

2 Representations of Finite Groups

2.1 Semisimple Representations

Throughout: Let A be an associative algebra over a field $\mathbb{F}$.

Exercise 1 (Week 1, Exercise 7). Suppose V is an A -module and W is a submodule. Then V/W is an A -module also.

Proof. Define a map $A \times V/W \rightarrow V/W$ by $(a, v + W) \mapsto av + W$.

Suppose $v + W = v' + W$. Then $v - v' \in W$, so $av - av' \in W$, and $av + W = av' + W$. Thus we have a well-defined representation. $\square$

Proposition 2.1.1. *Let $T : V \rightarrow W$ be a homomorphism of A -modules. Then $V / \ker(T) \cong \text{im}(T)$ (isomorphic as A -modules).*

Definition 2.1.2. *A **semisimple** (or **completely reducible**) A -module is a direct sum of irreducible submodules (i.e. simple submodules).*

Proposition 2.1.3. *Let A be finite dimensional. Then every irreducible A -module V is finite dimensional. (Dimensions are taken as vector space dimensions for both A and V .)*

Proof. Let V be an irreducible A -module. Let $v \in V$ be nonzero. Then $Av = \{av \mid a \in A\}$ is a submodule of V that is finite-dimensional. But V is irreducible and $Av \neq 0$, so $Av = V$. Thus V is finite dimensional. $\square$

Now we specialise to the case $A = \mathbb{F}[G]$ for some group G . [Note that if G is a finite group, then A is finite dimensional.]

Definition 2.1.4. *Let G be a group, and $A = \mathbb{F}[G]$. Then a **representation** of G over $\mathbb{F}$ is a representation of A , i.e., a map $\rho : A \rightarrow \text{End}(V)$ satisfying $\rho(ab) = \rho(a)\rho(b)$ and $\rho(1) = \text{Id}_V$. **Note:** This restricts to a group homomorphism $\rho : G \rightarrow \text{GL}(V)$, which is also referred to as the representation of the group G .*

Example 2.1.5. *Let $\mathbb{F} = \mathbb{C}$ and $G = D_4 = \langle r, f \mid r^4 = e = f^2, fr = r^{-1}f \rangle$ (the group of symmetries of a 4-gon, which is sometimes referred to as D_8 instead of D_4).*

1. $\rho : G \rightarrow \text{GL}(2, \mathbb{C})$ given by $\rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \rho(f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2. $\rho : G \rightarrow \text{GL}(n, \mathbb{C})$ given by $\rho(x) = \text{Id}_n$ for all $x \in G$ (degree n trivial representation)
3. $\rho : G \rightarrow \text{GL}(1, \mathbb{C}) \cong \mathbb{C}^\times$ given by $\rho(r) = \pm 1, \rho(f) = -1$.
4. $\rho : G \rightarrow \text{GL}(4, \mathbb{C})$ given by $\rho(r) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \rho(f) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Theorem 2.1.6 (Maschke's Theorem). *Let G be a finite group and let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Then every finite dimensional $\mathbb{F}[G]$ -module is semisimple.*

Non-example: $G = C_3 = \langle x \mid x^3 = e \rangle$, $\mathbb{F} = \mathbb{F}_3 \cong \mathbb{Z}/3\mathbb{Z}$ (field of order 3), $V = \mathbb{F}^2$.

$$\rho : G \rightarrow \mathrm{GL}(2, \mathbb{F}), \quad x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow x^j \mapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$$

$$U = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \text{ is a submodule of } V, \text{ but it has no complement.}$$

Proof of Maschke's Theorem. Let V be an $\mathbb{F}[G]$ -module and U a submodule.

[We wish to show that there exists an $\mathbb{F}[G]$ -submodule W of V such that $V = U \oplus W$.]

Let W_0 be a subspace (not necessarily a submodule!) of V such that $V = U \oplus W_0$. Define $\Phi : V \rightarrow V$ by $\Phi(v) = u$ where $v = u + w$ for $u \in U$, $w \in W_0$. We wish to modify Φ to a suitable $\mathbb{F}[G]$ -homomorphism. Let $\Psi : V \rightarrow V$ be given by

$$\Psi(v) = \frac{1}{|G|} \sum_{g \in G} g \Phi(g^{-1}v).$$

Then Ψ is a linear map such that $\mathrm{im}(\Psi) = U$ and in fact $\Psi(u) = u$ for all $u \in U$. To see that this is an $\mathbb{F}[G]$ -module homomorphism, we observe that for $h \in G$:

$$\Psi \circ h(v) = \Psi(hv) = \frac{1}{|G|} \sum_{g \in G} g \Phi(g^{-1}hv) = \frac{1}{|G|} \sum_{k \in G} hk^{-1} \Phi(k^{-1}v) = h\Psi(v).$$

Now let $W = \ker(\Psi)$. Then W is a submodule of V , and $V = \mathrm{im}(\Psi) \oplus \ker(\Psi) \cong U \oplus W$. □

Exercise 2. $G = C_3 = \langle x \rangle$, $V = \langle v_1, v_2, v_3 \rangle_{\mathbb{C}}$, $x \cdot v_i = v_{i+1 \bmod 3}$.

$$U = \langle v_1 + v_2 + v_3 \rangle, \quad W_0 = \langle v_1, v_2 \rangle$$

Let $\Phi : V \rightarrow V$ be given by $\Phi(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) \mapsto \lambda_3(v_1 + v_2 + v_3)$.

Find Ψ and W .

Corollary 2.1.7. *Let G be a finite group and $\mathbb{F}$ a field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then any irreducible $\mathbb{F}[G]$ -module is a direct summand of the regular $\mathbb{F}[G]$ -module (where $\mathbb{F}[G]$ acts on $\mathbb{F}[G]$ by left multiplication.)*

Proof. Let U be an irreducible $\mathbb{F}[G]$ -module. Let $u \in U$ be nonzero. Define an $\mathbb{F}[G]$ -module homomorphism $T : \mathbb{F}[G] \rightarrow U$ by

$$T \left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} \lambda_g g u, \quad \lambda_g \in \mathbb{F}.$$

We can verify that T is an $\mathbb{F}[G]$ -module homomorphism. Since U is irreducible and $\mathrm{im}(T) \neq 0$ (since e.g., $T(1) = T(e) = u \neq 0$), we must have $\mathrm{im}(T) = U$. Then by Maschke's Theorem, $\mathbb{F}[G] = \ker(T) \oplus W$ for some submodule W of V . Since we also know that $\mathbb{F}[G]/\ker(T) \cong \mathrm{im}(T) = U$, we have $U \cong W$. □

Corollary 2.1.8. *If G is a finite group, then there are only finitely many non-isomorphic irreducible $\mathbb{F}[G]$ -modules when $\text{char}(\mathbb{F}) \nmid |G|$.*

Natural Question: With what multiplicity does each irreducible module appear as a direct summand of $\mathbb{F}[G]$ (the regular $\mathbb{F}[G]$ -module)?

2.2 The space of $\mathbb{F}[G]$ -homomorphisms

Throughout: Let G be a finite group, let $\mathbb{F}$ be an algebraically closed field such that $\text{char}(\mathbb{F}) \nmid |G|$ (for example, $\mathbb{F} = \mathbb{C}$), and let $A = \mathbb{F}[G]$.

Proposition 2.2.1. *Suppose V, W are irreducible A -modules. Then*

$$\dim(\text{Hom}_A(V, W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Proof. If $V \not\cong W$, then $\text{Hom}_A(V, W) = 0$ by Schur's Lemma.

Suppose $V \cong W$. Let $\Phi : V \rightarrow W$ be an A -module isomorphism. Now consider any $\Psi \in \text{Hom}_A(V, W)$. Then $\Phi^{-1}\Psi$ is an A -module homomorphism $V \rightarrow V$, and hence $\Phi^{-1}\Psi = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{F}$ by the second version of Schur's Lemma. Therefore $\Psi = \lambda\Phi$, so $\text{Hom}_A(V, W) = \{\lambda\Phi \mid \lambda \in \mathbb{F}\}$. $\square$