Introduction to Subconvexity

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Problem (Subconvexity)

Prove non-trivial upper bounds for L-functions on the critical line.

1. What is an L-function?
2. What does it mean to be a non-trivial upper bound?

Basic examples of L-functions

- Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

in $\mathrm{Re}(s) > 1$. Analytic continuation and Functional Equation:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

- Dirichlet L-function (χ primitive modulo q, $\chi(-1) = 1$):

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

in $\mathrm{Re}(s) > 1$. Analytic continuation and Functional Equation:

$$(\frac{\pi}{q})^{-s/2}\Gamma(s/2)L(s,\chi) = (\frac{\tau(\chi)}{\sqrt{q}})(\frac{\pi}{q})^{-(1-s)/2}\Gamma((1-s)/2)L(1-s,\overline{\chi}).$$

L-functions in general

A meromorphic function $L(s,f)$ is called an L-function if:

1. Dirichlet series and Euler product: There exists $n \geq 1$ and complex numbers $\alpha_{i,f}(p)$, $i = 1, \dots n$ with $|\alpha_{i,f}(p)| < p$ s.t.

   $$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \frac{1}{(1-\alpha_{f,1}(p)p^{-s}) \cdots (1-\alpha_{f,1}(p)p^{-s})},$$

   absolutely convergent in $\mathrm{Re}(s) > 1$.
   For example: $\lambda_f(1) = 1$, $\lambda_f(p) = \alpha_{1,f}(p) + \cdots + \alpha_{n,f}(p)$, etc.
2. Meromorphic continuation to $\mathbb{C}$; poles only at $s = 0, 1$.
3. There exist $\mu_{i,f} \in \mathbb{C}$, $i = 1, \dots, n$ and $q = q(f) \in \mathbb{N}$ s.t.

   $$\Lambda(s,f) := q^{s/2} \left(\prod_{i=1}^{n} \pi^{-s/2} \Gamma\left(\frac{s+\mu_i}{2} \right) \right) L(s,f),$$

   $$\Lambda(s,f) = \epsilon(f) \overline{\Lambda(1-\overline{s},f)}, \quad |\epsilon(f)| = 1.$$

Analytic Conductor

Definition

The quantity

$$q(s, f) = q \prod_{i=1}^{n} (|s + \mu_j| + 1)$$

is called the analytic conductor of $L(s, f)$.

Measures the analytic "complexity" of $L(s, f)$.

Theorem (Approximate Functional Equation)

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{\sqrt{q}}\right) + \varepsilon(s, f) \sum_{n=1}^{\infty} \frac{\overline{\lambda_f(n)}}{n^{1-s}} V_{1-s}\left(\frac{n}{\sqrt{q}}\right)$$

Upshot: $L(s, f)$ is expressible by two sums of length $\sqrt{q(s, f)}$.

Estimating the AFE trivially (if $\lambda_f(n) \ll_\varepsilon n^\varepsilon$):

Theorem (Convexity Bound)

$$L(1/2+it,f) \ll_\varepsilon q(1/2+it,f)^{1/4+\varepsilon}.$$

Also possible without the hypothesis "$\lambda_f(n) \ll_\varepsilon n^\varepsilon$" by the Pragramen-Lindelöf convexity principle.

Problem (Subconvexity)

Show that there exists $\delta > 0$ such that for all $L(s,f)$ we have

$$L(1/2+it,f) \ll q(1/2+it,f)^{1/4-\delta}.$$

Theorem

Suppose that the Riemann Hypothesis holds for $L(s,f)$. Then

$$L(1/2+it,f) \ll_\varepsilon q(1/2+it,f)^{\varepsilon}.$$

Deep expression of conjectures of Langlands:

Conjecture

All L-functions of degree n arise as (products of) L-functions of cuspidal automorphic representations π *on* GL_n *over* $\mathbb{Q}$.

Theorem (Kaczorowski-Perelli)

The conjecture is true for $n < 2$.

Automorphic representations factor: $\pi \simeq \bigotimes_v \pi_v$,
where π_v is a certain representation of $\mathrm{GL}_n(\mathbb{Q}_v)$.

The analytic conductor $q(s, f)$ can be defined purely in terms of data attached to each π_v:

$$q(s, f) = \prod_v q(\pi_v),$$

where $q(\pi_\infty) = \prod_{i=1}^n (|s + \mu_j| + 1)$ and $q(\pi_v) = 1$ if $v \nmid q$.

Results

First subconvexity result: Hardy-Littlewood-Weyl (1920s):

$$\zeta(\frac{1}{2} + it) \ll (1 + |t|)^{\frac{1}{6}+\varepsilon}.$$

Based on the method of Weyl differencing;
invariance of continuous functions under translation.

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Burgess (1962) χ primitive modulo q:

$$L(1/2,\chi) \ll q^{\frac{3}{16}+\varepsilon}$$

Throws away correlation between character sums on many very short intervals, but uses Hölder and RH for *curves* over finite fields.

Subconvexity results for GL_2 developed by Duke, Friedlander and Iwaniec (1993-2002), culminating with:

Theorem (DFI)

Let f be a primitive cuspidal Maass waveform with Laplace eigenvalue $1/4 + t_f^2$ on $\Gamma_0(D)$ which is a eigenfunction of the Hecke algebra. Then

$$L(s, f) \ll (|t_j| + |s|)^{10} D^{\frac{1}{4} - \frac{1}{23041}}$$

for $\mathrm{Re}(s) = 1/2$.

DFI also prove subconvex bounds in all of the other aspects

Theorem

Michel-Venkatesh There exists δ such that for all π on GL_1 or GL_2 over a number field we have

$$L(1/2, \pi) \ll q(\pi)^{1/4-\delta}.$$

Han Wu: Specified $\delta < 1/1889$ (in worst case that $q(\pi) = q(\omega_\pi)$).

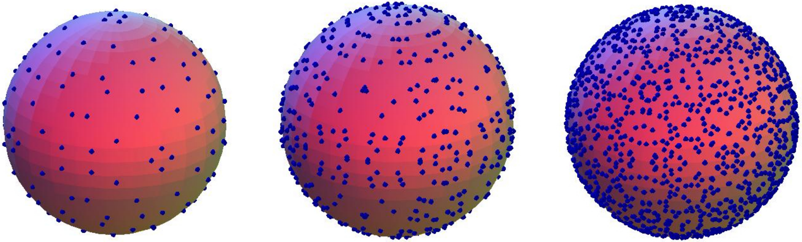
Why do we care?

Applications:

- Representation of integers by ternary quadratic forms
- Equidistribution of integral points on ellipsoids
- Equidistribution of Heegner points/cycles
- Quantum Unique Ergodicity
- Quantitative applications in many other problems...

Striking fact: in many of the above, convexity gives no applications, subconvexity for any small $\delta > 0$ implies equidistribution.

Example: Equidistribution of integer points on spheres:



Integer points on spheres of radii $101, 8011, 104851$, respectively

Image credit: Ellenberg, Michel, and Venkatesh.

Beyond GL_2

First subconvex result beyond GL_2:

Theorem (X. Li 2011)

We have

$$L(1/2+it, \mathrm{sym}^2 f) \ll (1+|t|)^{3/2-1/16+\varepsilon}.$$

Adapting ideas of Conrey-Iwaniec, in the vein of DFI results using moments/amplification technique.

Theorem (Munshi 2015)

We have for any π on $\mathrm{SL}_3(\mathbb{Z})$ (i.e. unramified)

$$L(1/2+it, \pi) \ll (1+|t|)^{3/2-1/16+\varepsilon}.$$

Uses δ-symbol method, circle method.

Theorem (Munshi)

There exists $\delta > 0$ *such that for* π *as above,* χ *conductor q we have*

$$L(1/2, \pi \otimes \chi) \ll q^{3/2-\delta}.$$

Many other such results!