

From last time:

Throughout, we set  $F = \mathbb{C}$ ,  $A = \mathbb{C}[G]$  for finite gp  $G$ .

Defn 1: Let  $G$  be a finite gp, and let  $F(G, \mathbb{C})$  be the space of functions  $G \rightarrow \mathbb{C}$ . Define an inner product on  $F(G, \mathbb{C})$  by:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Thm 1: Let  $U, V$  be irred  $\mathbb{C}[G]$ -modules with chars  $\chi, \psi$  resp.  
Then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{if } U \not\cong V. \end{cases} \quad [\chi = \psi \text{ in this.}]$$

Technical lemmas to prove Thm 1:

Lemma 1: Let  $U, V$  be irred  $\mathbb{C}[G]$ -mods  $\ncong$  repns  $\rho, \sigma$  resp.  
Let  $T: V \rightarrow U$  be a lin. map. Then  $\tilde{T}: V \rightarrow U$   
given by:

$$\tilde{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \circ T \circ \sigma(g).$$

is a  $\mathbb{C}[G]$ -hom. Furthermore,

i) if  $U \not\cong V$ ,  $\tilde{T}$  is the zero map

ii) if  $U \cong V, \rho = \sigma$ ,  $\tilde{T} = \lambda I_n$  ( $\lambda = \frac{1}{n} \text{tr}(T)$ )

Cor 1: Let  $\rho(g) = [r_{ij}(g)]$ ,  $\sigma(g) = [s_{ij}(g)]$  and  $T = [x_{ij}]$ . Then

$$\tilde{T}_{il} = \frac{1}{|G|} \sum_{g \in G} \sum_{j=1}^m \sum_{k=1}^n r_{ij}(g^{-1}) x_{jk} s_{kl}(g),$$

so if  $U \not\cong V$ , then

$$\frac{1}{|G|} \sum_{g \in G} r_{ij}(g^{-1}) s_{kl}(g) = 0 \quad \forall i, j, k, l$$

and if  $U \cong V$ ,  $\rho = \sigma$ , then:

$$\frac{1}{|G|} \sum_{g \in G} r_{ij}(g^{-1}) r_{kl}(g) = \frac{\delta_{il} \delta_{jk}}{n}.$$

Pf of Thm 1: IF  $\chi = \psi$ :

$$\begin{aligned}\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \left( \sum_i r_{ii}(g) \right) \left( \sum_j r_{jj}(g^{-1}) \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\delta_{ij}}{n} = 1\end{aligned}$$

IF  $\chi \neq \psi$  (and  $U \not\cong V$ ), then:

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \left( \sum_{i=1}^n r_{ii}(g) \right) \left( \sum_{j=1}^n s_{jj}(g^{-1}) \right) = 0.$$

□

Thm 2: Let  $V$  be a  $\mathbb{C}[G]$ -mod with char  $\psi$  s.t.  
 $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ , where each  $V_i$  is irred. Let  $W$  be an irred  $\mathbb{C}[G]$ -mod with char  $\chi$ . Then  $\langle \psi, \chi \rangle$  is an integer equal to the number of  $V_i$  isom to  $W$ .

Pf: Let  $\psi_i$  be the char of  $V_i$ . Then  $\psi = \psi_1 + \psi_2 + \dots + \psi_r$ , so  
 $\langle \psi, \chi \rangle = \langle \sum_i \psi_i, \chi \rangle = \sum_i \langle \psi_i, \chi \rangle$ .  
 But  $\psi_i, \chi$  are irred. chars, and by Thm 1,  $\langle \psi_i, \chi \rangle = 1$  iff  $V_i \cong W$  (and 0 otherwise), so  $\langle \psi, \chi \rangle$  equals the number of  $V_i$  isom to  $W$ . □

Cor 2: Let  $U$  and  $V$  be  $\mathbb{C}[G]$ -mods  $\forall$  chars  $\chi, \psi$  resp.  
 Then  $U \cong V$  iff  $\chi = \psi$ .

Pf: IF  $U \cong V$ , then  $\chi = \psi$  by Section 3.1, Prop 2.

IF  $U \not\cong V$  then  $\chi \neq \psi$  if  $U$  and  $V$  are irred (since  $\langle \chi, \psi \rangle = 0$  in that case). If  $U$  and  $V$  are not both irred, then there exists an irred mod.  $W$  with char  $\xi$  that appears with different multiplicity in  $U$  and  $V$ . Then  $\langle \chi, \xi \rangle \neq \langle \psi, \xi \rangle$ , so  $\chi \neq \psi$ . □

Thm 3: Let  $\chi$  be a char. of  $G$ . Then  $\langle \chi, \chi \rangle = 1$  iff  $\chi$  is irred.

Pf: ( $\Leftarrow$ .) By Thm 1.

( $\Rightarrow$ .) Suppose  $\chi$  is the char of a reducible module  $V$ . Then  $V = V_1 \oplus \dots \oplus V_r$  for some irred  $V_i$ . Let  $\chi_i$  be the char of  $V_i$ . Then  $\chi = \chi_1 + \dots + \chi_r$ , and so  $\langle \chi, \chi \rangle = \langle \sum_i \chi_i, \sum_i \chi_i \rangle \geq r$ . □

#### 4.1) The space of class functions on $G$

Throughout:  $G$  is a fin. gp.

$$\mathbb{F} = \mathbb{C}.$$

$$A = \mathbb{C}[G].$$

$F_c(G, \mathbb{C})$ : space of class functions on  $G$   
functions constant on conjugacy classes

Prop 1:  $\dim F_c(G, \mathbb{C}) = \#$  of conj. classes of  $G$ .

Pf: Let  $x_1, \dots, x_\ell$  be representatives of the distinct conj. classes of  $G$ .

Then  $f_i \in F_c(G, \mathbb{C})$  given by:

$$f_i(x) = \begin{cases} 1 & \text{if } x \text{ is conj. to } x_i \\ 0 & \text{otherwise} \end{cases}$$

is a basis for  $F_c(G, \mathbb{C})$ . □

Now, let  $\chi_1, \dots, \chi_k$  be a complete of distinct irred. chars of  $G$ .

Prop 2: The  $\chi_i$  are lin. indep.

Pf: Suppose  $\sum_{i=1}^k c_i \chi_i = 0$ . Then  $\forall j$ :

$$0 = \left\langle \sum_{i=1}^k c_i \chi_i, \chi_j \right\rangle = \sum_{i=1}^k c_i \langle \chi_i, \chi_j \rangle = c_j. \quad \square$$

Lemma 1: Let  $f \in F_c(G, \mathbb{C})$  and  $\rho$  be an irred. repr. of  $G$  on an  $n$ -dim. module  $V$  with char  $\chi$ . Then  $\mathcal{Q}: V \rightarrow V$  given by  $\mathcal{Q} = \sum_{g \in G} f(g) \rho(g)$  equals  $\lambda 1_V$  where  $\lambda = \frac{|G|}{n} \langle f, \bar{\chi} \rangle$ .

Pf: We observe  $\forall g \in G$ :

$$\rho(g)^{-1} \mathcal{Q} \rho(g) = \rho(g)^{-1} \sum_{x \in G} f(x) \rho(x) \rho(g) = \sum_{x \in G} f(x) \rho(g^{-1} x g)$$

$$= \sum_{y \in G} f(g y g^{-1}) \rho(y) = \sum_{y \in G} f(y) \rho(y) = \mathcal{Q}.$$

Thus,  $\mathcal{Q} \rho(g) = \rho(g) \mathcal{Q}$ , and hence  $\mathcal{Q}$  is an  $A$ -hom., so by Schur's Lemma,  $\mathcal{Q} = \lambda 1_V$  for some  $\lambda \in \mathbb{C}$ . In fact:  
 $\text{tr}(\mathcal{Q}) = n \lambda = \sum_{g \in G} f(g) \chi(g) = |G| \langle f, \bar{\chi} \rangle.$  □

Prop 3: Let  $\chi$  be a char. of  $G$ . Then  $\bar{\chi}$  is also a char of  $G$ , and  $\bar{\chi}$  is irred iff  $\chi$  is irred.

Pf: If  $\rho: G \rightarrow GL(n, \mathbb{C})$  is a repn, then so is  $\sigma(g) = \overline{\rho(g)}$ . □

Lemma 2: If  $f \in F_c(G, \mathbb{C})$  and  $\langle f, \chi_i \rangle = 0 \ \forall i$ , then  $f = 0$ .

Pf: Since  $\{\chi_1, \dots, \chi_k\} = \{\bar{\chi}_1, \dots, \bar{\chi}_k\}$ , we have  $\langle f, \bar{\chi}_i \rangle = 0 \ \forall i$ .  
Let  $\mathcal{Q}_\rho = \sum_{g \in G} f(g) \rho(g)$ .

If  $\rho$  is irred, then by Lemma 1,  $\mathcal{Q}_\rho = 0$ .

If  $\rho$  is reducible, then  $\mathcal{Q}_\rho$  acts by 0 on every irred submod, and so again  $\mathcal{Q}_\rho = 0$ .

Now, let  $\rho$  be the regular repn. Then for  $h \in G$ ,  
 $\mathcal{Q}_\rho(h) = \sum_{g \in G} f(g) \rho(g)h = \sum_{g \in G} f(g) gh = 0$

Since  $\{gh \mid g \in G\} = G$  forms a basis for the regular module, we have  $f(g) = 0 \ \forall g \in G$ . □

Prop 4: The  $\chi_i$  span  $F_c(G, \mathbb{C})$ .

Pf: Let  $f \in F_c(G, \mathbb{C})$ . Let  $\tilde{f} = f - \sum_{i=1}^k \langle f, \chi_i \rangle \chi_i$ . Then  
 $\langle \tilde{f}, \chi_j \rangle = \langle f, \chi_j \rangle - \sum_i \langle f, \chi_i \rangle \langle \chi_i, \chi_j \rangle = 0 \ \forall j$ ,  
and so  $\tilde{f} = 0$ . Thus,  $f = \sum_i \langle f, \chi_i \rangle \chi_i$ . □

Cor 1: The  $\chi_i$  form a basis for  $F_c(G, \mathbb{C})$  and the number of distinct irred. chars equals the number of distinct conjugacy classes.

Cor 2: Suppose  $g, h \in G$ . Then  $g$  is conj. to  $h$  iff  $\chi(g) = \chi(h)$  for all chars  $\chi$  of  $G$ .

Pf: ( $\Leftarrow$ ): Let  $f \in F_c(G, \mathbb{C})$  be the function s.t.  
 $f(x) = \begin{cases} 1 & \text{if } x \text{ is conj to } g \\ 0 & \text{otherwise.} \end{cases}$

Then, since  $\chi_i(g) = \chi_i(h) \ \forall$  irred chars  $\chi_i$  of  $G$ , we have  $f(h) = f(g) = 1$ , so  $h$  is conj. to  $g$ . □

Cor. 3: For all  $g \in G$ ,  $g$  is conj to  $g^{-1}$  iff  $\chi(g)$  is a real number for all chars.  $\chi$  of  $G$ .

## 4.2) Character Tables:

Let  $\chi_1, \chi_2, \dots, \chi_k$  be the irred chars of  $G$  and let  $g_1, \dots, g_k$  be reps of the distinct conj classes of  $G$ .

Defn 1: The  $k \times k$  <sup>array</sup> matrix whose  $(i, j)$ -entry is  $\chi_i(g_j)$  is called the character table of  $G$ .

Note: Since the  $\chi_i$  are lin. indep., this matrix is invertible.

Thm 1: i) Row orthogonality:

$$\sum_{i=1}^k \frac{\chi_r(g_i) \chi_s(g_i^{-1})}{|Z(g_i)|} = \delta_{rs} \quad (Z(g_i) = \{z \in G \mid zg_i = g_i z\} \text{ is called the centraliser of } g_i)$$

ii) Column orthogonality:

$$\sum_{i=1}^k \frac{\chi_i(g_r) \chi_i(g_s^{-1})}{|Z(g_s)|} = \delta_{rs}$$

PF: i)  $\delta_{rs} = \langle \chi_r, \chi_s \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_r(g) \chi_s(g^{-1})$

$$= \frac{1}{|G|} \sum_{i=1}^k \chi_r(g_i) \chi_s(g_i^{-1}) |Cl_G(g_i)|$$

where  $Cl_G(g_i)$  is the conjugacy class of  $g_i$ .  
By the Orbit-stabiliser Theorem,  $|G| = |Cl_G(g_i)| |Z(g_i)|$ .

ii) Let  $f_s \in F_C(G, \mathbb{C})$  s.t.  $f_s(g_r) = \delta_{rs}$ . Then  $f_s = \sum_{i=1}^k \langle f_s, \chi_i \rangle \chi_i$ , where

$$\langle f_s, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} f_s(g) \chi_i(g^{-1}) = \frac{1}{|G|} |Cl_G(g_s)| \chi_i(g_s^{-1}) = \frac{\chi_i(g_s^{-1})}{|Z(g_s)|}$$

This implies that

$$\delta_{rs} = f_s(g_r) = \sum_{i=1}^k \frac{\chi_i(g_r) \chi_i(g_s^{-1})}{|Z(g_s)|}$$

Examples:

1)  $G = C_3 = \langle x \mid x^3 = e \rangle = \{e, x, x^2\}$

| $\begin{matrix} g_i \\  Z(g_i)  \end{matrix}$ | $\begin{matrix} e \\ 3 \end{matrix}$ | $\begin{matrix} x \\ 3 \end{matrix}$ | $\begin{matrix} x^2 \\ 3 \end{matrix}$ |
|-----------------------------------------------|--------------------------------------|--------------------------------------|----------------------------------------|
| $\chi_1$                                      | 1                                    | 1                                    | 1                                      |
| $\chi_2$                                      | 1                                    | $\omega$                             | $\omega^2$                             |
| $\overline{\chi_2} = \chi_3$                  | 1                                    | $\omega^2$                           | $\omega$                               |

$$\omega := e^{2\pi i/3}$$

$$\Rightarrow \omega^3 = 1$$

Check orthog. of  $\chi_1$  and  $\chi_2$ :

$$\frac{1 \cdot 1}{3} + \frac{1 \cdot \omega^2}{3} + \frac{1 \cdot \omega}{3} = \frac{1}{3} (1 + \omega + \omega^2) = 0$$

Check orthog of  $\chi_2$  and  $\chi_3$ :

$$\frac{1 \cdot 1}{3} + \frac{\omega \cdot \overline{\omega^2}}{3} + \frac{\omega^2 \cdot \overline{\omega}}{3} = \frac{1}{3} + \frac{\omega^2}{3} + \frac{\omega}{3} = 0$$

2)  $G = D_3 = \langle r, f \mid r^3 = e, f^2 = e, fr = r^{-1}f \rangle$

Recall:  $|G| = \sum [\dim(V_i)]^2$

$$6 = 1 + 2^2 + 1$$

| $\begin{matrix} g_i \\  Cl_G(g_i)  \\  Z(g_i)  \end{matrix}$ | $\begin{matrix} e \\ 1 \\ 6 \end{matrix}$ | $\begin{matrix} r \\ 2 \\ 3 \end{matrix}$ | $\begin{matrix} f \\ 3 \\ 2 \end{matrix}$ |
|--------------------------------------------------------------|-------------------------------------------|-------------------------------------------|-------------------------------------------|
| $\chi_1$                                                     | 1                                         | 1                                         | 1                                         |
| $\chi_2$                                                     | 1                                         | 1                                         | -1                                        |
| $\chi_3$                                                     | 2                                         | $\boxed{-1}_x$                            | $\boxed{0}_y$                             |

$$r(\underline{f}r^{-1}) = r(rf)$$

$$= r^2f$$

$$r^2(\underline{f}r^{-2}) = r^2(r^2f)$$

$$= rf$$

$$1 + 1 + 2x = 0 \quad (\text{by col. orthog})$$

$$1 - 1 + 2y = 0$$

Prop 1: Suppose  $\chi$  is a char of  $G$  and  $\lambda$  is a linear char of  $G$ . Then the product  $\lambda\chi$  given by  $\lambda\chi(g) = \lambda(g)\chi(g)$  is also a char. If  $\chi$  is irred, then so is  $\lambda\chi$ .

Example

3)  $G = A_4$

| $g_i$ :<br>$ Cl_G(g_i) $ :<br>$ Z(g_i) $ : | id | $\pm 1$<br>$(12)(34)$<br>3<br>4 | $1, \omega, \omega^2$<br>$(123)$<br>3        | $(132)$<br>3 |
|--------------------------------------------|----|---------------------------------|----------------------------------------------|--------------|
| $\chi_1$                                   | 1  | 1                               | 1                                            | 1            |
| $\chi_2$                                   | 1  | $\boxed{1}_x$                   | $\omega$                                     | $\omega^2$   |
| $\overline{\chi_2} = \chi_3$               | 1  | 1                               | $\omega^2$                                   | $\omega$     |
| $\chi_4$                                   | 3  | $\boxed{-1}_y$                  | $\underbrace{0 \quad 0}_{\text{by Prop 1.}}$ |              |

$$\omega := e^{2\pi i/3}$$

$$1, 2 \text{ row orthog: } \frac{1 \cdot 1}{12} + \frac{1 \cdot x}{4} + \frac{\omega \cdot \omega}{3} + \frac{\omega^2 \cdot \omega^2}{3} = 0$$

$$\Rightarrow \frac{1}{12} + \frac{x}{4} + \frac{\omega^2}{3} + \frac{\omega}{3} = 0$$

$$\Rightarrow \frac{-1}{4} + \frac{x}{4} + \underbrace{\left( \frac{1}{3} + \frac{\omega}{3} + \frac{\omega^2}{3} \right)}_0 = 0$$

$$\Rightarrow x = 1$$

$$1, 4 \text{ row orthog: } \frac{3}{12} + \frac{y}{4} = 0 \Rightarrow y = -1$$