_i \rangle$ is the same for all i .

Pf: 1) Let V be a $\mathbb{C}[G]$ -mod corr to χ and U an irred $\mathbb{C}[H]$ -submod of $V \downarrow H$. Then $\forall g \in G$, the set $gU = \{gu \mid u \in U\}$ is a subsp. of V , and $\forall h \in H$:

$$h(gu) = (hg)u = g(g^{-1}hg)u \in gU \text{ (since } ghg^{-1} \in H).$$

So gU is also a $\mathbb{C}[H]$ -submodule of $V \downarrow H$.

Now suppose W is a H -submodule of gU . Then $g^{-1}W$ is a submodule of U , and hence $g^{-1}W = U$ or $\{0\}$, and so $W = gU$ or $\{0\}$. Therefore, gU is irred. Since g is an invertible linear transformation, $\dim gU = \dim U$.

Finally, note that $\bigoplus gU$ is a $\mathbb{C}[G]$ -submod of V . Since V is irred, $\bigoplus gU = V$, so V is the direct sum of these irred submodules of equal dim. Thus $V = g_1U \oplus \dots \oplus g_\ell U$.

2) Exercise $\square$

Prop 2: Let $H \leq G$, and let χ be an irred char of G , and let $\psi_1, \dots, \psi_r$ be the irred chars of H . Then $\chi \downarrow H = d_1\psi_1 + \dots + d_r\psi_r$ where $\sum d_i^2 \leq [G:H] = \frac{|G|}{|H|}$. Equality holds iff $\chi(g) = 0 \ \forall g \in G - H$.

Pf: • $\sum d_i^2 = \langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\chi(h)}$.

• $1 = \langle \chi, \chi \rangle_G = \frac{1}{|G|} \left(\sum_{h \in H} \chi(h) \overline{\chi(h)} + \sum_{g \in G-H} \chi(g) \overline{\chi(g)} \right)$
 $= \frac{|H|}{|G|} \left(\sum d_i^2 \right) + K$

where $K > 0$.

□

5.2) Induction:

Let $H \leq G$. Let $g_1, \dots, g_k$ be reps of the left cosets of H in G . Then given a $\mathbb{C}[H]$ -mod W , we wish to define a $\mathbb{C}[G]$ -mod V s.t. $V = g_1 W \oplus \dots \oplus g_k W$ [the most general module that restricts to W].

Defn 1: • Let U be an irred $\mathbb{C}[H]$ -module. Then, viewing it as an $\mathbb{C}[H]$ (the reg module), define

$$U \uparrow G = \{gu \mid u \in U, g \in G\} = \text{Ind}_H^G(U)$$

We call $U \uparrow G$ the $\mathbb{C}[G]$ -module induced from U .

• For a $\mathbb{C}[H]$ -mod U , suppose $U = \oplus U_i$ (direct sum of irred submods). Then $U \uparrow G = \oplus (U_i \uparrow G)$.

• If U has char ψ , then the char of $U \uparrow G$ is denoted $\psi \uparrow G$.

Example 1: $G = S_3 = \langle a, b \mid a^3 = b^2 = e, ba = a^{-1}b \rangle \cong D_3$

$$H = \langle (1 \ 2 \ 3) \rangle = \langle a \rangle = C_3$$

Irred $\mathbb{C}[H]$ -submods are:

$$W_1 = \text{sp}\{e + a + a^2\}$$

$$W_2 = \text{sp}\{e + \omega a + \omega^2 a^2\}$$

$$W_3 = \text{sp}\{e + \omega^2 a + \omega a^2\}$$

Then:

$$W_1 \uparrow G = \text{sp}\{e + a + a^2, b + ba + ba^2\} \cong U_1 \oplus U_2$$

$$W_2 \uparrow G = \text{sp}\{e + \omega a + \omega^2 a^2, b + \omega ba + \omega^2 ba^2\} \cong U_3$$

$$W_3 \uparrow G = \text{sp}\{e + \omega^2 a + \omega a^2, b + \omega^2 ba + \omega ba^2\} \cong U_3$$

Thm 1: (Frobenius Reciprocity Thm:) Let ψ be a char of H and let χ be a char of G . Then

$$\langle \psi \uparrow G, \chi \rangle_G = \langle \psi, \chi \downarrow H \rangle_H$$

Prop 1: Let ψ be a char of H . Then let $\dot{\psi}: G \rightarrow \mathbb{C}$ be given by:

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\text{Then } \psi \uparrow G(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(ygy^{-1})$$

Ch 6) Repn Theory of S_n [Bruce Sagan - The symmetric gp]

The conjugacy classes of S_n are given by cycle-type:
e.g. $(1\ 2\ 3)(4\ 5)(6\ 7)$
3, 2, 2

These are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ of n , where $\lambda_i \in \mathbb{N}$, $\lambda_1 + \dots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k$.

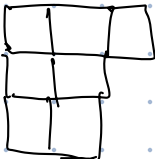
We write $|\lambda| = \sum \lambda_i$, and if $|\lambda| = n$, we write $\lambda \vdash n$.

$\uparrow$ is a partition of

These can be drawn as Young diagrams:

- λ_i boxes in row i
- row 1 at the top, row 2 just below, etc
- rows are left-justified.

Eg: $\lambda = (3, 2, 2) :$



$$\begin{aligned} \text{Let } S_\lambda \text{ be the subgroup of } S_n \text{ given by} \\ S_{\{1, 2, \dots, \lambda_1\}} \times S_{\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}} \times \dots \times S_{\{n-\lambda_k+1, n-\lambda_k+2, \dots, n\}} \\ \cong S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k} \end{aligned}$$

Now let M^λ be the module associated to $1 \uparrow_{S_\lambda}^{S_n}$. This is not generally irred. BUT we can order partitions of n such that:

- $M^{\lambda^{(1)}}$ is irred (call it $S^{\lambda^{(1)}}$)

- $M^{\lambda^{(2)}}$ contains copies of $S^{\lambda^{(1)}}$ and a unique copy of a new irred module, $S^{\lambda^{(2)}}$.

• etc

S^{λ} are called Specht module
 M^{λ} are called permutation modules corr to λ .

The ordering needed is reverse-lexicographic (revlex)
 (largest numbers to smaller numbers, starting with 1st component, and moving to later components to break ties).

Example:

Partions of 4:

$(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$

$M^{\lambda} = \mathbb{C}\text{-span} \{ \bar{t}_1, \dots, \bar{t}_k \}$ where $\bar{t}_i$ are distinct λ -tabloids:
 Fill the Young diagram of λ with $\{1, \dots, n\}$ (this is called a Young tableau t_i) and mark as equivalent if all corr. rows have same elts:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline 4 & & \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 4 & & \\ \hline \end{array} \sim \dots$$

This can be written as $\frac{1 \ 2 \ 3}{4}$

Examples:

1) IF $\lambda = (n)$, then
 $M^{(n)} = \text{sp} \{ \frac{1 \ 2 \dots n}{} \} \cong \mathbb{C}^1$
 S_n acts on this trivially.
 $1 \uparrow_{S_n}^{S_n} = 1$.

2) IF $\lambda = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}) = (1^n)$, then

$$M^{\lambda} = \text{sp} \left\{ \frac{i}{1 \dots i} \right\}_{i=1}^n \text{ } n \text{ rows}, \left\{ \frac{2}{1}, \frac{3}{1 \ 2}, \dots, \frac{n}{1 \ 2 \dots n-1} \right\}$$

($n!$ tabloids)

$$1 \uparrow_{S(1^n)}^{S_n} = 1_{S_1 \times S_1 \times \dots \times S_1}^{S_n} = \text{reg. char of } \mathbb{C}[S_n]$$

3) If $\lambda = (n-1, 1)$, then:

$$M^\lambda = \text{sp} \left\{ \overbrace{\quad}^{n-1 \text{ terms}} \right\}$$

$$= \text{sp} \left\{ \overline{\underline{1}}, \overline{\underline{2}}, \overline{\underline{3}}, \dots, \overline{\underline{n}} \right\}$$

so $1 \uparrow_{S_{n-1} \times S_1}^{S_n}$ gives $(n \text{ terms})$ the usual (defining) permutation repr.