Lecture 5

Harmonic functions and the Dirichlet principle

In this lecture we initiate a discussion about the existence of (non-constant) meromorphic functions on a compact Riemann surfaces, or equivalently holomorphic functions to the Riemann sphere $\mathbb{CP}^1$.

We start with the following very basic remark, which might be obvious to everyone but which helps with locating the difficulty. Since every point of a Riemann surface has a neighbourhood biholomorphic to a disk in $\mathbb{C}$, and that we know that the disk has *very very many* holomorphic functions, **the difficulty in building meromorphic functions is certainly not local**. What is difficult is to patch many holomorphic functions defined locally to a **global** meromorphic function.

Because the problem at hand is of a very different nature than everything we have done so far (we are leaving the world of *algebra* to do some *analysis*), we are going to take the long route, to recast the problem of the existence of meromorphic functions in its natural context. So instead of discussing meromorphic functions on Riemann surfaces, we are going to focus on *harmonic functions* on the disk, with boundary conditions, because it is the baby version of the problem. I think that if one understands this case well enough the (formally) more complicated case of meromorphic functions becomes a formality.

5.1 Holomorphic functions and harmonic functions

5.1.1 The analytic take on holomorphic functions

A holomorphic function is usually defined as a differentiable function seen as a function of one complex variable, *i.e.* f is holomorphic at a point z_0 if there exists $w \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + w \cdot h + o(|h|).$$

One could reproach to this definition that it is hiding a fair amount of differential calculus. A map $f : \mathbb{C} \longrightarrow \mathbb{C}$ is nothing but a map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

To say that f holomorphic at z_0 amounts to saying two things:

- f is differentiable as a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$;
- Df_{z_0} the differential of f at z_0 is a similarity, *i.e.* is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or if you prefer it $\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

The relation between the complex number w and a, b, λ, θ is given by $a = \mathrm{Re}(w)$, $b = \mathrm{Im}(w)$, $\lambda = |w|$ and $\theta = \arg(w)$. There is therefore a fair bit of complicated two-dimensional calculus hidden behind the derivative of a holomorphic function (given as a complex number).

If we write the variable $z = x + iy$ and $f = u + iv$ with u and v real-valued functions, the matrix of differential Df_{z_0} in coordinates (x, y) is just $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$. The fact that f is holomorphic thus translates to the following.

Proposition 5.1.1 (Cauchy-Riemann equations). *Let U be an open subset of $\mathbb{C}$. The function $f : U \longrightarrow \mathbb{C}$ is holomorphic on U if and only we have*

- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$;
- $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$;

where u and v are the real and imaginary parts of f.

In particular, using these two equations one easily shows the following statement.

PROPOSITION 5.1.2. *If $u : U \longrightarrow \mathbb{R}$ is the real (or imaginary) part of a holomorphic function $f : U \longrightarrow \mathbb{C}$ then we have*

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$$

on U.

The converse is also true, if $u : U \longrightarrow \mathbb{R}$ is a harmonic function (*i.e.* which satisfies $\Delta u = 0$), there exists a unique holomorphic function (up to constants) such that $u = \mathrm{Re}(u)$. Building holomorphic functions is therefore pretty much equivalent to building harmonic functions.

5.1.2 The Dirichlet problem

Here we introduce a problem, which although not directly related to the existence of meromorphic functions on Riemann surfaces, encapsulates all the analytic difficulty one will encounter.

Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$ and $g : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be a continuous function.

Dirichlet problem. Find a continuous function $u : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$, of class $\mathcal{C}^2$ on $\mathbb{D}$ such that

- the restriction of u to $\partial\mathbb{D}$ is equal to g;
- $\Delta u \equiv 0$ on $\mathbb{D}$.

This problem is very similar in nature to find meromorphic functions:

- harmonic functions (functions satisfying $\Delta u = 0$) are real parts of holomorphic functions;
- this problem really is a global problem, as the value of u at the boundary of the open set we are considering is prescribed.

One could be inclined to believe that cutting out a Riemann surfaces into discs along curves, and prescribing values on these curves, one could reduce the problem of finding holomorphic/meromorphic functions to a finite number of Dirichlet problems. It's not quite as simple, but not that far off, and understanding how to solve this Dirichlet problem is almost as good as finding merormorphic functions, as far as methods of proof are concerned.

5.2 Dirichlet principle

We dedicate the rest of this lecture to explaining how to prove the following theorem.

Theorem 5.2.1 (Dirichlet principle). *For any $g : \partial\mathbb{D} \longrightarrow \mathbb{R}$ continuous, there exists a unique continuous function $u : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^\infty$ on $\mathbb{D}$ such that*

- *u is harmonic on $\mathbb{D}$;*
- *the restriction of u to $\partial\mathbb{D}$ is equal to g.*

5.2.1 Basic calculus of variations

It is a very basic remark which does absolute wonders: if a differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as a local extremum at x, then $\nabla f(x) = 0$. So if you are looking to show the existence of solution to an equation of the form $\varphi(x) = 0$ with

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and you can find $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\Delta\Phi = \varphi$, all you've got to do is to find an extrem x_0 a for the function(nal) Φ.

5.2.2 Harmonic functions as minimisers

We are going to treat the case of harmonic functions over $\mathbb{R}$, which is technically trivial but which allows to see how things work without the burden of too heavy notation. Assume we try to solve

$$\Delta f = 0$$

on an interval $I = [a, b]$ with $f(a) = \alpha$ and $f(b) = \beta$ (which corresponds to fixing the value of f on the boundary. We are going to ignore for this discussion that $\Delta f = f''$ and that we can easily solve this differential equation to find that the affine map $f(x) := \frac{\beta}{b-a}(x - a) + \frac{\alpha}{a-b}(x - b)$ is the unique solution. Instead we consider

$$\begin{array}{rccc} \Phi & : & \mathcal{C}^2([a,b],\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & f & \longmapsto & \int_a^b |f'(x)|^2 dx \end{array}.$$

Proposition 5.2.2. *Let u be a function of class $\mathcal{C}^2$ such that*

$$\Phi(u) = \inf_{f \in \mathcal{C}^2([a,b],\mathbb{R}), f(a)=\alpha, f(b)=\beta} \Phi(f).$$

Then $\Delta u = 0$.

Proof: The proof is essentially: if Φ achieves its minimum at u, its differential at $\mathrm{D}\Phi(u)$ has to vanish, but $\mathrm{D}\Phi(u) = 0$ implies Δu.

Now, we don't want to bother introducing spaces where $\mathrm{D}\Phi$ actually makes sense. That would force us to introduce Banach manifolds, and we'd have to deal with the issue that a linear map in infinite dimension is not always continuous, which is a bummer. We are going to use this useful trick of restricting Φ to 1-dimensional families so we can use usual calculus. To all practical purposes its the same thing as considering $\mathrm{D}\Phi$, but at no point will we ever need to define $\mathrm{D}\Phi$.

So, if $\epsilon : [a, b] \longrightarrow \mathbb{R}$ is a $\mathcal{C}^2$ function such that $\epsilon(a) = \epsilon(b) = 0$, we have that $g_t = u + t \cdot \epsilon$ is $\mathcal{C}^2$ and $g_t(a) = \alpha$ and $g_t(b) = \beta$ for all parameters t. So the function

$$\varphi := t \mapsto \Phi(g_t)$$

achieves a minimum at 0. But

$$\varphi(t) = \Phi(g_t) = \int ||u'(x) + t \cdot \epsilon'(x)||^2 dx$$

$$\varphi(t) = \int ||u'(x)||^2 dx + 2t \int u'(x)\epsilon'(x) dx + t^2 \int ||\epsilon'(x)||^2 dx.$$

This way we see that φ is differentiable (it's a degree 2 polynomial) and $\varphi'(0) = \int u'(x)\epsilon'(x)dx$. Since φ is minimal at 0, we get $\int u'(x)\epsilon'(x)dx = 0$.

An important step which often goes unnoticed is that an **integration by parts** transforms $\int u'(x)\epsilon'(x)dx$ into $\int \Delta u(x)\epsilon(x)dx$. We have obtained the following statement.

If u is a minimiser of Φ, then for every $\epsilon : [a, b] \longrightarrow \mathbb{R}$ vanishing on $\{a, b\} = \partial[a, b]$ we have

$$\int_a^b \Delta u(x)\epsilon(x) dx = 0.$$

This is enough to conclude that $\Delta u = 0$ everywhere. ■

This Proposition and its proof generalise almost verbatim to the 2-dimensional case. We just replace $[a, b]$ with the unit disk $\mathbb{D} \subset \mathbb{R}^2$ and Φ with

$$\begin{array}{cccc} \Phi & : & \mathcal{C}^2(\mathbb{D}, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & f & \longmapsto & \int_a^b ||\nabla f(x, y)||^2 dx dy \end{array}.$$

PROPOSITION 5.2.3. *Let $g : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be a continuous function, and Let $u : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-function such that*

$$\Phi(u) = \inf_{f \in \mathcal{C}^2(\overline{\mathbb{D}},\mathbb{R}), f_{|\partial\mathbb{D}} = g} \Phi(f).$$

Then $\Delta u = 0$ on $\mathbb{D}$.

5.2.3 WHY DOES THE MINIMISING APPROACH WORK?

We have seen (thanks to Proposition 5.2.3) that solutions to the Dirichlet problem, if they exist, have to minimise the energy functional

$$\Phi(f) := \int_{\mathbb{D}^2} ||\nabla f||^2.$$

We are thus left with a way of proving that such a solution exists.

Find a function $u : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$ minimising

$$\int_{\mathbb{D}^2} ||\nabla u||^2$$

on the set of functions which are equal to $g : \partial\mathbb{D} \longrightarrow \mathbb{R}$ on the boundary.

Before setting out on our quest to find such a minimiser, we need to ask ourselves *why* we should stand a chance of succeeding. After all, a lot of functionals do not have a minimiser!

Key ingredient n°1 : the energy functional is bounded from below!

Ok, it sounds a bit silly said like that, but we can start with a minimising sequence $u_n \in \mathcal{C}^1(\mathbb{D}, \mathbb{R})$ such that

$$\Phi(u_n) \longrightarrow \inf_{f \in \mathcal{C}^1(\overline{\mathbb{D}},\mathbb{R}), f_{\partial\mathbb{D}} = g} \Phi.$$

Now our hope is that u_n should converge towards the minimiser we are after. Of course, the space of continuous functions is not compact so there is no abstract nonsense reason why (u_n) (or a subsequence) should converge.

Existence of a limit for the minimising sequence Let's try and visualise a sequence (u_n) which doesn't converge. You can try and draw pictures, but I am going to state a philosophical principle that is easily turned into a mathematical statement.

There are only two ways for a sequence of functions ($u_n : X \longrightarrow \mathbb{R}$) where X is a compact manifold (or whatever metric space well-behaved enough) not to converge (up to subsequences of course)

1. the very silly way: $\sup_X |u_n| \to +\infty$;
2. the slightly more subtle way: $\sup_X |u_n|$ remains bounded but u_n oscillates more and more.

You are already familiar with the mathematical formulation of this principle, it is the Arzela-Ascoli theorem. For our problem, try and imagine that our minimising sequence (u_n) oscillates more and more. Since u_n is $\mathcal{C}^1$, it is going to have to oscillate a lot, which in turn is going to force its derivative (or gradient ∇f) to be very big. **This is completely at odds with minimising $\int_{\mathbb{D}^2} ||\nabla u||^2$!!!**

If we do things properly, we will be able to show that a minimising sequence, because it satisfies $\int_{\mathbb{D}^2} ||\nabla u_n||^2 < A$ for some universal constant A, converges uniformly to a continuous function u_∞.

Existence of a $\mathcal{C}^1$-minimiser Now we should be happy, we have got our minimiser! Not so fast :) We've just found that any minimising sequence has a *continuous* limit, which has no reason *a priori* to be any more regular. Now again, imagine we start with a function u_∞ which is *not* differentiable. Is it possible that a sequence of differentiable functions (u_n) should converge to u_∞, without anything bad happening to ∇u_n?

That's were one would like to take out their catalogue of examples, see what's the worst that could happen. What is an example of a good non-differentiable function? Probably an example everyone has in mind is the absolute value $x \mapsto |x|$ which is not differentiable at 0. From an analysis point of view though, this is almost as good as differentiable as it is Lipschitz[1]. On the contrary, the *typical* non-differentiable function is differentiable *nowhere*, and oscillates pretty badly at all scales. Based on the simple fact that a differentiable function approximating a function that oscillates a lot at a point must have a pretty big derivative, one can prove something along the lines of

[1] An important theorem ensures that Lipschitz functions are differentiable almost everywhere

If (u_n) a sequence of differentiable functions converges to a limit u_∞ which is "badly" non-differentiable, then the gradients (∇u_n) will have to degenerate in a way that will make $\int_{\mathbb{D}^2} ||\nabla u_n||^2$ tend to infinity.

As such, any limit of a minimising sequence will have to be at worst "nicely" non-differentiable, maybe Lipschitz or differentiable away from a small set of points. There are technicalities to be overcome here, but to all practical purposes we could make these ideas work to give us

u_∞ is of class $\mathcal{C}^1$ and furthermore $\Phi(u_\infty) = \inf \Phi$ where the infimum is taken over $\mathcal{C}^1$ functions which are equal to g on the boundary.

Existence of a $\mathcal{C}^2$-minimiser Ok, but we still aren't done yet, remember that what we wanted was to find solutions to $\Delta u = 0$ (with the values of u prescribed on the boundary). We had found that if a minimiser u of Φ was of class $\mathcal{C}^2$, it had to satisfy $\Delta u = 0$, but this said nothing of a minimiser which was only $\mathcal{C}^1$.

We are going to explain that again, a sequence (u_n) of class $\mathcal{C}^2$ converging to $u_\infty \in \mathcal{C}^1(\mathbb{D}, \mathbb{R})$ has to have reasonably small second derivatives, for otherwise it would tend to make $\int_{\mathbb{D}^2} ||\nabla u_n||^2$ bigger than it should. This is probably the reasoning that is hardest to be made rigorous, but it seems to me that the following is fairly convincing. If $u_\infty \in \mathcal{C}^1(\mathbb{D}, \mathbb{R})$, and let's say that ∇u_∞ is badly non-differentiable, then ∇u_∞ oscillates pretty badly on very small balls. I take such a small ball B near a point p_0, and on B, we could replace u_∞ by a linear approximation

$$u_a(p_0 + \vec{h}) = u(x_0) + \vec{v} \cdot \vec{h}$$

where $\vec{v}$ is the average of ∇u_∞ on B, that is

$$\vec{v} := \frac{1}{\text{area}(B)} \int_B \nabla u_\infty.$$

This way, $\nabla u_\infty = \vec{v} + \vec{\epsilon}$ where $\vec{\epsilon}$ is a vector valued function of average 0. The main point is that now we have

$$\int_B ||\nabla u_a||^2 \leq \int_B ||\nabla u_\infty||^2$$

as

$$\int_B ||\nabla u_\infty||^2 = \int_B ||\vec{v} + \vec{\epsilon}||^2 = \int_B ||\vec{v}||^2 + 2\int_B \vec{v} \cdot \vec{\epsilon} + \int_B ||\vec{\epsilon}||^2.$$

We have $\int_B \vec{v} \cdot \vec{\epsilon} = \vec{v} \cdot (\int_B \vec{\epsilon}) = 0$ which proves $\int_B ||\nabla u_a||^2 \leq \int_B ||\nabla u_\infty||^2$.

The upshot of this tentative argument is that replacing a $\mathcal{C}^1$ function u by its linear approximation decreases the value of $\int ||\nabla u||^2$. By this token, we should expect $\mathcal{C}^1$-minimisers of $\int ||\nabla u||^2$ to be actually smooth.

WARNING: I don't think this last argument for $\mathcal{C}^2$-regularity of the minimiser can be made rigorous as it is. When one replaces u_∞ by its linear approximation on B, one changes the values of u_∞ on the boundary of B and so there would be a fair amount of patching things together to do. The punchline still is that minimisers will want to "push" towards more regularity.

5.3 "Rigorising" the proof

Now we go back to what a lot of people would consider to be a satisfactory proof of the Dirichlet principle. My view is that if one understands well enough the discussion of the previous paragraph, they should have become independent enough as to recover the technical details of the proof with a bit of work. The issue is that these technical details can be conveniently buried within the very neat theory of Sobolev spaces and whatnot which makes for very short proofs, but completely obscure the fairly simple ideas behind it.

In this paragraph we point to the technical concepts which allow for a fully rigorous proof of each of the steps explained above.

Existence of a limit for the minimising sequence - Sobolov spaces and Sobolev embeddings. This is done by considering the spaces

$$\mathrm{L}^2(\mathbb{D}, \mathbb{R}) := \{f \text{ measurable} \mid \int_{\mathbb{D}} ||f||^2\}$$

and

$$\mathrm{H}^1(\mathbb{D}, \mathbb{R}) := \{f \in \mathrm{L}^2(\mathbb{D}, \mathbb{R}) \mid \nabla f \text{ exists almost everywhere and } \int_{\mathbb{D}} ||\nabla f||^2 < +\infty\}.$$

Both are Hilbert spaces when endowed with the scalar product

$$\langle f, g \rangle_{\mathrm{L}^2} := \int_{\mathbb{D}} fg$$

for $\mathrm{L}^2(\mathbb{D}, \mathbb{R})$ and

$$\langle f, g\rangle_{\mathrm{H}^1} := \int_{\mathbb{D}} fg + \int_{\mathbb{D}} \nabla f \cdot \nabla g.$$

$\mathrm{L}^2(\mathbb{D},\mathbb{R})$ and $\mathrm{H}^1(\mathbb{D},\mathbb{R})$ must be thought of as versions of $\mathcal{C}^0(\mathbb{D})$ and $\mathcal{C}^1(\mathbb{D})$ respectively. The key theorem that allows for a rigorous proof of the convergence of the minimising sequence is the following adapted version of Arzela-Ascoli, and often goes under the daunting name of *Sobolev embedding theorem*.

THEOREM 5.3.1 (Arzela-Ascoli for L^2). *The unit ball of* $\mathrm{H}^1(\mathbb{D},\mathbb{R})$ *(for the distance induced by the scalar product* $\langle\cdot,\cdot\rangle_{\mathrm{H}^1}$ *is pre-compact in* $\mathrm{L}^2(\mathbb{D},\mathbb{R})$ *(for the distance induced by scalar product* $\langle\cdot,\cdot\rangle_{\mathrm{L}^2}$*).*

Existence of a $\mathcal{C}^1$-minimiser - Completeness of L^2 In this framework (replacing continuous and differentiable functions by $\mathrm{L}^2(\mathbb{D},\mathbb{R})$ and $\mathrm{H}^1(\mathbb{D},\mathbb{R})$), one easily show the following Proposition.

PROPOSITION 5.3.2. *If* $(u_n) \in \mathrm{H}^1$ *is a minimising sequence, then* (∇u_n) *is a Cauchy sequence in* L^2*.*

We leave the proof as an exercise, as it is a pretty straightforward calculation. (∇u_n) is thus shown to converge as L^2 is complete, and its limit is nothing but ∇u_∞, and furthermore

$$\Phi(u_\infty) = \min \Phi.$$

This statement is not completely straight forward, but is a consequence of basic properties of convergence of sequences of functions that we leave to our reader to check.

Existence of a $\mathcal{C}^2$-minimiser - Elliptic regularity The most annoying point is promoting the differentiability to something essentially once differentiable ($u_\infty \in \mathrm{H}^1$) to something at least twice differentiable.

The method that achieves this is often called *elliptic regularity* in the literature. The general principle, that one can start by accepting as a black box as it is so ubiquitous in geometric analysis, is the following

> Any solution u of class $\mathcal{C}^1$ to an equation
>
> $$P(u) = 0$$
>
> where P is a "Laplacian-like" operator is actually of class $\mathcal{C}^\infty$.

Here we explain how this can be proven in the particular case of harmonic functions, by a method which illustrates the tendency (discussed earlier) of functions minimising the energy function to want to be "almost linear" locally. The main fact upon which we are going to base our reasoning is the following:

THEOREM 5.3.3 (Mean property of harmonic functions). *If $u \in \mathcal{C}^1(\mathbb{D}, \mathbb{R})$ minimises the energy functionnal Φ, then for any point $p \in \mathbb{D}$, we have*

$$u(p) = \frac{1}{area(B(p,\epsilon))} \int_{B(p,\epsilon)} u(z)dxdy = \frac{1}{length(S(p,\epsilon))} \int_{S(p,\epsilon)} u$$

where $B(p,\epsilon)$ is the ball of radius ϵ about p and $S(p,\epsilon) = \partial B(p,\epsilon)$ the circle of radius ϵ around p.

In other words, at any point p, u is equal to its average on a small ball around p or a small circle around p. We first explain why, if we have Theorem 5.3.3, we can easily show that u is as regular as we like. This can be summed up with the following punchline:

Taking means is highly regularising.

Precisely we have the following Proposition.

PROPOSITION 5.3.4. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ which is smooth, non-negative and vanishes away from $[-r, r]$[2]. Then the function $\text{Average}_g^r : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by*

$$\text{Average}_g(p)^r := \int_{B(p,r)} \rho(|\mathbf{x}|)g(\mathbf{x})d\mathbf{x}$$

is of class $\mathcal{C}^\infty$.

We leave its proof as an exercise as it is pretty straightforward differentiating functions defined by integrals. With this Proposition, we see that Theorem 5.3.3 directly implies that a minimiser of the energy functional is as regular as we like.

Proof of Theorem 5.3.3 : We first give the proof in the case where u is $\mathcal{C}^2$. In this case we have $\Delta u \equiv 0$. For simplicity we assume that p is equal to 0. We consider the function

$$\psi(r) := \frac{1}{\text{length}(S(p,r))} \int_{S(p,\epsilon)} u.$$

[2]We are cheating ever so slightly here, what is need is such a ρ such that $x \mapsto \rho(|\mathbf{x}|)$ is smooth.

By definition we have (up to a constant which we will ignore)

$$\psi(r) = \frac{1}{r}\int_0^{2\pi} u(r,\theta)d\theta = \int_0^{2\pi} u(r,\theta)d\theta$$

in spherical coordinates. This gives use

$$\frac{\partial\psi}{\partial r} = \int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta)d\theta.$$

Note that $\frac{\partial u}{\partial r}(r,\theta) = \nabla u \cdot \vec{n}$ where $\vec{n}$ is the unit vector that is normal to the circle of radius r. We now kindly recall the *divergence theorem*, which is but one of the many specifications of Stokes' theorem.

Divergence Theorem If $\vec{X}$ is a vector field on a $U \subset \mathbb{R}^2$ where U is an open set whose boundary is $\mathcal{C}^1$. Then

$$\int_{\partial U} \vec{X}\cdot\vec{n} = \int_U \operatorname{div}(\vec{X}).$$

Applied to $\int_0^{2\pi} u(r,\theta)d\theta$ we get

$$\int_0^{2\pi} u(r,\theta)d\theta = \int_{S(p,r)} \nabla u\cdot\vec{n} = \int_{B(p,r)} \operatorname{div}(\nabla u) = \int_{B(p,r)} \Delta u = 0.$$

The only non-trivial point here is the equality $\operatorname{div}(\nabla u) = \Delta u$, which we invite our reader to check (it is actually often taken as the very definition of the laplacian, which survives to passing to the general context of Riemannian manifolds).

Thus $\frac{\partial\psi}{\partial r} \equiv 0$ which implies that ψ is constant equal to its value at 0. ■

Baby distributions. Now it is time to object to the fact that we only gave the proof for $\mathcal{C}^2$-functions, when what we are trying to do is to prove that $\mathcal{C}^1$-minimisers of the energy functional are actually $\mathbb{C}^2$ *using* this Proposition.

The important point is that this proof works exactly the same if we do it for arbitrary functions, performing computation in "the weak sense". In practical terms, we use the following general fact

A continuous function f is equal to zero if and only if for all $g \in \mathcal{C}^\infty$, $\int f\cdot g = 0$.

We have seen earlier that if u is a $\mathcal{C}^1$-function that minimises the energy functional, then

$$\int_{\mathbb{D}} < \nabla u, \nabla \epsilon > = 0$$

for all $\epsilon \in \mathcal{C}^1(\mathbb{D}, \mathbb{R})$ which extends continuously to the boundary to the zero function. An integration by parts will give

$$\int_{\mathbb{D}} u \Delta \epsilon = 0$$

when ϵ is at least $\mathcal{C}^2$. The calculation that we have performed in the proof of Theorem 5.3.3 can be turned into the following exercise.

Exercise 5.3.5. Using the fact that $\int_{\mathbb{D}} u \Delta \epsilon = 0$ for all ϵ of class $\mathcal{C}^2$, show that for any $r > 0$ and we have

$$\int_{\mathbb{D}} (u(p) - \psi(r,p)) \cdot \epsilon(p) dp = 0$$

with $\psi(r,p) = \int_{B(p,r)} u(x,y) dxdy$.

5.4 Exercises

Exercise 5.4.1. Show that $u : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is harmonic if and only if for all open Euclidean ball B centred at a point $z \in U$ and contained within U, we have

$$u(z) = \frac{1}{\text{vol}(B)} \int_B u(x,y) dxdy.$$

Exercise 5.4.2. Build an example of function $f : (0,1) \longrightarrow \mathbb{R}$ which is differentiable nowhere.

Exercise 5.4.3. Using the fact that $\int_{\mathbb{D}} u \Delta \epsilon = 0$ for all ϵ of class $\mathcal{C}^2$, show that for any $r > 0$ and we have

$$\int_{\mathbb{D}} (u(p) - \psi(r,p)) \cdot \epsilon(p) dp = 0$$

with $\psi(r,p) = \int_{B(p,r)} u(x,y) dxdy$.