Lecture 1

Examples of curves

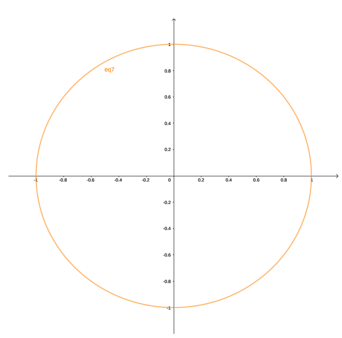
In this first lecture, we deal with basic examples of algebraic curves over $\mathbb{C}$. We concretely illustrate, on two basic (but arguably difficult) examples, conics and cubics, some of the important concepts that we are going to be dealing with during the first half of this course.

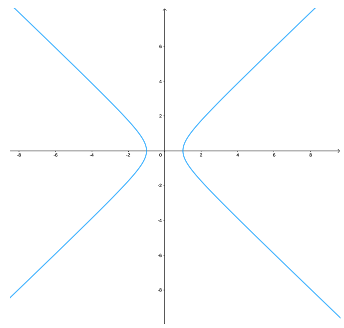
I drew a lot of inspiration from the book [1], which I wholeheartedly recommend to anyone who would like to learn more about the (pre-)history of algebraic curves and Riemann surfaces.

1.1 Two basic questions about curves.

1.1.1 Real algebraic curves

A very amusing game one can play is to consider a real polynomial in two variables (such as $P(x,y) = x^2 - y^2 - 3$ or $P(x,y) = y^2 - x^3 - x - 17$) and draw its set of zeroes in $\mathbb{R}^2$. At the very least it produces intriguing (some might say beautiful?) pictures. For instance, if one takes $P(x,y) = x^2 + y^2 - 1$, the set of zeroes is a familiar object

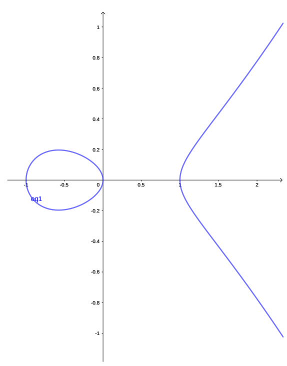




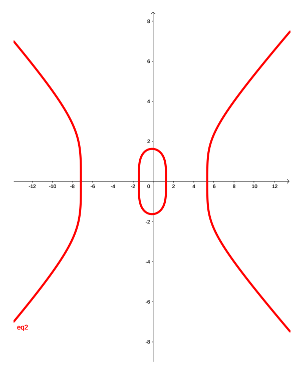
We can modify P ever so slightly by changing a sign to get a curve that looks significantly different: for $P(x, y) = y^2 - x^2 + 1$ we obtain

This *curve* (as it is what we call the set of zeroes of a polynomial in $\mathbb{R}^2$) is often called a *hyperbola*. Despite being defined by a polynomial very similar to that used to define a circle, we get a curve which is **non-compact** and **disconnected**.

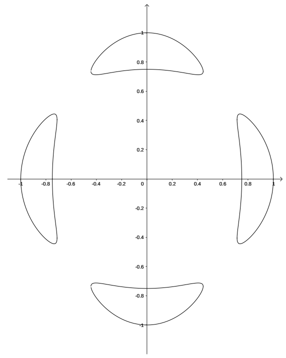
Playing a bit more, we can produce other examples of such algebraic curves with two, three and even four connected components, see below. The examples with three and four connected components are given by polynomial of degree 4.



(a) $y^2 = 0.1(x^3 - x)$



(b) $y^4 = 0.1x^2 + 0.2x^3 - 4x^2 - x + 7)$



(c) $144(x^4 + y^4) - 225(x^2 + y^2) + 350(xy)^2 + 81 = 0$, the Trott curve

If you were to toy a bit more, you'd soon realise that in order to create curves with many connected components, you'd have to increase the degree of your polynomial. We are therefore left with the following question.

> Let $d \in \mathbb{N}$ be a positive integer. How many connected components a curve defined by a polynomial $P \in \mathbb{R}[x, y]$ of degree d can have?

This question, on top of being fun, is a good question, because to find an answer to it one has to considerably extended the scope of the theory. It will be required to consider the set of complex-valued solutions to the equation $P(x, y) = 0$. This is a topological surface in $\mathbb{C}^2$, and the maximum number of connected components of the real curve will be the genus of the complex curve $+1$.

1.1.2 Parametrisations

Another line of questions pertains to parametrisations of curves. For instance, if we consider the circle $\mathcal{C} = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$, we find the following two interesting ways of parametrising it.

- The map $t \mapsto (\cos t, \sin t)$.
- The map $t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$, which maps $\mathbb{R}$ to $S^1 \setminus \{(-1, 0)\}$.

The first one is interesting from a *topological* perspective, it is a covering map, and more even more the universal covering map of S^1. The second one is even more intriguing, it is an *algebraic* parametrisation. It invites the following question, which will act as a guide for the rest of this lecture.

> When can an algebraic curve be parametrised algebraically?

1.2 Conics

Definition 1.2.1 (Conic). Let $\mathbf{k}$ be a field. A conic (or quadratic curve) in $\mathbf{k}^2$ (respectively $\mathrm{P}^2(\mathbf{k})$) is an algebraic set defined by an equation of the form

$$\mathcal{C} := \{(x, y) \in \mathbf{k}^2 \mid P(x, y) = 0\}$$

where P is a polynomial of degree 2

$$(\text{respectively } \mathcal{C} := \{(x, y) \in \mathbf{k}^2 \mid P([x : y : z]) = 0\}$$

where P is a homogeneous polynomial of degree 2.)

1.2.1 Real conics

The terminology *conic* is actually short for *conic section*. It derives from the fact that a conic in $\mathbb{R}^2$ can always be obtained as the intersection of cone in $\mathbb{R}^3$ with an arbitrary plane.

One shows this way that up to an affine change of coordinates (of the form $X \mapsto AX + B$ with $A \in \mathrm{GL}(2, \mathbb{R})$ and $B \in \mathbb{R}^2$), a (regular) conic is one of the three models.

- The ellipse, given by the equation $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$. It has **1** connected component.
- The parabola, given by the equation $y - x^2 = 0$. It also has **1** connected component.
- a hyperbola, given by the equation $y^2 - x^2 - 1 = 0$. It has **2** connected components.

Exercise 1.2.2. Classify regular conics in $\mathbb{RP}^2 = \mathrm{P}^2(\mathbb{R})$, up to projective automorphisms of $\mathbb{RP}^2$.

1.2.2 Complex conics

Theorem 1.2.3. *Let $\mathcal{C}$ be a non-degenerate conic in $\mathbb{CP}^2$. Then there exists a biholomorphism*

$$\psi : \mathbb{CP}^1 \longrightarrow \mathcal{C}$$

and furthermore, ψ, seen as a function $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$, is rational.

First proof. Recall that, by definition, a non-degenerate conic is a curve defined by an equation of the form

$$Q(z_1, z_2) = 1$$

where Q is a non-degenerate quadratic form. Up to a linear change of coordinates, we can assume that

$$Q(z_1, z_2) = z_1^2 + z_2^2.$$

Define

$$\mathcal{C}_a := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 1\}.$$

We want to find a parametrisation of $\mathcal{C}$, and we draw inspiration from the real case. f be the following map

$$f \ := \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathcal{C}_a \\ t & \longmapsto & (\cos t, \sin t) \end{array}.$$

Recall that for t a complex number, cos and sin are defined by the following formulae: $\cos(t) = \frac{1}{2}(e^{it} + e^{|it})$ and $\sin(t) = \frac{1}{2i}(e^{it} - e^{|it})$.

This map is surjective; it is actually a covering map. It satisfies $f(t+2\pi) = f(t)$ for all $t \in \mathbb{C}$ and therefore induces a map $\bar{f} : \mathbb{C}/\mathbb{Z} \longrightarrow \mathcal{C}_a$. One easily checks that $\bar{f}$ is a biholomorphism.

Now we promised a biholomorphism between $\mathcal{C} \subset \mathbb{CP}^2$, the closure of $\mathcal{C}_a \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ in $\mathbb{CP}^2$. Note that $\mathbb{C}/\mathbb{Z}$ is biholomorphic to $\mathbb{CP}^1 \setminus \{0, \infty\}$ (via the exponential map). We can thus think of $\bar{f}$ as a map

$$\bar{f} : \mathbb{CP}^1 \setminus \{0, \infty\} \longrightarrow \mathcal{C}_a.$$

It now suffices to show that $\bar{f}$ extends to 0 and ∞. In the identification given by the exponential map, 0 corresponds limits of points $z \in \mathbb{C}$ tending to infinity with $\mathrm{Im}(z) \to -\infty$, whereas ∞ corresponds limits of points $z \in \mathbb{C}$ tending to infinity with $\mathrm{Im}(z) \to +\infty$. When $\mathrm{Im}(z) >> 1$, $\sin(z) \simeq \cos(z)$ whereas when $\mathrm{Im}(z) >> 1$, $\sin(z) \simeq -\cos(z)$. This way one sees that in $\mathbb{CP}^2$, $\bar{f}$ tends to $[1, 1, 0]$ and $[1, -1, 0]$ in 0 and ∞ respectively. Since $[1, 1, 0]$ and $[1, -1, 0]$ are exactly the points points at infinity of the conic $\mathcal{C} = \overline{\mathcal{C}_a}$, $\bar{f}$ induces a continuous bijection ψ between $\mathbb{CP}^1$ and $\mathcal{C}$, which is holomorphic on $\mathbb{CP}^1 \setminus \{0, \infty\}$. By the theorem on removable singularities, ψ is a biholomorphism.

We had promised that on top of that, ψ was rational function. With this approach, this a bit cumbersome to establish. We therefore suggest a different, more geometric approach which gives an easier proof of the whole theorem.

Second proof. This second proof does not require any result on the classification of quadratic forms. Assume $\mathcal{C}$ is closure in $\mathbb{CP}^2$ of

$$\mathcal{C}_a = \{(z_1, z_2) \in \mathbb{C}^2 \mid Q(z_1, z_2) = 1\}.$$

Consider $\mathcal{L}$ an arbitrary line in $\mathbb{C}^2$ (which compactifies to a $\mathbb{CP}^1$ in $\mathbb{CP}^2$). Choose an arbitrary point p_0 on $\mathcal{C}$. Let $t \in \mathcal{L}$, we defined $\psi(t)$ to be the second intersection point between the line through p_0 and t and $\mathcal{C}$, as in the Figure below.

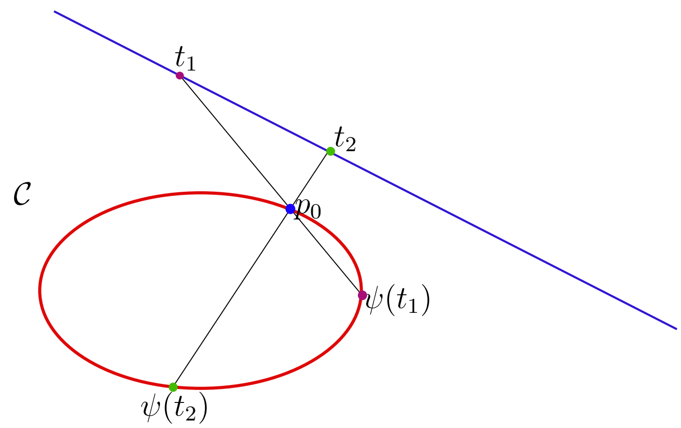


Figure 1.2.1: The geometric construction of the map ψ.

We leave the following Proposition as an exercise.

Proposition 1.2.4. *ψ defines a rational biholomorphism between $\mathcal{L} \simeq \mathbb{CP}^1 \subset \mathbb{CP}^2$ and $\mathcal{C}$, which maps the point at infinity in $\mathcal{L}$ to p_0.*

1.3 Cubics

We now move to degree 3 curves, which are commonly referred to as *cubics*.

1.3.1 Real cubics

We just say a few words about real cubics defined by the formula $y^2 = x^3 + ax + y$ where $a, b \in \mathbb{R}$. When experimenting on examples, one easily finds two types of examples:

- cases with one unbounded component;
- cases with two components, one bounded homeomorphic to a circle and one unbounded one.

We leave the following question open: are these two cases the only ones that can occur? An answer will be within reach after having treated the complex case.

1.3.2 Elliptic curves

We restrict our attention to curves in $\mathbb{C}^2$ given by an equation of the form

$$y^2 = x^3 + ax + b.$$

Such a curve can be compactified by adjoining it a point at infinity. Formally, considering $\mathbb{C}^2$ as a open subset of $\mathbb{CP}^2$, we see that by adjoining the point $[0 : 1 : 0] \in \mathbb{CP}^2$ to $\mathcal{C}$ we get a compact holomorphic curve in $\mathbb{CP}^2$. We are going to focus on the following questions.

1. What does $\mathcal{C}$ look like?
2. When are two cubics isomorphic?
3. Can $\mathcal{C}$ be parametrised in a nice way?
4. When a and b are real, how many connected components does the curve $y^2 = x^3 + ax + b$ have?

1.3.3 The Weierstrass $\mathcal{P}$ function

A bit of history. Lack of time and space in these notes prevents us from exploring the theory of special functions, which is a shame if one wants to truly appreciate the beauty of what is to come.

At the turn of the 19th century, a rigorous theory of continuous (or analytic) functions is not in sight, and mathematicians work only with functions they know how to define (either with a formula or whatever "natural" process). That leaves them with

- polynomials;
- trigonometric functions;
- the exponential and its inverse, the logarithm.

(One might well argue that we could merge the last two categories into one). At any rate, any function cropping up in nature is expected to be constructed using these building blocks, using usual algebraic operations and/or composition of functions. For mathematicians, it is therefore a massive problem when a function can be shown to exist in nature, but can't easily be shown to belong to the above category, as it challenges the mental image that they have formed of what a *reasonable* function should be. Two classes of such functions are the following.

1. Functions which give the length of an arc of an ellipse as a function of the interior angle of the arc. These are often referred to as *elliptic functions*, and can be shown to be integrals of the form

$$F(x) = \int_0^x \frac{dt}{\sqrt{P(t)}}$$

   where P is a real polynomial of degree 3.

2. Solutions to certain differential equations coming from physics and astronomy.

Of course, after many failed attempts to show that these functions belonged to the category of functions obtained from "usual" functions using basic operations, people started to suspect that they may have to add an extra entry to the list of *reasonable* functions in order to be able to include these very real examples that they would come across in nature.

It is in this context, of trying to understand the different classes of special functions which mathematicians kept coming across, that the theory of algebraic curves and Riemann surfaces took off. One cannot really appreciate the light that the work of Riemann and his successors shone on the world of mathematics in the nineteenth century if one does not appreciate what were the challenges of the time.

Back to maths. With all this in mind, we should think how what is coming next as the introduction of a remarkable new function.

Let ω_1 and ω_2 be two complex numbers which are linearly independent over the real numbers. If $\Lambda = \mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2$ the additive group they generate, we have the following facts

- Λ is isomorphic to $\mathbb{Z}^2$;
- Λ is discrete;
- the quotient $\mathbb{C}/\Lambda$ is homeomorphic to a torus $S^1 \times S^1$.

We leave this last point as an important exercise that I strongly recommend one does.

Exercise 1.3.1. With the assumption that ω_1 and ω_2 be two linearly independent complex numbers over the real numbers, show that $\mathbb{C}/\Lambda$ is homeomorphic to a torus $S^1 \times S^1$.

Definition 1.3.2. The following formula

$$\mathcal{P}(z) := \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

defines a meromorphic function on $\mathbb{C}$, which has a pole of order 2 at each $\omega \in \Lambda$.

We leave it to our reader to verify that the series defining $\mathcal{P}$ converges uniformly absolutely (locally) on $\mathbb{C} \setminus \Lambda$.

You are probably wondering at this stage what is the connection between this meromorphic function and everything that we have discussed so far. The first answer, which we cannot justify just yet, is the following.

> The function $\mathcal{P}$ belongs to the new class of functions whose introduction is needed to account for functions of the form $x \mapsto \int_0^x \frac{dt}{\sqrt{P(t)}}$ with P a degree 3 polynomial.

As we said above, we can't yet justify this point *but* we can note that $\mathcal{P}$ is not just any meromorphic function:

- $\mathcal{P}$ is meromorphic over the whole complex plane;
- but most importantly, $\mathcal{P}$ is *doubly periodic*: we have $\mathcal{P}(z) = \mathcal{P}(z+\omega_1) = \mathcal{P}(z+\omega_2)$ where ω_1 and ω_2 are generators of Λ.

This periodicity is remarkable in itself (how many entire functions do you know which are just 1-periodic?), and will serve as a base for our next endeavour.

For historical reasons that one can guess or research on the internet or elsewhere, the function $\mathcal{P}$ goes by the name of **Weierstrass function**.

1.3.4 Uniformisation of some cubics

The other reason why we introduced the function $\mathcal{P}$ is because it is going to allow us to prove the following important theorem, which essentially achieves a *good parametrisation* of cubics.

Theorem 1.3.3. *The function*

$$\begin{array}{ccc} \mathbb{C} \setminus \Lambda & \longrightarrow & \mathbb{C}^2 \\ z & \longmapsto & (\mathcal{P}(z), \mathcal{P}'(z)) \end{array}$$

maps $\mathbb{C} \setminus \Lambda$ *onto a cubic* $\{y^2 = x^3 + ax + b\}$.

Before proving this theorem, we are going to derive from it some important consequences. The first one is a projective version of it which will allow us to account for the full cubic $\mathcal{C} := \{[x : y : z] \in \mathbb{CP}^2 \mid zy^2 = x^3 + axz^2 + bz^3\}$.

Corollary 1.3.4. *The map defined above extends to a holomorphic map*

$$\psi : \mathbb{C} \longrightarrow \mathbb{CP}^2$$

whose image is exactly $\mathcal{C}$. Moreover, ψ is locally injective i.e. $\psi'(z) \neq 0$ for all $z \in \mathbb{C}$.

Keeping in mind that we built ψ out of a periodic function $\mathcal{P}$, we get that ψ above also enjoys the same periodic properties, namely

$$\forall \omega \in \Lambda, \forall z \in \mathbb{C},\ \psi(z + \omega) = \psi(z).$$

This implies, with ever so slightly more work, the following theorem.

Theorem 1.3.5. *The map ψ above induces a biholomorphism:*

$$\phi : \mathbb{C}/\Lambda \longrightarrow \mathcal{C}.$$

This theorem (whose proof is a good exercise, all that remains to be proven is the injectivity) answers one of the questions that we had initially posed:

> There is a cubic curve $\mathcal{C}$ which is homeomorphic to a torus $S^1 \times S^1$. In particular, it is *essentially* different from a conic!

1.3.5 Proof of Theorem 1.3.3

We now give a (sketch of) proof of Theorem 1.3.3. We are going to try an explain why *a priori* we could expect it to be true. The theorem is equivalent to proving the existence of $a, b \in \mathbb{C}$ such that

$$\forall z \in \mathbb{C} \setminus \Lambda,\ \mathcal{P}'(z)^2 = \mathcal{P}(z)^3 + a\mathcal{P}(z) + b.$$

> **Warning for the reader.** The proof that we are about to give is a modern proof that you will find in most textbooks. In my opinion, it is very flawed because it hides the lengthy process whereby people were led to guess that the function $\mathcal{P}$ was indeed a solution of this particular equation. It is a long story that involves those attempts to compute $\int_0^x \frac{dt}{\sqrt{t^3+at+b}}$ that we have mentioned earlier.

What puts us in a good position to find that $\mathcal{P}$ satisfy a reasonably simple differential equation is the three following facts

1. $\mathcal{P}$ and therefore all its derivatives are doubly-periodic.
2. $\mathcal{P}$ and its derivatives have poles at the same points.
3. Up to translating, $\mathcal{P}$ and its derivatives have only one pole.

It is therefore sufficient to find an algebraic combination of $\mathcal{P}$ and its derivatives where poles **at 0** compensate to give an entire function. By double-periodicity it'll have to be bounded and hence constant by Liouville's theorem.

Now that we have laid out this strategy, we leave to our reader that the function

$$z \mapsto \mathcal{P}'(z)^2 - \mathcal{P}(z)^3 - a\mathcal{P}(z)$$

does not have a pole at 0 if a is chosen to be equal to $60 \sum_{\omega \neq 0 \in \Lambda} \frac{1}{\omega^2}$. This will complete the proof of Theorem 1.3.3.

1.3.6 Further results

The above discussion leaves a few questions hanging. We have shown that give a lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, there exists a cubic in $\mathbb{CP}^2$, which we denote by $\mathcal{C}_\Lambda$ such that

$$\mathcal{C}_\Lambda \text{ is biholomorphic to } \mathbb{C}/\Lambda.$$

A natural question at this stage is whether all cubics can be made biholomorphic to a torus $\mathbb{C}/\Lambda$. It is indeed the case.

Theorem 1.3.6. *Let $\mathcal{C}_{a,b} \subset \mathbb{CP}^2$ a regular*[1] *conic defined by an equation of the form $y^2 = x^3 + ax + b$. Then there exists a unique (up to similarity) lattice Λ such that $\mathcal{C}_{a,b}$ is biholomorphic to $\mathbb{C}/\Lambda$.*

We actually have all the ingredients to prove this result, as the construction that we have given is very explicit (it gives the coefficients a and b associated to a lattice Λ). We leave it as interesting exercise for the motivated reader.

[1]The regularity condition can be shown to be equivalent to the equation $4a^3 + 27b^2 \neq 0$.

1.3.7 The Riemann surface point of view

Riemann surfaces. We now interpret the results that we have proven in a more topological and complex analytic way. We recall the following definition.

Definition 1.3.7 (Riemann surface). A Riemann surface is a topological surface endowed with an atlas of charts whose transitions maps are holomorphic.

Regular conics and cubics that we have studied are prime examples of compact Riemann surfaces. So are the more "geometric" examples given by quotients $\mathbb{C}/\Lambda$ for Λ a lattice, or even the basic Riemann sphere $\mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$.

Universal cover of cubics. When one is given a Riemann surface S, its universal cover $\tilde{S}$ inherits a Riemann surface structure (from the usual pull-back). In general, this is no reason $\tilde{S}$ should be a nice example of simply-connected Riemann surface which we already know of. The case of conics is quite remarkable in this view. The following theorem is just a restatement of the results form paragraph 1.3.4.

Theorem 1.3.8 (Uniformisation of cubics). *Recall that $\mathcal{C} \subset \mathbb{CP}^2$ is the cubic associated with a lattice Λ, and $\mathcal{P}$ is the associated Weierstrass function. The map*

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathcal{C} \\ z & \longmapsto & (\mathcal{P}(z), \mathcal{P}'(z)) \end{array}$$

is a covering map, and since $\mathbb{C}$ is simply-connected, $\mathbb{C}$ is the universal cover of $\mathcal{C}$.

In other words, cubics are *uniformised* by the complex plane.

1.4 A few words about a quartic

What happens for *quartics*, curves defined by polynomials of degree 4? It is a more difficult question than for cubics.

- What is the genus of a quartic?
- Can we parametrise (uniformise?) a quartic using the complex plane too?
- What is the type of the universal cover of a quartic, as a Riemann surface?

We mention an important example of historical importance. Felix Klein, a German mathematician of the late 19^{th} century, considers the complex curve $\mathcal{Q}$ (in $\mathbb{CP}^2$) defined by the following equation:

$$\{x^3y + y^3z + z^3x = 0\}.$$

He proved the following remarkable parametrisation theorem. Recall that $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

THEOREM 1.4.1. *$\mathcal{Q}$ is a genus 3 surface. Up to removing a set S consisting of 24 points*[2] *from it, the following map $\tau \mapsto (x(\tau), y(\tau))$ (with the notation $q = e^{2i\pi\tau}$)*

$$x(\tau) = q^{-\frac{1}{7}} \frac{\sum_{k\in\mathbb{Z}} (-1)^k (q^{\frac{1}{2}(21k^2+37k+16} - q^{\frac{1}{2}(21k^2+19k+4)})}{\sum_{k\in\mathbb{Z}} (-1)^{k+1} q^{\frac{1}{2}(21k^2+7k}}$$

and

$$y(\tau) = q^{\frac{4}{7}} \frac{\sum_{k\in\mathbb{Z}} (-1)^{k+1} q^{\frac{1}{2}(21k^2+7k}}{\sum_{k\in\mathbb{Z}} (-1)^k (q^{\frac{1}{2}(21k^2+25k+8} - q^{\frac{1}{2}(21k^2+31k+12)})}$$

defines a holomorphic parametrisation of $\mathcal{Q}$ S, which is also its universal covering map.

We don't comment much on the proof; in spirit it is similar to the way we have dealt with cubics: one starts finding special functions on $\mathbb{H}$, analogue to the $\mathcal{P}$-functions, which are good candidates to satisfy an algebraic equation of degree 4. These functions are called θ-functions, and are defined using a formula of the form

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{in^2z}$$

which converges for $z \in \mathbb{H}$.

1.5 Brief summary

We briefly summarise some of the result that we have proven (or not) to try and take a step back. We summarise below the different characteristic of curves depending on the degree of the polynomials defining them.

Type of curves	**Genus**	**Universal cover**
Conics	0	$\hat{\mathbb{C}} = \mathbb{CP}^1$
Cubics	1	$\mathbb{C}$
Quartics	3 and ?	maybe $\mathbb{H}$?
Degree > 4	?	?

[2] These points are the inflections points of $\mathcal{Q}$, that is those points where the Hessian of $x^3y + y^3z + z^3x$ is degenerate.

As we can see, they enjoy strikingly different geometric properties. We can start guess the uniformisation theorem, which in its strongest form will tell us that apart from conics and cubics which are covered by the Riemann sphere and the plane respectively, algebraic curves a parametrised, by way of their universal cover, by the upper-half plane $\mathbb{H}$.

1.6 Exercises

I have found that a lot of people know a lot about Riemann surfaces, algebraic curves and other advanced topics, but can be a bit shaky on the foundations in complex analysis and algebraic topology (myself and other specialists included). Here is handful of exercises that one should know how to do if one is serious about learning Riemann surfaces. Be careful, they are harder than they look!

Exercise 1.6.1.

1. Let P and Q be two complex polynomials and assume that Q is non-constant. Show that the meromorphic function $z \mapsto \frac{P(z)}{Q(z)}$ extends to an holomorphic function $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$. Such a function is called a *rational* map.
2. Show that the set of holomorphic maps from the Riemann sphere to itself is exactly the set of rational maps.

Exercise 1.6.2. Show that the group of biholomorphisms of the complex $\mathbb{C}$ is exactly the set of affine maps.

Exercise 1.6.3. Show that the group of biholomorphisms of the Riemann sphere $\mathbb{CP}^1$ is exactly the set of homographies.

Exercise 1.6.4. Let Λ_1 and Λ_2 be two lattices such that $\mathbb{C}/\Lambda_1$ is biholomorphic to $\mathbb{C}/\Lambda_2$. Show that there exists $a \in \mathbb{C}^*$ such that $\Lambda_2 = a\Lambda_1$.

Exercise 1.6.5. Let $\mathcal{C}$ be the cubic defined by the equation $y^2 = x^3 + ax + b$.

1. Show that $\mathcal{C}$ is a complex curve (*i.e.* a non-singular variety) if and only if $4a^3 \neq 27b^3$.
2. How many points at infinity does $\mathcal{C}$ have ?

Exercise 1.6.6. Let Σ be compact Riemann surface whose universal cover is biholomorphic to $\mathbb{C}$. Show that there exists a lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ such that Σ is biholomorphic to $\mathbb{C}/\Lambda$.

Exercise 1.6.7. Show that non open subset of regular cubic $\mathcal{C}$ defined by the equation $y^2 = x^3 + ax + b$ can be parametrised algebraically. In other words, show that there can't be a map $\psi : U \subset \mathbb{C} \longrightarrow \mathcal{C} \subset \mathbb{C}^2$ such that the coordinates of ψ are (restrictions to U of) rational functions.

Exercise 1.6.8. Show that a real cubic in $\mathbb{R}^2$, defined by an equation of the form $y^2 = x^3 + ax + b$, has at most two connected components.

Exercise 1.6.9. Show that two cubics in $\mathbb{C}^2$, defined by the equations $y^2 = x^3 + ax + b$ and $y^2 = x^3 + \alpha x + \beta$ are biholomorphic if and only if $(a, b) = (\alpha, \beta)$.

Exercise 1.6.10. Show that any quartic in $\mathbb{CP}^2$ has at most 24 points of inflection.

Bibliography

[1] Henri Paul de Saint-Gervais, *Uniformization of Riemann surfaces*, Heritage of European Mathematics, European Mathematical Society (EMS), Zürich, 2016. Revisiting a hundred-year-old theorem, Translated from the 2010 French original [MR2768303] by Robert G. Burns, The name of Henri Paul de Saint-Gervais covers a group composed of fifteen mathematicians: Aurélien Alvarez, Christophe Bavard, François Béguin, Nicolas Bergeron, Maxime Bourrigan, Bertrand Deroin, Sorin Dumitrescu, Charles Frances, Étienne Ghys, Antonin Guilloux, Frank Loray, Patrick Popescu-Pampu, Pierre Py, Bruno Sévennec and Jean-Claude Sikorav. MR3494804