LECTURE 4

THE ALGEBRAIC STRUCTURE OF RIEMANN SURFACES

In this lecture, we discuss algebraic aspects of Riemann surfaces. By that we mean two things.

- When a surface is known to be *algebraic*, we see that a lot of (actually all the) geometric information is contained within the algebraic structure of the field of its *meromorphic* functions.
- Armed with this new viewpoint, we prove the fact that a Riemann surface which has at least one (or maybe just a few more) meromorphic function(s) is actually algebraic.

4.1 HOLOMORPHIC FUNCTIONS ON ALGEBRAIC SURFACES

4.1.1 EXISTENCE OF MEROMORPHIC FUNCTIONS FOR RIEMANN SURFACES

Before we start discussing in greater detail the case of *algebraic* Riemann surfaces, I would like to make the following comment. An important remark is that the following question is completely non-trivial, and is one of the key tension points of the theory.

> Let Σ be a compact Riemann surface. Does Σ admit a non-constant meromorphic function $f : \Sigma \longrightarrow \mathbb{C}$?

Most of what we are going to do in paragraph 4.2 will only work on the assumption the Riemann surfaces we are working with have meromorphic functions. We want to insist that **this is no free assumption!** We will prove in the next lecture that every Riemann surface indeed has enough meromorphic functions for the arguments of paragraph 4.2 to carry over to the general case, but the price will be to do a fair bit of analysis (fret not young algebraic geometer, everything is going to be alright).

4.1.2 THE FIELD OF MEROMORPHIC FUNCTIONS AND ALGEBRAIC FUNCTIONS

We now consider an *algebraic* Riemann surface $\mathcal{C}$. The terminology "algebraic" is used here in a slightly non-standard way, here we just mean that $\mathcal{C}$ is biholomorphic to (the compactification in $\mathbb{CP}^2$ of)

$$\{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\}$$

for some irreducible polynomial $P \in \mathbb{C}[X, Y]$. In this case, it is easy to find a lot of non-constant meromorphic functions, any polynomial $Q \in \mathbb{C}[X, Y]$ defines by restriction to $\mathbb{C}$ a meromorphic function whose poles are exactly the points of $\mathcal{C}$ at infinity. We actually have a bit more.

PROPOSITION 4.1.1 (Nullstellensatz). *Let P and $\mathcal{C}$ as above. $Q \in \mathbb{C}[X, Y]$ induces the zero function on $\mathcal{C}$ if and only Q is a multiple of P.*

This result is an important structural result, which we will not prove here. You can think of it as a generalisation of the fundamental theorem of algebra to polynomials with two (or more) variable.

In particular, implies that the ring $A = A(\mathcal{C}) = \mathbb{C}[X, Y]/(P)$ is a subring of the field of meromorphic functions. We are going to call it the *ring of polynomial functions* of $\mathcal{C}$. From the Riemann surface perspective, this ring is not that special.

PROPOSITION 4.1.2. *$A(\mathcal{C})$ is exactly the ring of meromorphic functions on $\mathcal{C}$ that only have poles at infinity.*

From a Riemann surface perspective, the points at infinity play *a priori* (and as we will see *a posteriori*) no particular role. Now, if one takes the *inverse* of a polynomial, one just gets a new meromorphic function which has nothing special from the Riemann surface point of view. This invites us to introduce the following.

DEFINITION 4.1.3. Let $\mathcal{C}$ an algebraic curve and A its ring of polynomial functions. Its **field of rational functions** $K = K(\mathcal{C})$ is by definition the field of fractions of A.

It is pretty straightforward that $K(\mathcal{C})$ is a subfield of the field of meromorphic functions. The important point that we are going to discuss in the next paragraph is that if one knows $K(\mathcal{C})$ (as a purely abstract field), one can reconstruct $\mathcal{C}$. We state the following theorem (whose proof should become an exercise after this lecture) to give one a sense of the extent of the connection between $K(\mathcal{C})$ and $\mathcal{C}$.

Theorem 4.1.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two smooth algebraic curves in $\mathbb{CP}^2$. Assume that there is a field isomorphism between $K(\mathcal{C}_1)$ and $K(\mathcal{C}_2)$ which is $\mathbb{C}$-linear. Then this isomorphism is induced by a biholomorphism between $\mathcal{C}_1$ and $\mathcal{C}_2$.*

4.1.3 Abstract fields

Definition 4.1.5 (Transcendence degree). Let K be an extension of a field k. The transcendence degree of K over k is the minimal number of elements that are not algebraic over k needed to generate K over k.

With this definition, we have the following.

Proposition 4.1.6. *The field of meromorphic functions of an algebraic curve $\mathcal{C} \subset \mathbb{CP}^2$ is of transcendence degree 1 over $\mathbb{C}$. Conversely, any field of transcendence degree 1 over $\mathbb{C}$ is the meromorphic function field of a (possibly singular) algebraic curve in $\mathbb{CP}^2$.*

This proof of this Proposition is pretty straightforward, and we leave it as an exercise. The important point being that the fields which can appear as function fields of algebraic curves can be characterised by the two following simple algebraic properties.

1. They contain a copy of $\mathbb{C}$ (corresponding to the constant functions).
2. They are *almost* generated by one non-constant holomorphic function (all others are obtained as solution to an algebraic equation on this first function, like taking the square root or a higher degree root of that one).

4.2 Riemann surfaces are algebraic

In this paragraph we prove the following important theorem.

Theorem 4.2.1. *Every compact Riemann surface which has a non-constant meromorphic function is algebraic.*

We will see in the next lecture that assuming the existence of a non-constant meromorphic function is superfluous: every compact Riemann surface does indeed have one. The proof of this fact, as we have already said before, can only be obtained by analytic means.

4.2.1 MEROMORPHIC FUNCTIONS OF $\mathbb{CP}^1$

THEOREM 4.2.2. *Let* $f : \mathbb{CP}^1 \longrightarrow \mathbb{C}$ *be a meromorphic function. Then there exists* $P, Q \in \mathbb{C}[X]$ *such that* $f = \frac{P}{Q}$.

In other words, the field of meromorphic functions of $\mathbb{CP}^1$ is isomorphic to the field of rational function of $\mathbb{CP}^1$ seen as an algebraic curves. This can be reformulated this way

> The only meromorphic functions of $\mathbb{CP}^1$ the Riemann surface are the algebraic ones.

The importance of this result cannot be overstated. It is the only non-trivial fact of this discussion. The rest, however how confusing, is just packaging. The fact that any Riemann surface is biholomorphic to an algebraic curve can be seen, once one is used to all of these objects, as a mere consequence of this theorem.

4.2.2 THE FUNCTION FIELD OF AN ARBITRARY RIEMANN SURFACE

In this paragraph we give a description of the function field of an arbitrary Riemann surface which has at least one non-constant meromorphic function. We prove the following Proposition.

PROPOSITION 4.2.3. *Let* Σ *be a compact complex curve admitting non-constant meromorphic functions. Then its function field* $K(\Sigma)$ *has transcendency degree* 1 *and is therefore isomorphic to*

$$\mathbb{C}[X, Y]/(P)$$

for some irreducible polynomial P.

The curve $\{P = 0\}$ will then be a natural candidate to be biholomorphic to Σ. We will discuss this point later (you will see that from Proposition 4.2.3, it will again just be a matter of packaging the result as most of the hard work is the proof of Proposition 4.2.3).

The first meromorphic function Start with a non-constant meromorphic function, which we denote by φ. We use this notation to indicate that we will be thinking of it as a map

$$\varphi : \Sigma \longrightarrow \mathbb{CP}^1$$

which is a ramified cover. We are going to use φ to *compare* the field of meromorphic functions of Σ to that of $\mathbb{CP}^1$ (which is just $\mathbb{C}(x)$).

Practically, one just notices that φ defines an embedding of $K(\mathbb{CP}^1) = \mathbb{C}(x)$ into $K(\Sigma)$ via pulling back functions

$$\begin{array}{ccc} K(\mathbb{CP}^1) & \longrightarrow & K(\Sigma) \\ f & \longmapsto & f \circ \varphi \end{array}$$

This shows that $\mathbb{C}(x)$ is a subfield of $\mathbb{C}(x)$. The following Proposition gives a precise geometric description of these meromorphic functions which "came" from $\mathbb{CP}^1$.

PROPOSITION 4.2.4. *Let $g : \Sigma \longrightarrow \mathbb{CP}^1$ be a meromorphic function. There exists a meromorphic function $f \in \mathbb{C}(x) = K(\mathbb{CP}^1)$ such that $g = f \circ \varphi$ if and only if g is constant on preimages of φ.*

Proof: We only prove the direction of the equivalence which is not (completely) obvious. If g is constant on the pre-images, we build f the following way. For $x \in \mathbb{CP}^1$, we set $f(x) = g(y)$ for an arbitrary $y \in \varphi^{-1}(\{x\})$. The only thing that is not completely obvious is that f is meromorphic. If x is a regular value, φ has an holomorphic local inverse $\varphi^{-1} : U_x \longrightarrow V_y$ from a small neighbourhood around x to a neighbourhood of y; we see that on U_x, $f = g \circ \varphi^{-1}$ which proves that f is holomorphic away from critical values of φ.

This shows that $f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ is well-defined, continuous and holomorphic away from a finite number of points. By the Removable Singularity theorem, f is holomorphic everywhere. ■

Some algebraic abstract nonsense. Now if $g : \Sigma \longrightarrow \mathbb{CP}^1$ is an arbitrary holomorphic function (we're using repeatedly the fact that a meromorphic function is a holomorphic function taking values in $\mathbb{CP}^1$. We want to use the following fact from algebra.

PROPOSITION 4.2.5. *Let K be a extension of a field k. If every element of K is the root of a polynomial of degree at most d, then K is a finite extension and its degree over k is at most d.*

So if we can prove that any function $g \in K(\Sigma)$ is the solution of a degree d polynomial, we will be in a good position. From this point onwards, we could just present a five line computation which constructs such a polynomial, as it is done in most books. But we won't, and will take a little (but very important) detour.

The analogy with Galois theory. In this paragraph, K is finite degree extension of $\mathbb{Q}$ (number theorists call such a field K a *number field*). We explain how one can build a very fruitful analogy between such number fields and (ramified) coverings of Riemann surfaces over $\mathbb{CP}^1$. Once this analogy is understood, the proof of Theorem 4.2.1 is more easily understood.

For the expository purposes, we are going to restrict ourselves to *Galois* extension on the number theoretic side and *Galois* coverings on the other.

DEFINITION 4.2.6 (Galois extension). A Galois extension K of $\mathbb{Q}$ is an extension having the following property. If $x \in \mathbb{K}$ is the root of an irreducible polynomial $P \in \mathbb{Q}[X]$, then all the other roots of P are in K.

This definition is there to avoid jokes like $\mathbb{Q}(\sqrt[3]{2})$, which only contains one of the three roots of $X^3 - 2$. One can always Galoisify a non-Galois extension by adding some missing roots.

DEFINITION 4.2.7 (Galois cover). Let X and $\widehat{X}$ be two Riemann surfaces. A cover $\pi : \widehat{X} \longrightarrow X$ is Galois if $X = \widehat{X}/\Gamma$ where Γ is a group acting on $\widehat{X}$ properly discontinuously without fixed points and π is the natural projection on the quotient.

We now sum up in a table how the analogy works, and leave on entry empty to invite further reflection.

	Number theory	**Rieman**
Base field	$\mathbb{Q}$	$\mathbb{C}$
Extension	K a number field	$K(\Sigma) :=$ meromorphic functions
Points	$x \in K$ root of $P \in \mathbb{Q}[X]$	g meromorphi
Automorphisms	$\mathrm{Gal}(K/\mathbb{Q})$	Deck transformati
Galois conjugates	Other roots of the minimal polynomial of x	Translates of g by
Geometric object	???	Rieman

4.2.3 PROOF OF THEOREM 4.2.1

Going back to where we had left it before this little Galois interlude, we wanted to show that given

$$\varphi : \Sigma \longrightarrow \mathbb{CP}^1$$

and the inclusion of $\mathbb{C}(x) = K(\mathbb{CP}^1)$ in $K(\Sigma)$ that it induced, every other meromorphic function $g : \Sigma \longrightarrow \mathbb{C}$, we could find a polynomial $P \in (\mathbb{C}(x))[Y]$ such that $P(g) = 0$.

Recovering the minimal polynomial of an element of a number field We take a bit of a step back again, and think how we'd reconstruct a polynomial in $\mathbb{Q}[X]$ cancelling $x_1 \in K$, where K is a finite (Galois) extension of $\mathbb{Q}$. Well we consider all the Galois conjugates of x_1, which we denote by $x_2, \cdots, x_d$ and we'd form

$$P(X) = \prod_{i=1}^{d} (X - x_i).$$

If the extension K is Galois, then the set $\{x_1, \cdots, x_d\} = \{\sigma(x_1) \mid \mathrm{Gal}(K/\mathbb{Q})\}$. This shows that P is invariant under $\mathrm{Gal}(K/\mathbb{Q})$ which proves that it is in $\mathbb{Q}[X]$ (this is because we have the general fact, for a Galois extension, that $\mathrm{Fix}(\mathrm{Gal}(K/\mathbb{Q})) = \mathbb{Q}$).

The coefficients of $P(X) = \sum_{i=0}^{d} a_i X^i$ are precisely the symmetric functions in the roots of P, $a_{d-1} = \sum x_i$, $a_{d-2} = \sum_{i,j} x_i x_j, \cdots, a_0 = \prod x_i$, which is another way to see that the coefficients of P are invariant under $\mathrm{Gal}(K/\mathbb{Q})$ and are therefore in $\mathbb{Q}$.

Recovering the minimal polynomial of an element of a $K(\Sigma)$, assuming $\varphi : \Sigma \longrightarrow \mathbb{CP}^1$ is a Galois cover If $\varphi : \Sigma \longrightarrow \mathbb{CP}^1$ is a Galois cover, we can copy this proof almost line by line. If $g : \Sigma \longrightarrow \mathbb{C}$ is a meromorphic function, the group of deck transformations $\mathrm{Aut}(\Sigma, \varphi)$ acts on functions and we can form

$$P(Y) := \prod_{\sigma \in \mathrm{Aut}(\Sigma,\varphi)} (Y - g \circ \sigma)$$

which is a polynomial *a priori* in $K(\Sigma)[Y]$. But since it is invariant under the action of of $\mathrm{Aut}(\Sigma, \varphi)$, it shows that its coefficients are functions in $K(\Sigma)$ that are constant on the fibers of $\varphi : \Sigma \longrightarrow \mathbb{CP}^1$, which proves that they actually were in $K(\mathbb{CP}^1) = \mathbb{C}(x)$. This terminates the proof.

Sparing the Galois cover hypothesis Now, we need to convince ourselves that we can spare the Galois hypothesis. Notice the fact that this approach is only made possible in the case of function fields, because we have a geometric object (the Riemann surface) associated to our field which is lacking in the case of number fields.

Let's set the stage. In the case were our cover is Galois, what is the geometric description of the Galois conjugates of a meromorphic function $g : \Sigma \longrightarrow \mathbb{C}$? If we take $x_1 \in \Sigma$, it projects onto a point $y \in \mathbb{CP}^1$ and the pre-images of y by φ can be described as follows

$$\varphi^{-1}(\{y\}) = \{x_1, \cdots, x_d\} = \{\sigma(x_1) \mid \sigma \in \mathrm{Aut}(\Sigma, \varphi)\}.$$

Thus the value of $g \circ \sigma(x_1)$ is just that value of g at one of the other pre-images of y by φ. That's the key point.

> The values of g at pre-images of $y = \varphi(x_1)$ are well-defined even if the cover $\varphi : \Sigma \longrightarrow \mathbb{CP}^1$ is NOT Galois.

Again, in the Galois case, the coefficients of the polynomial P cancelling P constructed earlier are symmetric functions of the Galois conjugates.

> The value of such a coefficient at a point $x_1 \in \Sigma$ is just the symmetric function of the values of g at the $\{x_1, \cdots, x_d\}$ the pre-images of $y = \varphi(x_1)$ by φ. This is well-defined regardless of whether the cover is Galois.

The coefficients thus defined are obviously constant over preimages of φ, they are thus seen to be meromorphic functions of $\mathbb{C}(x)$. This proves the following Proposition.

Proposition 4.2.8. *In $K(\Sigma)$, every element is a root of a degree at most d polynomial in $\mathbb{C}(x)[Y]$, where d is the degree of the (ramified) cover $\varphi : \Sigma \longrightarrow \mathbb{CP}^1$.*

4.2.4 Wrapping things up.

From Proposition 4.2.8, one easily derives that the degree of $K(\Sigma)$ over $\mathbb{C}(x)$ is at most d. By the Primitive Element theorem (which says that a finite degree extension of a characteristic 0 field is generated by one element), we get that $K(\Sigma)$ is isomorphic to

$$\mathbb{C}(x)[Y]/(P)$$

where P is a polynomial in $\mathbb{C}[X, Y]$. Precisely, it means the following thing.

Proposition 4.2.9. *There are two meromorphic functions $g_1, g_2 \in K(\Sigma)$ such that*

- *g_1 and g_2 generate $K(\Sigma)$ as a field;*
- *there exists a polynomial $P \in \mathbb{C}[X, Y]$ such that $P(g_1, g_2) = 0$*

The final claim should be that Σ is biholomorphic to the (closure in $\mathbb{CP}^2$ of the) curve $\mathcal{C} := \{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\}$. What should be the isomorphism should be given by

$$\begin{array}{rcrcl}\Phi & := & \Sigma & \longrightarrow & \mathbb{CP}^2 \\ & & p & \longmapsto & (g_1(p), g_2(p))\end{array}.$$

This for sure defines a map from Σ to the algebraic curve $\mathcal{C}$. There, one essentially bad thing could happen though, this map is a degree $d \geq 2$ cover (after all, if we just wanted a map to an algebraic curve, the original meromorphic function φ provided one). The main difference that we have is the following key fact.

The meromorphic functions on Σ are exactly those of $\mathcal{C}$, pulled-back by the map Φ.

If Φ were to not be of degree 1, it would mean that *all* meromorphic functions on Σ would have an extra-symmetry. In particular they would be constant over pre-images of Φ. At this stage we have proved the following Theorem (and we won't be able to do much better with algebraic tools).

THEOREM 4.2.10. *A Riemann surface Σ whose meromorphic functions* ***separate points*** *(which means that for every $p \neq q \in \Sigma$, there exists a meromorphic function g such that $g(p) \neq g(q)$) is biholomorphic to an algebraic curve.*

A very last gap to fill in the proof If you are not too tired by now, you will have noticed that we cheated a bit: the curve $\mathcal{C}$ need not be smooth and what we have constructed here is technically a *birational isomorphism*. This can be remedied by considering instead

$$\begin{array}{rcrcl}\Phi & := & \Sigma & \longrightarrow & \mathbb{CP}^3 \\ & & p & \longmapsto & (g_1(p), g_2(p), g_3(p))\end{array}$$

for a well-chosen $g_3 \in K(\Sigma)$, we leave this as an exercise to our reader. The image of Φ will instead be a smooth curve, but the price to pay will be that it'll be an algebraic curve embedded in $\mathbb{CP}^3$.

4.3 HIGHER DIMENSIONAL COMPLEX MANIFOLDS

For this short discussion I drew most of the material from the superb blog post

https://sbseminar.wordpress.com/2008/02/14/complex-manifolds-which-are-not-algebraic/

If you have time to spare, you might as well go straight there rather that read the following paragraph.

The main result

THEOREM 4.3.1. *There exists a complex surface which is not biholomorphic to a complex algebraic surface.*

We might as well kill the suspense, the complex surface which does the trick is $X = (\mathbb{C}^2 \setminus \{(0,0)\})/((z,w) \sim (2z,2w))$. Understanding why though, is a longer an interesting process. The structural reason could be summed up the following way:

> Algebraic surfaces have strong restrictions on their topology, which general complex surfaces have not.

We can actually make this rather vague statement into many precise ones. We just give one criterion which will already be enough.

PROPOSITION 4.3.2. *The second homology group* $\mathrm{H}_2(X,\mathbb{Z})$ *of an algebraic surface is always non-trivial.*

An algebraic surface in this context is a complex submanifold of $\mathbb{C}^3$ defined by an equation $P(X,Y,Z) = 0$ where $P \in \mathbb{C}[X,Y,Z]$ is irreducible, whose closure in $\mathbb{CP}^3$ is still a complex submanifold. (We could have chosen to consider surfaces in $\mathbb{CP}^n$ defined by $n-2$ polynomial equations,it wouldn't make a huge difference).

We give a sketch of proof for Proposition 4.3.2.

- This first remarkable fact is that an **algebraic surface $\mathcal{S}$ has a lot of complex submanifolds of dimension** 1: it suffices to take the intersection of such a surface with a generic plan.

- By carefully choosing such hyperplanes, one will find two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ embedded in $\mathcal{S}$ which **intersect positively**.

- If $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect non-trivially, so do their homology classes in $\mathrm{H}_2(\mathcal{S},\mathbb{Z})$. This is a consequence of Exercise 4.4.6. This implies that $\mathrm{H}_2(\mathcal{S},\mathbb{Z})$ cannot be trivial.

In particular, since $X = (\mathbb{C}^2 \setminus \{(0,0)\})/((z,w) \sim (2z,2w))$ is homeomorphic to $S^1 \times S^3$ (Exercise 4.4.4), its homology group is trivial (Exercise 4.4.5). In particular we get

THEOREM 4.3.3. $X = (\mathbb{C}^2 \setminus \{(0,0)\})/((z,w) \sim (2z,2w))$ *is not biholomorphic (and even homeomorphic) to an algebraic surface.*

4.4 EXERCISES

EXERCISE 4.4.1.
1. Show that polynomials, as functions $\mathbb{C} \longrightarrow \mathbb{C}$ define meromorphic functions on $\mathbb{CP}^1$.
2. Show that any meromorphic function on $\mathbb{CP}^1$ is the ratio of two polynomials.

EXERCISE 4.4.2. Show that the extension $\mathbb{Q}[\sqrt[3]{2}]$ over $\mathbb{Q}$ is not Galois.

EXERCISE 4.4.3. Construct a covering map of a compact surface onto another which is NOT Galois.

EXERCISE 4.4.4. Show that $X = (\mathbb{C}^2 \setminus \{(0,0)\})/((z,w) \sim (2z,2w))$ is homeomorphic to $S^1 \times S^3$.

EXERCISE 4.4.5. Show that $\mathrm{H}_2(S^3 \times S^1, \mathbb{Z})$ is trivial.

EXERCISE 4.4.6. Let $\mathcal{S}$ be a complex surface and $\mathcal{C}_1$ and $\mathcal{C}_2$ two complex curves in $\mathcal{S}$. Assume that where they intersect, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect transversally. Show that the intersection between the homology classes of $\mathcal{C}_1$ and $\mathcal{C}_2$ is equal to $\#\{\mathcal{C}_1 \cap \mathcal{C}_2\}$.