

Representation Theory of Finite Groups

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1 Fundamentals

1.1 Algebras

Throughout: Let $\mathbb{F}$ be a field.

Definition 1.1.1 (Recall). A **vector space** V over $\mathbb{F}$ is a set V together with operations:

- (i) $+: V \times V \rightarrow V$ (*addition*)
- (ii) $\cdot: \mathbb{F} \times V \rightarrow V$ (*scalar multiplication*)

such that

- (a) V is an abelian group under $+$
- (b) $\forall u, v \in V, \forall \lambda, \mu \in \mathbb{F}$:
 - 1. $\lambda(u + v) = \lambda u + \lambda v$
 - 2. $(\lambda + \mu)v = \lambda v + \mu v$
 - 3. $(\lambda\mu)v = \lambda(\mu v)$
 - 4. $1_{\mathbb{F}}v = v$

Definition 1.1.2. An **associative algebra** A over $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a bilinear map $A \times A \rightarrow A, (a, b) \mapsto ab$, such that $(ab)c = a(bc)$.

Definition 1.1.3. A **unit** in an associative algebra A is an element $1_A \in A$ such that

$$1_A a = a 1_A = a.$$

Proposition 1.1.4. *If A has a unit, then it must be unique.*

Proof. Suppose $1, 1'$ are units in A . Then $1 = 1 \cdot 1' = 1'$. □

Henceforth, by an *algebra* we will mean an associative algebra with a unit.

Example 1.1.5.

1. $A = \mathbb{F}$
2. $A = \mathbb{F}[x_1, x_2, \dots, x_n]$ (*polynomial algebra* over $\mathbb{F}$)
3. Let V be a vector space over $\mathbb{F}$. Then $A = \text{End}(V)$ is an algebra, where $\text{End}(V)$ is the set of all linear maps $V \rightarrow V$.
4. The *free algebra* $A = \mathbb{F}\langle x_1, \dots, x_n \rangle$ has basis consisting of words in $x_1, \dots, x_n$, and multiplication given by concatenation, e.g., $(x_1x_2x_1) \cdot (x_3x_1) = x_1x_2x_1x_3x_1$.
5. Let G be a group. The *group algebra* $A = \mathbb{F}[G]$ has basis indexed by elements of G , and multiplication given by the binary operation on G . (Elements have the form $\lambda_1g_1 + \lambda_2g_2 + \dots + \lambda_ng_n, g_i \in G, \lambda_i \in \mathbb{F}$.)

Definition 1.1.6. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$.

Definition 1.1.7. A **subalgebra** of an algebra A is a vector subspace B of A such that $b_1b_2 \in B$ for all $b_1, b_2 \in B$.

Definition 1.1.8. A **homomorphism of algebras** $T : A \rightarrow B$ is a linear map such that $T(xy) = T(x)T(y)$ for all $x, y \in A$, and $T(1_A) = 1_B$.

1.2 Representations

Definition 1.2.1. A **representation** of an algebra A on a vector space V is an algebra homomorphism $\rho : A \rightarrow \text{End}(V)$. In this case, we call V a (left) **A -module**.

Notation: We will often write $\rho(a)v$ as simply av for $a \in A, v \in V$. In this way, we treat $\rho : A \rightarrow \text{End}(V)$ as a map $A \times V \rightarrow V$. Then the homomorphism property is given by $(ab)v = a(bv)$.

Example 1.2.2.

1. A any algebra, $V = 0, \rho$ zero map.
2. $V = A, \rho : A \rightarrow \text{End}(A)$ given by $\rho(a)b = ab$. This is called the **regular representation** of A .
3. $A = \mathbb{F}\langle x_1, x_2, \dots, x_n \rangle$. Then any left A -module is a vector space V over $\mathbb{F}$ with a collection of arbitrary linear maps $\rho(x_1), \rho(x_2), \dots, \rho(x_n) \in \text{End}(V)$.

Definition 1.2.3. A **submodule** of an A -module V is a subspace $W \leq V$ which is invariant under $\rho(a) \in \text{End}(V)$ for all $a \in A$, i.e. $\rho(a) : W \rightarrow W$.

The homomorphism $\rho_W : A \rightarrow \text{End}(W)$ is a **subrepresentation** of ρ .

Definition 1.2.4. An A -module $V \neq 0$ (and its associated representation) is **irreducible** (or **simple**) if its only submodules are 0 and V .

Definition 1.2.5. Let $\rho : A \rightarrow \text{End}(V)$ and $\sigma : A \rightarrow \text{End}(W)$ be two representations of A . Then a **homomorphism** (or **intertwining operator**) $\Phi : V \rightarrow W$ is a linear map which commutes with the action of A , i.e.,

$$\Phi \circ \rho(a) = \sigma(a) \circ \Phi$$

for all $a \in A$. Therefore the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho(a)} & V \\ \Phi \downarrow & & \downarrow \Phi \\ W & \xrightarrow{\sigma(a)} & W \end{array}$$

- A homomorphism of representations is said to be an **isomorphism** (or **equivalence**) of representations if it is bijective.
- The set of all homomorphisms of representations $V_1 \rightarrow V_2$ is denoted $\text{Hom}_A(V_1, V_2)$.

Definition 1.2.6. Let V_1 and V_2 be A -modules. Then $V_1 \oplus V_2$ is also an A -module under the map $a(v_1, v_2) \mapsto (av_1, av_2)$. This is called the **direct sum** of V_1 and V_2 .

Definition 1.2.7. A nonzero A -module V is said to be **indecomposable** if it is not isomorphic to a direct sum of two nonzero modules.

Note. Irreducible $\Rightarrow$ indecomposable, but irreducible $\not\Rightarrow$ indecomposable.

Example 1.2.8. Let $A = \mathbb{F}[x]$ and $V = \mathbb{F}^2$, and $\rho : A \rightarrow \text{End}(V)$ be given by $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $\rho(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is a submodule.

Hence V is not irreducible. But if $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \notin W$, then $v_2 \neq 0$, so

$$\rho(x) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix} \neq \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ for any } \lambda \in \mathbb{F}.$$

So there is no other one-dimensional submodule for V besides W , and therefore $V \neq W \oplus U$ for any submodule U . Thus, V is indecomposable.

Proposition 1.2.9 (Schur's Lemma). *Let V_1, V_2 be modules of an algebra A over a field $\mathbb{F}$ (not necessarily algebraically closed). Let $\Phi : V_1 \rightarrow V_2$ be a nonzero homomorphism of representations. Then:*

- (i) *If V_1 is irreducible, then Φ is injective.*
- (ii) *If V_2 is irreducible, then Φ is surjective.*

Thus, if V_1 and V_2 are both irreducible, then Φ is an isomorphism.

Exercise 1.2.10.

- $\ker \Phi = \{v \in V_1 \mid \Phi(v) = 0\}$ is a submodule of V_1 .
- $\operatorname{im} \Phi = \{\Phi(v) \in V_2 \mid v \in V_1\}$ is a submodule of V_2 .

Proof of Schur's Lemma. (i) We observe $\ker \Phi$ is a submodule of V_1 . Since $\Phi \neq 0$ and V_1 is irreducible, we must have $\ker \Phi = 0$. Thus Φ is injective.

(ii) The image I of Φ is a submodule of V_2 . Since $\Phi \neq 0$ and V_2 is irreducible, we must have $I = V_2$. Thus Φ is surjective. $\square$

Corollary 1.2.11 (Schur's lemma for algebraically closed fields). *Let V be a finite-dimensional irreducible A -module over an algebraically closed field $\mathbb{F}$, and let $\Phi : V \rightarrow V$ be an intertwining operator. Then $\Phi = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$.*

Proof. Let λ be an eigenvalue of Φ , which exists because $\mathbb{F}$ is algebraically closed. Then $\Phi - \lambda \operatorname{Id}$ is also an intertwining operator which is not an isomorphism. Then by Proposition 1.2.9, the operator is the 0 map. Thus $\Phi = \lambda \operatorname{Id}$. $\square$

Corollary 1.2.12. *Let A be a commutative algebra over an algebraically closed field $\mathbb{F}$. Then every finite-dimensional irreducible A -module V is 1-dimensional.*

Proof. Let V be irreducible. Then for all $a \in A$, $\rho(a) : V \rightarrow V$ is an intertwining operator. Thus by Schur's lemma, $\rho(a)$ is a scalar operator for all $a \in A$. Hence every subspace of V is a submodule. So if V is irreducible, it cannot have any subspaces, and hence $\dim V = 1$. $\square$