

From last time:

2.2) The space of $F[G]$ -homs:

Throughout: G fin. gp.

F alg. closed field s.t. $\text{char}(F) \nmid |G|$ (think: $F = \mathbb{C}$)

$A = F[G]$ (group alg.)

Prop 1: Suppose V, W are irred A -modules. Then

$$\dim(\text{Hom}_A(V, W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Defn 1: If V is an A -module and U is an irreducible A -module, we say U is a composition factor of V if V has an A -submodule that is isomorphic to U .

Prop 2: Let V, W be fin. dim. A -modules and suppose $\text{Hom}_A(V, W) \neq \{0\}$. Then V and W have a common composition factor.

Pf: Let $Q \in \text{Hom}_A(V, W)$ be nonzero. Then $V = \ker Q \oplus U$ for some submodule U of V . Let X be an irred. submodule of U . Then $Q(X) \neq \{0\}$, and so by Schur's Lemma, $\text{im}(Q) \cong X$ and $\text{im}(Q)$ is a submodule of W . $\square$

Prop 3: Let V_i, W_j be irred A -modules. Then:

($1 \leq i \leq n, 1 \leq j \leq m$)

$$\dim(\text{Hom}_A(\bigoplus_{i=1}^n V_i, \bigoplus_{j=1}^m W_j)) = \sum_{i=1}^n \sum_{j=1}^m \dim(\text{Hom}_A(V_i, W_j))$$

(PF: HW)

Cor 1: Let V be an A -module with $V = U_1 \oplus \dots \oplus U_r$, where each U_i is an irred. submodule. Let W be any irred. A -mod. Then $\dim(\text{Hom}_A(V, W))$ and $\dim(\text{Hom}_A(W, V))$ both equal the number of submodules U_i s.t. $U_i \cong W$.

Prop 4: If U is an A -module, then $\dim(\text{Hom}_A(A, U)) = \dim U$. regular repr

Pf: Let $d = \dim U$. Let $u_1, \dots, u_d$ be a basis for U . Define

$$\begin{aligned} \varphi_i : A &\rightarrow U && \text{by} \\ \varphi_i(a) &= au_i \end{aligned}$$

Then these are linear maps s.t. $\forall a, b \in A$, we have

$$\varphi_i(ab) = (ab)u_i = a(bu_i) = a(\varphi_i(b)),$$

so $\varphi_i \in \text{Hom}_A(A, U)$.

We now wish to show the φ_i form a basis.
Let $\psi \in \text{Hom}_A(A, U)$. Then:

1e, 1eFF, ceG
 $\psi(1) = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_d u_d$ for some $\lambda_i \in \mathbb{F}$.

Since ψ is an A -hom, we have

$$\psi(a) = \psi(a1) = a\psi(1) = a[\sum \lambda_i u_i] = \sum \lambda_i au_i = \sum \lambda_i \varphi_i(a).$$

Thus, the φ_i span $\text{Hom}_A(A, U)$.

Now suppose $\sum \lambda_i \varphi_i = 0$. [WTS: $\lambda_i = 0 \forall i$]. Then, by evaluating both sides at $1 \in A$, we get:

$$\sum \lambda_i u_i = 0.$$

Since the u_i are a basis for U , we have $\lambda_i = 0 \forall i$.
Thus, the φ_i are lin. indep, and hence a basis.

Therefore, $\dim(\text{Hom}_A(A, U)) = d = \dim U$. □

Thm 1: Suppose $A = \mathbb{F}[G] = U_1 \oplus \dots \oplus U_r$, where each U_i is an irred A -submod. If U is any irred. A -module, then the number of A -modules U_i with $U_i \cong U$ is equal to $\dim U$.

Pf: By Prop 4, $\dim U = \dim(\text{Hom}_A(A, U))$, and by Cor 1, $\dim(\text{Hom}_A(A, U))$ equals the number of submodules U_i isom to U . □

Thm 2: Let $V_1, \dots, V_k$ form a complete set of non-isomorphic irred $\mathbb{F}[G]$ -modules. Then:

$$|G| = \sum_{i=1}^k (\dim V_i)^2.$$

Example 1: Let G be a gp of order 8. Then the possible dimensions d_i of irred. modules of G are:

1, 1, 1, 1, 1, 1, 1, 1 (abelian)

1, 1, 1, 1, 2 (non-abelian)

Note: For all algebras $\mathbb{F}[G]$, we can define a repn on a 1-dim vect. sp V (so $V \cong \mathbb{F}$) given by:

$$G \times V \rightarrow V$$

$$(g, v) \mapsto v$$

This is irred, since V has dim 1, and is called the trivial repn (correspondingly, V is the trivial module).

3.1) Characters:

Repeat assumptions:

Throughout: G fin. gp.

$\mathbb{F}$ alg. closed field s.t. $\text{char}(\mathbb{F}) \nmid |G|$ (think: $\mathbb{F} = \mathbb{C}$)

$A = \mathbb{F}[G]$ (group alg.)

Defn 1: Let A be an alg. with repn. ρ on a fin. dim. vect. sp V . Then the character of ρ (or of V) is the linear map $\chi: A \rightarrow \mathbb{F}$ given by:

$$\chi(a) = \text{tr}(\rho(a)).$$

↓
sum of eigenvalues

G , rather than $\mathbb{F}[G]$

For repns. of finite gps, we often take the character to be $\chi: G \rightarrow \mathbb{F}$.

Recall: $\cdot \operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
 $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
 $\Rightarrow \operatorname{tr}(BAB^{-1}) = \operatorname{tr}(A) \quad \star$

Defn 2: A function $f: G \rightarrow \mathbb{F}$ satisfying $f(sts^{-1}) = f(t) \quad \forall s, t \in G$ is called a class function. [These are constant on conjugacy classes.]

Defn 3: $\cdot$ We say χ is a char. of G if χ is the char of some repn of G .

$\cdot$ We say χ is irred if it is the char of an irred repn.

Defn 4: The character of a 1-dim repn. is called a linear character. [In this case, the character is the repn.]

Prop 1: Let χ be the char of a repn ρ of G on V . Then for $s, t \in G$:

i) $\chi(e) = \dim V$. [This is called the degree of V .]

ii) $\chi(sts^{-1}) = \chi(t)$.

Prop 2: If V and W are isom. A -modules, then they have the same character.

$$\rho: A \rightarrow \operatorname{End}(V)$$

$$\sigma: A \rightarrow \operatorname{End}(W)$$

$$T: V \rightarrow W \quad \text{s.t.} \quad \sigma(a) \circ T = T \circ \rho(a) \quad \forall a \in A.$$

$\downarrow$
isom

$$\sigma(a) = \underbrace{T \circ \rho(a) \circ T^{-1}}$$

This preserves trace.

Prop 3: Let $\mathbb{F} = \mathbb{C}$, let χ be a char of G , let $g \in G$ be an elt. of order m . Then:

i) $\chi(g)$ is a sum of m^{th} roots of unity

ii) $\chi(g^{-1}) = \overline{\chi(g)}$

$\hookrightarrow$ complex conjugate.

Pf: i) Let $H = \langle g \rangle$. Then ~~H is cyclic and~~ any $\mathbb{C}[G]$ -mod. V is also a $\mathbb{C}[H]$ -module. Since H is cyclic, we can decompose $V = U_1 \oplus \dots \oplus U_n$, where the U_i are irred $\mathbb{C}[H]$ -modules, and hence 1-dim. Taking $u_i \in U_i$ to be nonzero, we have $u_1, \dots, u_n$ is a basis $\mathcal{B}$ of V and $[g]_{\mathcal{B}}$ is a diagonal matrix.

↳ g written as a matrix relative to $\mathcal{B}$.

Since $g^m = e$, for each i , $g u_i = \omega_i u_i$ where $\omega_i \in \mathbb{C}$ is an m^{th} root of unity. Thus, the trace of $[g]_{\mathcal{B}}$ is the sum of the ω_i .

$$\text{ii) } [g^{-1}]_{\mathcal{B}} = \text{diag} \{ \omega_1^{-1}, \omega_2^{-1}, \dots, \omega_n^{-1} \} = \text{diag} \{ \overline{\omega_1}, \overline{\omega_2}, \dots, \overline{\omega_n} \}.$$

$$\text{So } \chi(g^{-1}) = \overline{\chi(g)}.$$

□

3.2) Inner products of characters

Throughout: $\mathbb{F} = \mathbb{C}$
 G : fin. gp
 $A = \mathbb{C}[G]$.

Defn 1: Let G be a finite gp, and let $F(G, \mathbb{C})$ be the space of functions $G \rightarrow \mathbb{C}$. Define an inner product on $F(G, \mathbb{C})$ by:

$$\underbrace{\langle f_1, f_2 \rangle}_{\substack{\downarrow \\ \text{inner prod of } f_1 \text{ and } f_2}} = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Prop 1: Let G be a fin. gp and let ϕ and ψ be characters of G . Then $\langle \phi, \psi \rangle$ is a real number.

$$\begin{aligned} \text{Pf: } \overline{\langle \phi, \psi \rangle} &= \overline{\frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}} = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \phi(g^{-1}) \psi(g) = \frac{1}{|G|} \sum_{g \in G} \phi(g) \psi(g^{-1}) = \langle \phi, \psi \rangle. \end{aligned}$$

↳ Sum over all elts
 = sum over all inverses

□

Thm 1: Let U, V be irred $\mathbb{C}[G]$ -mods with chars χ, ψ resp.

Then:

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{if } U \not\cong V. \end{cases} \quad [\chi = \psi \text{ in this case}]$$

To prove this, we need some preliminary results:

Lemma 1: Let U, V be irred $\mathbb{C}[G]$ -mods with reps ρ, σ resp.

Let $T: V \rightarrow U$ be a lin. map. Then the map $\tilde{T}: V \rightarrow U$ given by:

$$\tilde{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \circ T \circ \sigma(g)$$

is a $\mathbb{C}[G]$ hom. Furthermore:

- i) if $U \not\cong V$, $\tilde{T}$ is the zero map.
- ii) if $U \cong V$ as $\mathbb{C}[G]$ -mods (so $\rho = \sigma$), then $\tilde{T} = \lambda I_n$ where $\lambda = \frac{1}{n} \text{tr}(T)$.