LECTURE 2

TOPOLOGICAL SURFACES

Since the topological background to a lot of things that we are going to discuss are *surfaces* (that is 2-dimensional manifolds), we are going to dedicate an entire lecture to their topology. In particular, this is a purely **topological** lecture.

There are more reasons to study topological surfaces:

- they are incredibly rich topological objects which are arguably interesting in their own right;
- no-one can seriously claim to do higher-dimensional topology if one does not have a good command of this lower dimensional case;
- a lot of basic techniques of algebraic topology can be introduced, tested and almost fully understood by restricting oneself to the case of surfaces.

History I am less sure about the precise history of the theory of topological surfaces. The classification was achieved by the end of the 19^{th} century, but one would have to look up precise references. The algebraic topology aspect of it is probably to be credited to Poincaré (who invented the fundamental group and the universal cover, and provided the first explicit such constructions via hyperbolic geometry which will see in next lecture). In his article one will find traces of the mapping class group, and the fact that surfaces have non-trivial symmetries had been a known fact since then. However the study of the mapping class group really took of in the 70s and 80s, when Thurston gave geometric and dynamical description of all classes of homeomorphisms; and show their central role in the classification of 3-manifolds.

2.1 DEFINITION, EXAMPLES, AND CLASSIFICATION

DEFINITION 2.1.1 (Surface). A surface is a 2-dimensional topological manifold.

We say a surface is **closed** if it is compact. The reason for this odd terminology is that it is often convenient to allow in the definition of surface 2-manifolds *with boundary*, in which case a closed surface is a compact 2-manifold with boundary whose boundary is empty (I know, lol).

The sphere Our first example of closed surface is the sphere S^2, which is defined as

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

PROPOSITION 2.1.2. *S^2 thus defined is a closed, orientable and simply-connected 2-manifold.*

EXERCISE 2.1.3. Show that S^2, endowed with the topology induced by $\mathbb{R}^3$, is a 2-manifold.

EXERCISE 2.1.4. Show that S^2 is simply-connected.

The torus

DEFINITION 2.1.5. The torus is the product of two circles,

$$\mathbb{T}^2 = S^1 \times S^1.$$

The torus

- is compact;
- orientable;
- NOT simply-connected.

The fundamental group of S^1 can be easily computed:

$$\pi_1(\mathbb{T}^2) = \pi_1(S^1) \oplus \pi_1(S^1) = (\mathbb{Z}^2, +).$$

Non-orientable examples: the projective plane and the Klein bottle.

Definition 2.1.6 (The projective plane). The (real) projective plane $\mathbb{RP}^2$ is the set of lines in $\mathbb{R}^3$. It is endowed with the topology given by the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \setminus \{0\})/\mathbb{R}^*$$

where $\mathbb{R}^*$ acts by multiplication on $\mathbb{R}^3$.

Exercise 2.1.7.

1. Show that $\mathbb{RP}^2$ is a topological surface.
2. Show that $\mathbb{RP}^2$ is NOT orientable.
3. Show that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

Connected sums and higher genus surfaces I think we can all draw a genus g surface (it is the surface of a doughnut with g holes, embedded in $\mathbb{R}^3$). But if provoked into constructing it formally, how many people could actually do that? I can think of two ways to go about it.

1. Either found a set of equations in $\mathbb{R}^3$ whose solutions is what we want a genus 2 surface to be (some sort of tubular neighbourhood of a wedge of two circles). This is reasonably is to do for the torus (genus 1), probably much harder for higher genus.
2. Use the notion of *connected sum*, which is a notion of interest in topology whose utility goes beyond the case of surfaces. That's what we are going to do.

We explain the general construction of connected sums. Let M_1 and M_2 two (smooth) manifolds of the same dimension $d \geq 2$. Consider B_1 and B_2 two trivially embedded **open** balls in M_1 and M_2 respectively. [1] Note the following facts

- These two balls are bounded by spheres S_1 and S_2 of dimension $d-1$. Consider any homeomorphism $i = S^1 \longrightarrow S^2$. Note that.
- $M_1 \setminus B_1$ and $M_2 \setminus B_2$ are two manifolds with boundary S_1 and S_2 respectively.

Definition 2.1.8 (Connected sum). The connected sum of two connected manifolds M_1 and M_2 is the topological space obtain by the following quotient

$$M_1 \# M_2 := (M_1 \setminus B_1) \cup (M_2 \setminus B_2)/(x_1 \in S_1 \sim i(x_1) \in S_2).$$

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[1] By trivially embedded, we mean that we have taken a chart and considered a ball in of small radius in these coordinates.

To show that the connected sum is well-defined, one should check that:

- the result of the construction does not depend on the choice of a homeomorphism $i : S_1 \longrightarrow S_2$;
- it does not depend on the choice of balls B_1 and B_2.

We can now define the surface with g handles. Let Σ_1 be the torus $S^1 \times S^1$.

DEFINITION 2.1.9. For $g \geq 2$ The surface Σ_g is defined inductively by

$$\Sigma_g := \Sigma_{g-1} \# \Sigma_1.$$

2.1.1 THE GENUS

We recall the rigorous definition of the genus.

DEFINITION 2.1.10 (Genus). The genus of a topological surface Σ, denoted $g = g(\Sigma)$ is the maximal number of pairwise disjoint, non-homotopically equivalent simple closed curves that one can remove without disconnecting Σ.

With this definition, we have the following Proposition.

PROPOSITION 2.1.11. *The surface with g-handles Σ_g has genus g.*

Proof: (Difficult) exercise! ■

2.1.2 POLYGONAL REPRESENTATION(S) OF A COMPACT SURFACE

Another interesting way to build compact (orientable surfaces is to consider a (topological) polygon with an even number of sides, pair sides together and glue them together in a way that is compatible with the orientation.

PROPOSITION 2.1.12. *The quotient space obtained with the identifications suggested above is a topological surface.*

As a lot of things in this lecture, we leave the proof as an exercise. It is not a completely straightforward fact, as the 3-dimensional equivalent of this Proposition is false. If you take a polyhedra and glue pairs of faces together in an orientation-preserving way, you don't always get a 3-manifold.

EXERCISE 2.1.13. Find a way of glueing pairs of faces of a cube such that the quotient space is NOT a 3-manifold.

Figure 2.1.1: Two different polygonal models for the genus 2 surface.

2.2 Algebraic topology of surfaces

We now cover some fundamental results about the algebraic topology of **compact, orientable** surfaces. There is a theory for compact non-orientable surfaces which is very similar to the orientable case, which we will not cover. The theory for non-compact surfaces is much much more involved, and the classification in this case makes use of *fractal* spaces. We start with the classification theorem.

Theorem 2.2.1 (Classification of compact and orientable surfaces). *Let S be a compact, orientable surface and let g be its genus. S is homeomorphic to Σ_g the surface with g-handles.*

We do not give a proof of this theorem here. It is not necessarily extremely complicated, but it would take us too much time. The important thing is that it tells us that it makes sense to talk of *the genus g surface*, as the genus classifies orientable surfaces up to homeomorphisms.

2.2.1 Universal cover of closed surfaces

We now give a description of universal covers of orientable, compact surfaces.

Theorem 2.2.2 (Universal cover of closed surfaces). *Let Σ_g be a closed, orientable surface of genus $g \geq 0$. Its universal cover is*

- *S^2 if $g = 0$ (in which case Σ_0 is simply connected and is homeomorphic to S^2);*
- *$\mathbb{R}^2$ if $g \geq 1$ (in which case Σ_0 is simply connected and is homeomorphic to S^2).*

There are essentially two ways of going about proving this theorem. A very good one and a worse one. We give the good one for $\Sigma_0 \simeq S^2$ and $\Sigma_1 \simeq \mathbb{T}^2$, and the worse one for Σ_g, $g \geq 2$.

Visualising and constructing the universal covers of S^2 and $\mathbb{T}^2$. We go very quickly on the case of S^2. It is simply-connected (prove it!) and therefore is its own universal cover.

We move to the torus. Consider $\mathbb{R}^2$ and the two transformations $T_a : (x, y) \mapsto (x + 1, y)$ and $T_b : (x, y + 1) \mapsto (x, y + 1)$. We note the two following facts.

- T_a and T_b commute and generate a group $\Gamma < \mathrm{Homeo}(\mathbb{R}^2)$ that is isomorphic (as a group) to $\mathbb{Z}^2$.
- Γ acts properly discontinously on $\mathbb{R}^2$, without fixed points.

It is easy to check that $\mathbb{R}^2/\Gamma$ is isomorphic to $S^1 \times S^1$. We can now call upon the following Proposition (which is standard algebraic topology and which we will not prove).

Proposition 2.2.3. *Let X be a manifold, and Γ a subgroup of* $\mathrm{Homeo}(X)$ *such that Γ acts properly discontinuously on X and without fixed points. Then the projection*

$$\pi : X \longrightarrow X/\Gamma$$

is a covering map.

This in particular implies that $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma \simeq \mathbb{T}^2$ is a covering map. Since $\mathbb{R}^2$ is simply-connected, $\mathbb{R}^2$ is (homeomorphic to) the universal cover of $\mathbb{T}^2$.

An overkill proof for Σ_g when $g \geq 2$. We are (at this stage) unable to offer anything in way of an explicit construction of the universal cover of Σ_g (but that will come with hyperbolic geometry). We therefore resort to the following hammer.

Theorem 2.2.4. *Every simply-connected* 2 *manifolds is homeomorphic to either S^2 or $\mathbb{R}^2 \simeq \mathbb{D}^2$.*

This theorem is a fairly difficult one, I don't know of a simple proof of it. In particular I am not aware of a proof which doesn't use in a crucial way some geometry(Edit, Mathoverflow has found a topological proof for me: see there). That's why it seems reasonable to call it a hammer. Now, the universal cover of Σ_g is either the sphere S^2 or $\mathbb{R}^2$. Here I see two ways to conclude (pick the one your most comfortable with).

- Since S^2 is compact, any group acting properly discontinously on it would have to be finite. As we will see shortly, $\pi_1(\Sigma_g)$ is infinite, which precludes S^2 being the universal cover if Σ_g.

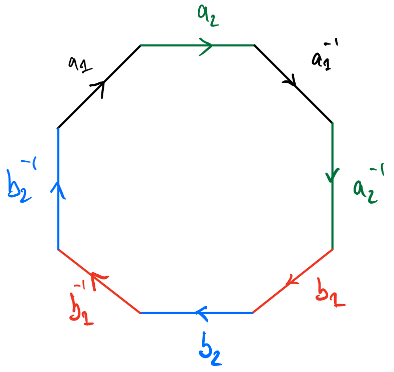
- Σ_g being orientable, if S^2 were its universal cover $\pi_1(\Sigma_g)$ would have to act on S^2 via orientation-preserving maps. But it is known that an orientation preserving homeomorphism of S^2 must have a fixed point. So $\pi_1(\Sigma_g)$ couldn't possibly act on S^2 without fixed points, unless it were trivial. But Σ_g is not homeomorphic to S^2 for $g \neq 0$.

2.2.2 FUNDAMENTAL GROUP

THEOREM 2.2.5 (Fundamental group of a closed surface). *Let Σ_g be a closed, orientable surface of genus $g > 0$. Its fundamental group is isomorphic to the group given by the following presentation*

$$\langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle.$$

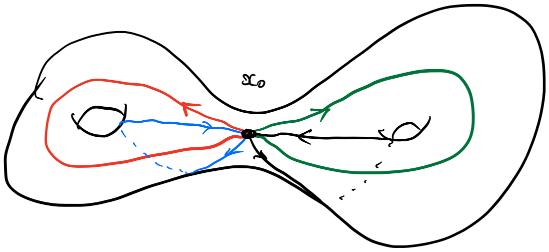
The surface of genus 2 is the quotient space obtained from an octogon P with the identifications of its side as indicated in the Figure below.



We set by definition

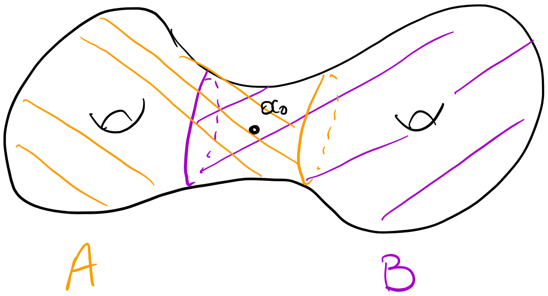
$$\Sigma = P/_{\sim}.$$

We are going to compute the fundamental group of Σ, using Van Kampen theorem. We consider a point x_0 as in Figure below and closed paths based at x_0, a_1, a_2, b_1 and b_2.



First method : Σ as the union of two holed tori.

We write Σ as a union of two holed tori A and B as in Figure below. Let δ a curve of $A \cap B$, based at x_0 that is *freely* homotopic to the boundary curve of both A and B.



Working out the algebra. We apply Van Kampen theorem to $X = A \cup B$, we obtain that

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0)$$

where the amalgamation is relative to the induced map of the injections $i_A : A \cap B \to A$, $i_B : A \cap B \to B$. Precisely, that means that $\pi_1(X, x_0)$ is the quotient of $\pi_1(A, x_0) * \pi_1(B, x_0)$ by the subgroup

$$H =<< (i_A)_*(h) \cdot (i_B)_*(h)^{-1} \mid h \in \pi_1(A \cap B, x_0) >>$$

which the NORMAL subgroup generated by elements of the form $(i_A)_*(h) \cdot (i_B)_*(h)^{-1}$.

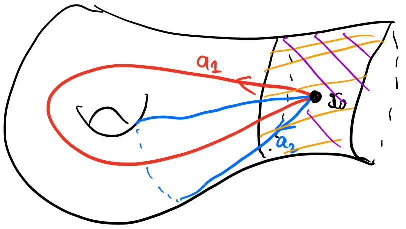
In our case $A \cap B$ is homeomorphic to $S^1 \times (0,1)$ (which deformation retracts onto S^1), thus $\pi_1(A \cap B, x_0)$ is isomorphic to $\mathbb{Z}$ and is generated by the class of δ. The subgroup H is therefore the normal subgroup generated by $(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1}$. In other words,

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1}$$

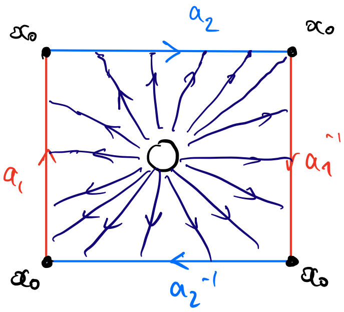
is the only extra relation to be added to $\pi_1(A, x_0) * \pi_1(B, x_0)$ to obtain $\pi_1(X, x_0)$.

That's us done with the algebra. We are now left to computing (that is giving a presentation of) $\pi_1(A, x_0)$, $\pi_1(B, x_0)$ and computing $(i_A)_*([\delta])$ and $(i_B)_*([\delta]))^{-1}$.

Presentation of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$. We only treat the case of A as B is exactly the same. A is the one holed torus as in Figure below, where the dashed area is $A \cap B$.



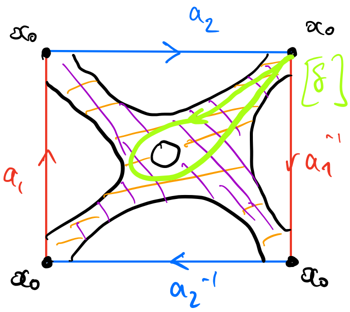
Cutting along a_1 and b_1, we can realise A as the quotient of the following square (where the four vertices project onto x_0).



On this Figure one sees that A deformation retracts on $a_1 \vee b_1 \simeq S^1 \vee S^1$. Thus, $\pi_1(A, x_0)$ is free generated by a_1 and a_2. Same holds for $\pi_1(B, x_0)$, which is freely generated by b_1 and b_2. We have obtained the following Proposition

PROPOSITION 2.2.6. *$\pi_1(A, x_0) * \pi_1(B, x_0)$ is a free group, freely generated by a_1, a_2, b_1 and b_2.*

Computation of $(i_A)_*([\delta])$ and $(i_B)_*([\delta])$ We just add δ to the Figure below



Here one sees that δ can be homotoped (in A!) to $a_1 a_2 a_1^{-1} a_2^{-1}$. This implies that

$$(i_A)_*([\delta]) = a_1 a_2 a_1^{-1} a_2^{-1}.$$

A similar reasoning shows that δ can be homotoped (in B this time) to $b_2^{-1} b_1^{-1} b_2 b_1$ (check that it is consistent with the orientation chosen on the Figure!) to obtain

$$(i_B)_*([\delta]) = b_2^{-1}b_1^{-1}b_2b_1.$$

Therefore

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1} = a_1a_2a_1^{-1}a_2^{-1}b_1b_2b_1^{-1}b_2^{-1}.$$

We use the notation $[a, b] = aba^{-1}b^{-1}$ to write

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1} = [a_1, a_2] \cdot [b_1, b_2].$$

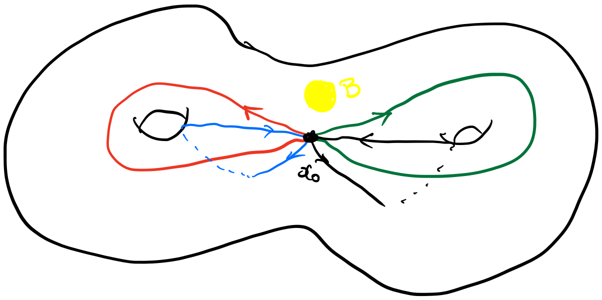
Putting everything together we obtain the following

Theorem 2.2.7. *The fundamental group of* Σ *is isomorphic to*

$$< a_1, a_2, b_1, b_2 \mid [a_1, a_2] \cdot [b_1, b_2] = 1 > .$$

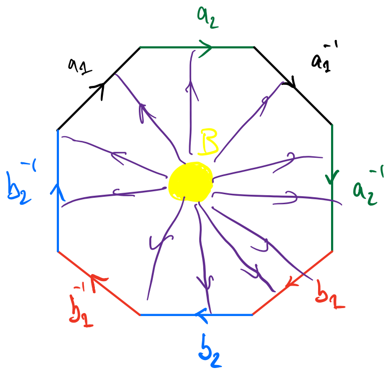
Second method : $\Sigma \setminus \{p\}$ deformation retracts on a wedge of 4 circles.

We sketch the proof of a second method. Take a point p different from x_0 and let B be a little disc around p and let C be a small disk whose closure is contained in the interior of B. Let $A := \Sigma \setminus C$. We put x_0 in $A \cap B$.

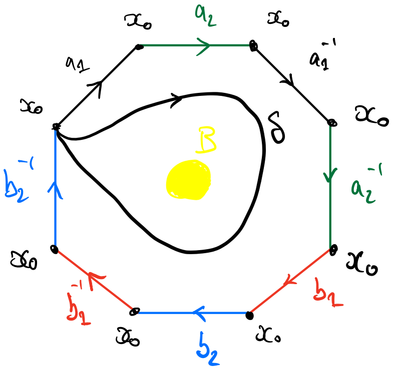


We apply Van Kampen to $X = A \cup B$. We have

- $A \cap B$ is homeomorphic to $S^1 \times (0, 1)$ with generator δ as in Figure below.
- $\pi_1(B, x_0)$ is trivial.



We consider A and cut along a_1, a_2, b_1 and b_2 to see A as quotient as Figure below, with all the vertices of the octogon projecting onto x_0 in the quotient.



On can see thus that A deformation retract onto $a_1 \vee a_2 \vee b_1 \vee b_2$ the wedge of four circles. The fundamental group of A is therefore isomorphic to the free group generated by a_1, a_2, b_1 and b_2.

Adding δ in the picture, one show that $(i_A)_*(\delta) = [a_1, a_2] \cdot [b_1, b_2] \in \pi_1(A, x_0)$ (keeping the notations of the first method) and $(i_B)_*(\delta) = 1 \in \pi_1(B, x_0)$. Working out the algebra of Van Kampen yields another proof of Theorem 3.2.7.

2.2.3 HOMOLOGY

We conclude with a short discussion about the homology groups of Σ_g. Unlike higher-dimensional manifolds, there isn't any extra information in the homology that is not already contained in the fundamental group.

- Since Σ_g is connected, $\mathrm{H}_0(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$.
- We have that $\mathrm{H}_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$. This can be seen in two ways.
    1. Either using the fact that $\mathrm{H}_1(M, \mathbb{Z})$ is the abelianisation of $\pi_1(M)$ for any manifold M.
    2. Or using the van Kampen formula for the decomoposition of Σ_g as the connected sum of g tori.
- By virtue of Σ_g being compact and orientable, we have $\mathrm{H}_2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$.

2.3 A SHORT INTRODUCTION TO THE MAPPING CLASS GROUP

One of the features that makes surfaces very rich manifolds from the topological viewpoint is that, **unlike most manifolds**, they have a lot of *topological* symmetries.

2.3.1 ORIENTATION-REVERSING HOMEOMORPHISMS

We will discuss in more detail what should count as a non-tivial *topological symmetry* a bit later. Here we give first examples of things that will definitely qualify as such.

THEOREM 2.3.1. *Every compact orientable surface admits an **orientation-reversing** homeomorphism.*

Proof: The key is to realise Σ_g as a submanifold of $\mathbb{R}^3$ which is symmetric with respect to a plane. Then taking the reflection of in this plane gives a homeomorphism which reverses the orientation.

For the Σ_0 and Σ_1, such realisation are easy to find, for $g \geq 2$ one must think a bit harder :-) ■

2.3.2 DEHN TWISTS

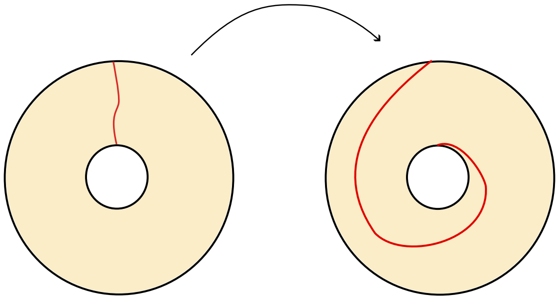
We continue toying with concrete examples. We consider the annulus

$$\mathcal{A} = [0, 1] \times S^1.$$

We consider the map

$$\begin{array}{cccc} D := & \mathcal{A} & \longrightarrow & \mathcal{A} \\ & (\theta, x) & \longmapsto & (\theta, x+\theta) \end{array}.$$

Note that D restricted to the two boundary components of $\mathcal{A}$ is the identity? Let's look at what happens to a curve drawn on $\mathcal{A}$ under the action of D.



Now what happens if we shoehorn this annulus into a compact surface Σ_g. Formally what we do is the following:

- We consider a simple closed curve γ, and we thicken it to get an annulus with boundary homeomorphic to $\mathcal{A}$.
- We define a homeomorphism of the whole Σ_g by letting it be D on the copy of $\mathcal{A}$ embedded in Σ_g and the identity elsewhere (the mapping this defined is continuous as it is the identity on the boundary of $\mathcal{A}$).

Definition 2.3.2 (Dehn twist). The map T_γ is called the **Dehn twist** along the curve γ.

Let try and convince ourselves that T_γ is definitely "non-trivial" from a topological perspective. Consider an arbitrary curve α which intersects γ once.
We see that $T_\gamma(\alpha)$ is a new curve that is NOT homotopic to α. This leads us to say that T_γ is non-trivial topologically.

2.3.3 THE MAPPING CLASS GROUP

Let M be an arbitrary manifold. In this paragraph we define an important group associated to it: its *Mapping Class Group*[2]. The mapping class group (MCG for short hereafter) is the group of topological symmetries of a manifold.

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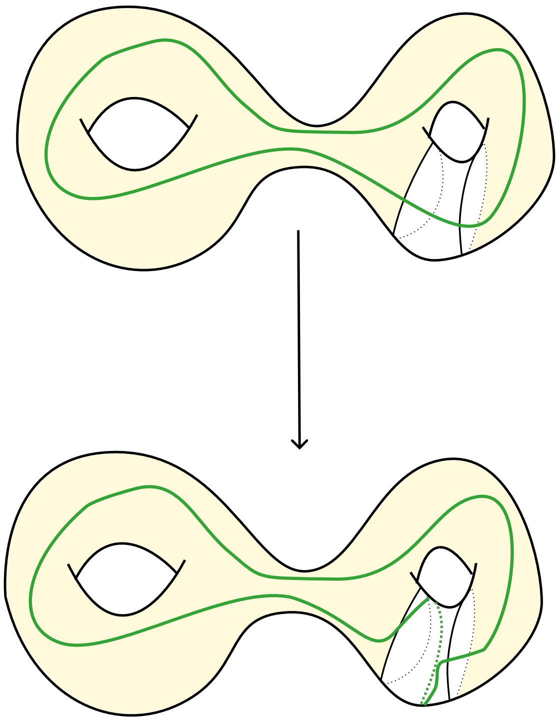
[2] The historical terminology, introduced by Poincaré, was *modular group*, which was progressively replaced by *mapping class group*, a terminology introduced by the American school of topology in the 70s.



What is a topological symmetry? Strictly speaking, it should be an isomorphism of the space under consideration which preserves the topology *i.e.* a homeomorphism. It would be fair to just consider the group Homeo(M). The problem with Homeo(M) is that it is absolutely massive. Now, it is not a problem in itself but the following might convince you that there might be a bit too much in it.

Consider $M = \mathbb{R}^2/\mathbb{Z}^2$ the torus. Consider the translation by a small vector $\vec{\epsilon}$, it induces a homeomorphism T_ϵ which is non-trivial but which, as far as the **global** topology of M is concerned, does nothing:

2.3.4 THE TORUS

As for many other topological questions, the mapping class group can be computed completely explicitly in the case of the torus. We explain how briefly.

If we think of the torus as $\mathbb{R}^2/\mathbb{Z}^2$, we can consider the action of integer-valued matrices on $\mathbb{R}^2$. If $A \in \mathrm{SL}(2, \mathbb{Z})$ we can check the following.

FACT 2.3.1. The action of A on $\mathbb{R}^2$ induces a diffeomorphism

$$T_A : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2.$$

Exercise 2.3.3. Check that when $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, T_A is a Dehn twist along a curve to be specified.

One important point is the following: T_A and T_B are isotopic if and only if $A = B$. This in particular implies that $\mathrm{MCG}(\mathbb{T}^2)$ contains a group isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. We have thus proved the following meta theorem.

Theorem 2.3.4. *The group of topological symmetries of* $\mathbb{T}^2$ *is big.*

2.3.5 The symplectic representation

We conclude with a first-order approximation of the Mapping Class Group of a surface of arbitrary genus.

A reminder from algebraic topology I take the risk of stating the obvious: the point of algebraic topology is to substitute the study of topological objects by that of associated algebraic objects. The main reason for that is that algebra is often simpler, but one shouldn't lose sight of the fact that **a lot of information is lost in the process**. There's a reason why it gets simpler! That being said, for many purposes it is often enough (but not always!).

When one is given a manifold M, one can associate to it its homology groups $H_\bullet(M, \mathbb{Z})$. We think of these as a combinatorial simplification of M. Now take a continuous map $f : M \longrightarrow M$, it will induce a series of linear maps on the $H_\bullet(M, \mathbb{Z})$. We denote such a map

$$f_* : H_k(M, \mathbb{Z}) \longrightarrow H_k(M, \mathbb{Z})$$

for all $k \in \mathbb{N}$. It has the following properties.

- If f is invertible, then f_* is invertible.
- If f and g are isotopic, $f_* = g_*$. In particular, if f is isotopic to the identity, f_* is the identity.
- For every two f, g, we have $(f \circ g)_* = f_* \circ g_*$.

The symplectic representation Recall that $\mathrm{H}_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$. Any diffeomorphism f of Σ_g thus induces an invertible linear map

$$f_* : \mathbb{Z}^{2g} \longrightarrow \mathbb{Z}^{2g}$$

in other words we can think of f_* as an element of $\mathrm{GL}(2g, \mathbb{Z})$. We therefore have defined a group homomorphism

$$\rho : \mathrm{Diff}(\Sigma_g) \longrightarrow \mathrm{GL}(2g, \mathbb{Z}).$$

Moreover, ρ maps $\mathrm{Diff}(\Sigma_g)_0$ onto the identity, we therefore have:

Proposition 2.3.5. *ρ induces a representation*

$$\rho := \mathrm{MCG}(\Sigma_g) \longrightarrow \mathrm{GL}(2g, \mathbb{Z}).$$

This representation is called the *symplectic representation*. Why so? Because of the theorem below. We recall that the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ is the set of matrices which preserves a bilinear form which non-degenerate and anti-symmetric. We use the following notation $\mathrm{Sp}(2g, \mathbb{Z}) = \mathrm{Sp}(2g, \mathbb{R}) \cap \mathrm{GL}(2g, \mathbb{Z})$.

Theorem 2.3.6. *The image of ρ is exactly* $\mathrm{Sp}(2g, \mathbb{Z})$.

This theorem essentially tells us that the action of a diffeomorphism of the homology preserves a symplectic form, and that up to this obstruction any linear symplectic map represents a diffeomorphism. I leave you with the following question.

Question. *Can you work out the geometric meaning of this symplectic form and thereby prove*

$$\rho(\mathrm{MCG}(\Sigma_g)) \subset \mathrm{Sp}(2g, \mathbb{Z}).$$

The fact that $\rho(\mathrm{MCG}(\Sigma_g)) = \mathrm{Sp}(2g, \mathbb{Z})$ is a deeper theorem. A first step towards it would be to solve the following exercise.

Exercise 2.3.7. Compute the matrix of the action of an arbitrary Dehn twist on $\mathrm{H}_1(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ in a basis of your choosing.

2.4 Higher dimensions

We conclude with a short discussion about some things that happens in higher dimensions.

2.4.1 A WARNING ABOUT REGULARITY

In this lecture, we have been a bit sloppy about the regularity of the surfaces that we were considering, or in the way we defined the mapping class group. You will find that many references do that, and that people doing low-dimensional topology often behave as if considering things up to homeomorphisms or diffeomorphisms were the exact same thing (they don't even bother specifying the regularity of the diffeomorphism). This is mostly because it is entirely justified.

THEOREM 2.4.1. *Let Σ be a topological surface and $\mathcal{D}_1$ and $\mathcal{D}_2$ two differentiable structure on Σ compatible with the topological structure of Σ. Then there is homeomorphism of Σ that is isotopic to the identity mapping $\mathcal{D}_1$ onto $\mathcal{D}_2$.*

In other words, the differentiable structure of a surface is unique. We also have the two following facts:

- every homeomorphism of a surface Σ is isotopic to a diffeomorphism;
- if two diffeomorphisms are isotopic via homeomorphisms, then they are isotopic via diffeomorphisms.

It implies in particular that

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma) = \mathrm{Diffeo}^+(\Sigma)/\mathrm{Diffeo}_0(\Sigma).$$

These facts, in particular Theorem 3.4.1, are **highly non-trivial**, as is illustrated by the theorem below. The casual approach to regularity taken by low-dimensional topologists shouldn't blind you to the existence of this difficulty, although in dimension 2 and 3 most of the time it can be ignored.

THEOREM 2.4.2. *There are infinitely many non-diffeomorphic differentiable structures on $\mathbb{R}^4$.*

It is really hard to imagine where these structures come from, and I have very little to offer in the way of visualising any such structure. I would invite you to consider the simpler question to begin:

QUESTION. *What are the most compelling examples of objects that you can think of/construct/imagine which exist in the topological class but can't be made smooth?*

The best example I can think of are topological embeddings of the two dimensional disks in $\mathbb{R}^4$ which are not isotopic to a differentiable one. These are beautiful examples, and I will leave it to you to see if you can guess what they look like.

2.4.2 CLASSIFICATION OF 3-MANIFOLDS

2.4.3 4-MANIFOLDS AND FINITELY GENERATED GROUPS

2.5 EXERCISES

EXERCISE 2.5.1. Find a polynomial equation for the genus 2 surface.

EXERCISE 2.5.2. Show that every compact orientable surface has an orientation-reversing homeomorphism.

EXERCISE 2.5.3. Show that $\mathbb{CP}^2$ does not have any orientation-reversing homeomorphisms.

EXERCISE 2.5.4. Show that a compact embedded surface of $\mathbb{R}^3$ is orientable.

EXERCISE 2.5.5. Compute the homology groups of $\mathbb{T}^2$.

EXERCISE 2.5.6. Using the van Kampen theorem and the previous theorem, compute the homology groups of Σ_g.

EXERCISE 2.5.7. Find a 3-manifold with infinite mapping class group.

EXERCISE 2.5.8 (Knot theory). Find a topological embedding of a 2-dimensional disk in $\mathbb{R}^4$ which is not homotopic to a smooth one. (Hint: consider the intersection of such an embedding $i : \mathbb{D}^2 \longrightarrow \mathbb{R}^4$ with the sphere of small radius centred at $i(0)$).