Illinois Journal of Mathematics
Volume 50, Number 4, Winter 2006, Pages 763–790
S 0019-2082

ON THE RADIUS OF CONVERGENCE OF THE LOGARITHMIC SIGNATURE

TERRY J. LYONS AND NADIA SIDOROVA

ABSTRACT. It has recently been proved that a continuous path of bounded variation in $\mathbb{R}^d$ can be characterised in terms of its transform into a sequence of iterated integrals called the signature of the path. The signature takes its values in an algebra and always has a logarithm. In this paper we study the radius of convergence of the series corresponding to this logarithmic signature for the path. This convergence can be interpreted in control theory (in particular, the series can be used for effective computation of time invariant vector fields whose exponentiation yields the same diffeomorphism as a time inhomogeneous flow) and can provide efficient numerical approximations to solutions of SDEs. We give a simple lower bound for the radius of convergence of this series in terms of the length of the path. However, the main result of the paper is that the radius of convergence of the full log signature is finite for two wide classes of paths (and we conjecture that this holds for all paths different from straight lines).

1. Overview

Consider a controlled differential equation of the form

$$dy_t = \sum_i a_i(y_t)d\gamma_t^i = A(y_t)d\gamma_t, \tag{1.1}$$

where the a_i are vector fields, $\gamma_t \in V$ represents some controlling multi-dimensional signal, and $y_t \in W$ represents the response of the system. These differential equations arise in stochastic analysis as well as in many deterministic problems from pure and applied mathematics. In many of these settings it is not natural to assume that the control γ is differentiable on normal timescales.

Over the last few years a general theory of Rough Paths has been developed to give meaning to differential equations without assuming differentiability of γ and which allows one to integrate them (see [8] and [9] for references). In particular, it provides a new pathwise foundation for Itô stochastic differential

Received June 1, 2005; received in final form June 7, 2006.
2000 *Mathematics Subject Classification.* 60H10, 34A34, 93C35.

equations, and extends the possible stochastic "driving noise" or controls to include those like Fractional Brownian motion, that are not semi-martingales and so are outside the Itô framework ([1], [4]).

One of the key techniques in the Rough Path Theory comes from the description of the control γ through its iterated integrals. Collectively, these integrals (which are polynomials on path space) capture the time ordered nature of the control process. For example, in the case where y is finite dimensional and A is linear in y, the solution of (1.1) can be represented as a series

$$y_t = \sum_{n=0}^{\infty} A^{*n}(y_0) \int_{0<u_1<\cdots<u_n<t} \cdots \int d\gamma_{u_1} \cdots d\gamma_{u_n} \tag{1.2}$$

of iterated integrals of γ where

$$A^{*n}(y)d\gamma_{u_1} \cdots d\gamma_{u_n} := A(A(\ldots A(y_0)d\gamma_{u_n} \ldots)d\gamma_{u_2})d\gamma_{u_1}$$

is the natural multi-linear extension of $A : V \to \mathrm{Hom}_{\mathbb{R}}(W, W)$ to a map A^{*n} from the n-th tensor power of the space V. The representation (1.2) goes back to Chen ([2]) and will be described in more precision later in the introduction. The representation makes it clear that the solution to any linear differential equation at time T can be determined by examining the series of iterated integrals

$$\left(\int_{0<u_1<\cdots<u_n<T} \cdots \int d\gamma_{u_1} \cdots d\gamma_{u_n} \right)_{n=1}^{\infty} \tag{1.3}$$

for γ.

This transformation of a path γ into a sequence of algebraic coefficients which together characterise the response of any linear system to γ already seems interesting. But it is deterministic and this perspective offers new insights in the stochastic setting as well. For example, for random γ one may consider the random variable (1.3). Its expectation can completely characterise the process γ even in cases where it is not Markovian and is a sort of fully non-commutative Laplace transform. In [5] Fawcett proves, under slight restrictions, that the law of the signature of Brownian Motion on $[0, 1]$ is characterised by its expectation.

We are interested in understanding the behaviour of the iterated integrals of general rough paths. However, in this paper, we will focus on the case where the driving signal is of bounded variation. Following [6] we interpret the whole collection of iterated integrals as a single algebraic object, known as the signature, living in the algebra of formal tensor series. This representation exposes the natural algebraic structure on the signatures of paths induced by the analytic structure as rough paths.

The logarithm (in the sense of the tensor algebra) of the signature (see (2.1), (2.2) for definition) is in some sense the optimal object to describe the control γ, and its convergence or divergence is essential for the existence or non-existence of the logarithm of the flow for equation (1.1) and seems worth of study in its own right.

2. Introduction

Let V be a real Banach space. Let $\|\cdot\|$ be cross-norms on the algebraic tensor products $V^{\otimes n}$, which means that $\|x \otimes y\| = \|x\| \, \|y\|$ for all $x \in V^{\otimes k}$, $y \in V^{\otimes m}$ for all k, m and in the case $n = 1$ the norm coincides with the specified Banach norm on V. Of course, such norms are not uniquely defined by the norm on V but do exist. For example, if $V = \mathbb{R}^d$ with the p-norm

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}$$

with respect to a basis (e_i) then the p-norms

$$\|u\|_p = \left(\sum_{1 \le i_1, \dots, i_n \le d} |x_{i_1 \cdots i_n}|^p \right)^{1/p}$$

on the tensor products $(\mathbb{R}^d)^{\otimes n}$ with respect to the bases $(e_{i_1} \otimes \cdots \otimes e_{i_n})$ are cross-norms. The same is true for the ∞-norms.

Denote by $V^{\widehat{\otimes} n}$ the completions of the spaces $V^{\otimes n}$ under the cross-norms $\|\cdot\|$. Further, denote by T the tensor algebra generated by V and the cross-norms, i.e.,

$$T = \mathbb{R} \oplus V \oplus V^{\widehat{\otimes} 2} \oplus \cdots \oplus V^{\widehat{\otimes} n} \oplus \cdots$$

and by $\widehat{T}$ its completion with respect to the augmentation ideal (i.e., the set of formal infinite sums).

Let $\gamma : [0, \theta] \to V$ be a continuous path of bounded variation. Following [8], we define the n-th iterated integral

$$\mathcal{S}^{(n)}(\gamma) = \int \cdots \int_{0 < u_1 < \cdots < u_n < \theta} d\gamma(u_1) \widehat{\otimes} \cdots \widehat{\otimes} d\gamma(u_n) \in V^{\widehat{\otimes} n}$$

and we call

$$\mathcal{S}(\gamma) = 1 + \mathcal{S}^{(1)}(\gamma) + \cdots + \mathcal{S}^{(n)}(\gamma) + \cdots \in \widehat{T}$$

the signature of γ.

Recent work [6] shows that, up to tree-like paths, a path is fully described as a control by its signature in a similar way to a function on a circle being determined, up to Lebesgue null-sets, by its Fourier coefficients. However, there are many algebraic dependencies between different iterated integrals and so one has a lot of redundancy in the whole sequence $\mathcal{S}(\gamma)$. This can already be seen when $V = \mathbb{R}^2$ with the Euclidean norm. Denote by γ_1 and γ_2 the coordinates of γ with respect to the standard basis (e_1, e_2) and assume for simplicity that γ is continuously differentiable and starts at zero. As the first iterated integral is just the increment of the path, we have

$$\mathcal{S}_i^{(1)} = \gamma_i(\theta), \quad i = 1, 2.$$

Further, integrating by parts we obtain

$$\begin{aligned} \mathcal{S}_{ii}^{(2)} &= \frac{1}{2}\gamma_i(\theta)^2 = \frac{1}{2}(\mathcal{S}_i^{(1)})^2, \quad i = 1, 2, \qquad \text{and} \\ \mathcal{S}_{12}^{(2)} + \mathcal{S}_{21}^{(2)} &= \gamma_1(\theta)\gamma_2(\theta) = \mathcal{S}_1^{(1)}\mathcal{S}_2^{(1)}. \end{aligned}$$

Thus, the only new information about the path contained in the second iterated integral (compared with the information available from the first one) is the difference $\mathcal{S}_{12}^{(2)} - \mathcal{S}_{21}^{(2)}$, which is the coordinate of the tensor $\mathcal{S}^{(2)}$ in the direction $[e_1, e_2] = e_1 \otimes e_2 - e_2 \otimes e_1$. Hence it is only one-dimensional while the dimension of $V \otimes V$ is equal to four.

The latter observation and the appearance of Lie brackets motivate the consideration of the tensor Lie algebra. As $\widehat{T}$ is an associative algebra we can define the Lie bracket $[u, v] = u \otimes v - v \otimes u$ of $u, v \in \widehat{T}$ and consider the Lie subalgebra

$$L = V \oplus [V, V] \oplus \cdots \oplus [V, \ldots [V, V] \ldots] \oplus \cdots \subset T$$

generated by V as well as its augmentation $\widehat{L}$. In fact, *the superfluity of information contained in* $\mathcal{S}(\gamma)$ *can be avoided* by injecting the signatures of paths into $\widehat{L}$. A natural mapping is $\log : \widehat{T} \to \widehat{T}$ defined (on a subset of $\widehat{T}$) by the corresponding power series

$$\log(1 + u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^{\widehat{\otimes} n} \tag{2.1}$$

whenever $u \in V \oplus V^{\widehat{\otimes} 2} \oplus \cdots$. It is injective, and the inverse is given by exp:

$$\exp u = \sum_{n=0}^{\infty} \frac{1}{n!} u^{\widehat{\otimes} n}.$$

These two functions are intimately connected with the signatures of paths. In fact, the signature of a path is always the exponential of an element in L (which must be unique because of the existence of log as an inverse function). Noting that our paths γ take their values in a vector space and that the signature $\mathcal{S}(\gamma)$ is invariant under reparametrisation of paths it is natural to consider the operation of concatenation $\cup$. $\mathcal{S}$ is a homomorphism from paths with $\cup$ to $\widehat{T}$ with $\otimes$. It is an easy exercise to show that the range of the map is a group and so the logs of signatures can be regarded as some sort of formal

Lie algebra for this group. Moreover, concatenations of paths correspond to the Campbell-Baker-Hausdorff formula.

The logarithm of the signature

$$\mathcal{LS}(\gamma) = \log \mathcal{S}(\gamma) \in \widehat{L} \tag{2.2}$$

is called the logarithmic signature of γ.

It follows from the classical Rashevski-Chow Theorem (see [3] and [11]) that there are no algebraic dependencies between the coefficients of $\mathcal{LS}(\gamma)$ and so, in contrast to the usual signature, there is no redundancy in the logarithmic signature.

It is also natural to scale a path in V and consider the map $f_\gamma : \lambda \to \mathcal{S}(\lambda\gamma)$. Notice that

$$\mathcal{S}^{(n)}(\lambda\gamma) = \lambda^n \mathcal{S}^{(n)}(\gamma)$$

and so

$$f_\gamma(\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathcal{S}^{(n)}(\gamma).$$

It is an easy exercise for γ with bounded variation, that

$$\|\mathcal{S}^{(n)}(\gamma)\| \leq \frac{l(\gamma)^n}{n!},$$

where $l(\gamma)$ is the length of γ in our chosen norm, and a factorial estimate holds for any rough path ([8]). In particular, $f_\gamma(\lambda)$ is not only a formal power series, and extends to an entire analytic function of exponential type.

In fact, it is interesting to know that our path γ defines a family of entire functions over several and indeed infinitely many complex variables. Suppose the path γ is defined on $[0, \theta]$, $P = \{0 = t_0 < t_1 < \cdots < t_n = \theta\}$ is a partition, $\gamma^i = \gamma|_{[t_i, t_{i+1}]}$ denotes the restriction of γ to $[t_i, t_{i+1}]$, and $\lambda_1, \ldots, \lambda_n$ are positive real numbers and consider the concatenation of the paths $\lambda_i \gamma^i$. Its signature is given by $f_{\gamma^1}(\lambda_1) \otimes \cdots \otimes f_{\gamma^n}(\lambda_n)$. One can regard this operation as a primitive integral and observe that in general, if τ is a smooth $\mathrm{Hom}_{\mathbb{R}}(V, V)$-valued function defined on $[0, \theta]$ then $\tau \to \mathcal{S}\left(\int_0^\cdot \tau d\gamma\right)$ extends to an analytic function mapping complex paths τ to the tensor algebra.

We leave further discussion of the several complex variable setting.

In this paper, the main question we will be discussing is the radius of convergence of the series $\mathcal{LS}(\gamma)$ defined by

$$R(\gamma) = \limsup_{n \to \infty} \|\mathcal{LS}^{(n)}(\gamma)\|^{-1/n},$$

where $\cdot^{(n)}$ is the natural projection of $\widehat{T}$ onto $V^{\widehat{\otimes} n}$. We will see that the map $f_\gamma(\lambda)$ already gives a great deal of information (as $\lambda \to +\infty$).

Definition 2.1. $R(\gamma)$ is called the radius of convergence of the logarithmic signature of γ.

One of the examples showing the importance and the origin of this problem is the construction of the logarithm of a flow. Consider a differential equation

$$dy = A(y)d\gamma, \tag{2.3}$$

where A is a linear map from V to a space of linear vector fields and suppose the fields form a Lie algebra. There is a flow corresponding to this equation and the goal is to find a fixed vector field which, if we flow along it for unit time, gives the same homeomorphism as solving the inhomogeneous differential equation over the whole time interval $[0, \theta]$. Integrating the equation (2.3) one obtains the classical expansion of the solution y

$$\begin{aligned} y_\theta &= y_0 + \int_0^\theta dy_{u_1} = y_0 + \int_0^\theta A(y_{u_1})d\gamma_{u_1} \\ &= y_0 + A(y_0) \int\limits_{0<u_1<\theta} d\gamma_{u_1} + \iint\limits_{0<u_1<u_2<\theta} A(A(y_{u_2})d\gamma_{u_2})d\gamma_{u_1} = \dots \\ &= \Big[I + A\ \mathcal{S}^{(1)}(\gamma) + AA\ \mathcal{S}^{(2)}(\gamma) + \cdots \Big] (y_0) \end{aligned} \tag{2.4}$$

into a series of iterated integrals of γ. Because of the universal property of the tensor Lie algebra L the map A extends to a unique Lie map A^∞, that is, a Lie-homomorphism from L into the Lie algebra of vector fields, and the logarithm of the flow should now be given by $A^\infty(\log \mathcal{S}(\gamma)) = A^\infty(\mathcal{LS}(\gamma))$. However, this formula only makes sense if the series $A^\infty(\mathcal{LS}(\gamma))$ converges, which depends on the relationship between $R(\gamma)$ and the norm of A. Namely, the series converges if $\|A\| \le R(\gamma)$. In Section 7 we will give an example of a flow on the circle which has no logarithm.

It is easy to see that if γ is a segment of a straight line, i.e., $\gamma(t) = tv$ for some $v \in V$, then $\mathcal{S}(\gamma) = \exp(\theta v)$ and so $\mathcal{LS}(\gamma) = \theta v$. This means that $\|\mathcal{LS}^{(n)}(\gamma)\| = 0$ for all $n \ge 2$ and therefore $R(\gamma) = \infty$. We conjecture that this is the only case when $\mathcal{LS}(\gamma)$ is an entire function.

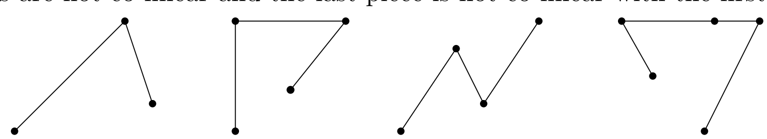
In this paper we show $R(\gamma) < \infty$ for two wide classes of paths: for 1-monotone paths (i.e., for paths which are monotone at least in one direction) but different from a straight line and for non-double piecewise linear paths (in the sense of the definitions below).

Definition 2.2. A continuous path of bounded variation γ is said to be 1-*monotone* (or monotone in one direction) if there is a bounded linear functional $f \in V^*$ such that $f \circ \gamma$ is strictly monotone.

Example 2.3. Let γ_0 be a straight line parametrised at unit speed and γ be a differentiable path in $\mathbb{R}^d$ with the Euclidean norm. If $\langle \dot\gamma(t), \dot{\gamma_0}(t)\rangle > 0$ for all t or, in particular, if $\sup_t \|\dot\gamma(t) - \dot{\gamma_0}(t)\| < 1$ then γ is 1-monotone. In particular, the statement that $R(\gamma) < \infty$ for 1-monotone paths implies that

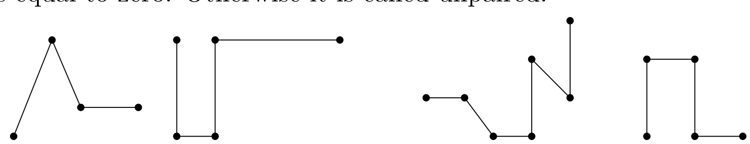
R has an isolated singularity whenever the path is a segment of a straight line, with respect to the gradient norm on the space of paths.

DEFINITION 2.4. A piecewise linear path is called *generic* if its adjacent pieces are not co-linear and the last piece is not co-linear with the first one.



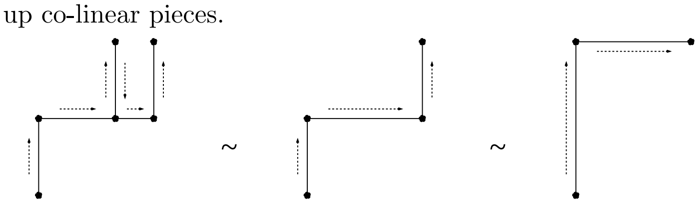
generic paths non-generic paths

A generic path is called paired if one can divide the set of its pieces into pairs such that the pieces from the same pair are either equal to each other or their sum is equal to zero. Otherwise it is called unpaired.



unpaired paths paired paths

Two piecewise linear paths are said to be equivalent if one can be transformed into the other by cyclic permutation of the linear pieces, by adding or removing equal pieces traversed in opposite directions and following each other, and by joining up co-linear pieces.



equivalent paths

Finally, a piecewise linear path in V is *non-double* if there are two bounded linear functionals $f, g \in V^*$ such that the path $(f \circ \gamma, g \circ \gamma)$ in $\mathbb{R}^2$ is equivalent to an unpaired path.

EXAMPLE 2.5. Any unpaired path in V (and any path equivalent to it) is a non-double path. In particular, any generic path in V consisting of an odd number of linear pieces (and any path equivalent to it) is a non-double path.

The strategy of the proof of $R(\gamma) < \infty$ in both classes of paths is based on finding a flow driven by the path γ that has no logarithm (which would be impossible if $\mathcal{LS}(\gamma)$ was an entire function).

More precisely, the idea is to find a matrix Lie algebra L_0 and a bounded linear mapping $F : V \to L_0$ such that the Cartan development $Y^{F,\gamma}$ of the image $X^{F,\gamma} = F \circ \gamma$ of the path γ into the corresponding Lie group G_0 would not lie in $\exp(L_0)$ at the terminal time θ. The development $Y^{F,\gamma}$ is defined using the right-invariant vector field corresponding to $X^{F,\gamma}$ as the solution of the equation

$$dY^{F,\gamma} = Y^{F,\gamma} dX^{F,\gamma}, \qquad Y^{F,\gamma}(0) = I. \tag{2.5}$$

It turns out that both for 1-monotone paths and for non-double piecewise linear paths it is worthwhile to take

$$L_0 = \mathfrak{sl}(2,\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr} A = 0\} \quad \text{and}$$
$$G_0 = \mathrm{SL}(2,\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det A = 1\}$$

since $\mathrm{SL}(2,\mathbb{R})\backslash\exp(\mathfrak{sl}(2,\mathbb{R}))$ is large enough to enable us to find an appropriate linear operator F.

In Section 3, we construct a suitable mapping F for 1-monotone paths, using the family of mappings F_h such that F_0 is easy to construct and brings the development to the boundary $\partial(\exp(\mathfrak{sl}(2,\mathbb{R})))$ at the terminal time θ. By varying the small parameter h one can bring the endpoint of the development outside $\exp(\mathfrak{sl}(2,\mathbb{R}))$.

In Section 4, we find an appropriate F for non-double piecewise linear paths. The key point of the construction is to use different scalings in the direction of the edge which occurs an odd number of times and in another direction, and to let the latter go to infinity.

Among all sequences of cross-norms on $V^{\otimes n}$ generated by the norm on V there are two special ones called the injective and projective cross-norms (see [13]). The injective norms $\varepsilon(\cdot)$ are the smallest and the projective norms $\pi(\cdot)$ are the greatest cross-norms, i.e.,

$$\varepsilon(x) \le \|x\| \le \pi(x)$$

for all $x \in V^{\otimes n}$ for all n. For example, the p-norms on the tensor products $(\mathbb{R}^d)^{\otimes n}$ are the injective cross-norms corresponding to the space $\mathbb{R}^d$ with the p-norm.

In order to prove the finiteness of $R(\gamma)$ we will look for lower bounds on the norms of the coefficients of $\mathcal{LS}(\gamma)$, which leads us to consider the injective norms. Denote by $R_\varepsilon(\gamma)$ the radius of convergence of the logarithmic signature of γ with respect to $\varepsilon(\cdot)$. Obviously,

$$R_\varepsilon(\gamma) \ge R(\gamma) \tag{2.6}$$

and so it will be sufficient to show that $R_\varepsilon(\gamma) < \infty$.

In Section 5 we prove the main results of the paper (Theorems 5.5 and 5.6). We give the precise definition of the injective norms and prove that the existence of the mapping F implies $R_\varepsilon(\gamma) < \infty$ and so $R(\gamma) < \infty$. Namely, the assumption that $\mathcal{LS}(\gamma)$ is an entire function leads to the convergence of the series $F^\infty(\mathcal{LS}(\gamma))$, where F^∞ denotes the natural extension of F to the tensor algebra T_ε corresponding to the injective norms. This means that the value of the development of $F \circ \gamma$ at the terminal time is given by $\exp(F(\mathcal{LS}(\gamma)))$, which would lead to a contradiction.

In Section 6, we find a lower bound for the radius of convergence (Theorem 6.1). Using mainly the factorial decay of the iterated integrals, we show that for any continuous path of bounded variation, $R(\gamma) \geq l(\gamma)^{-1}p$, where $l(\gamma)$ is the length of the path and p is such that $\log p = -p$.

Finally, in Section 7 we illustrate with an example the effect of the finiteness of the radius of convergence. If $R(\gamma) = \infty$ then the diffeomorphism (taken at terminal time) corresponding to the equation $dy = A(y)d\gamma$ always has a logarithm. We give an example of a diffeomorphism which, though produced by such an equation, has no logarithm. This is a direct consequence of $R(\gamma) < \infty$ as that can only happen if $\|A\| \geq R(\gamma)$. Further, we show explicitly how to change this diffeomorphism (and the operator A) in such a way that $\|A\|$ becomes less than $R(\gamma)$ and the new diffeomorphism has a logarithm.

3. The map $V \to \mathfrak{sl}(2,\mathbb{R})$ for 1-monotone paths

In this section we construct the operator $F : V \to \mathfrak{sl}(2,\mathbb{R})$ for 1-monotone paths which are not a concatenation of two equal pieces. First, we easily find a family of linear operators $F_0^{k,\lambda}$ such that whilst the Cartan developments of the images $F_0^{k,\lambda} \circ \gamma$ at the terminal time still have a logarithm, they lie on the boundary of $\exp(\mathfrak{sl}(2,\mathbb{R}))$ for all k and λ. Then, introducing a small parameter h we construct a small perturbation $F_h^{k,\lambda}$ of $F_0^{k,\lambda}$ which brings the development outside $\exp(\mathfrak{sl}(2,\mathbb{R}))$ unless a very strong condition on γ is satisfied. In the case of 1-monotone paths this condition turns out to be equivalent to the condition of being a concatenation of two equal pieces.

PROPOSITION 3.1. *Let $\gamma : [0,\theta] \to V$ be a 1-monotone path. Suppose it starts at zero and is not a concatenation of two equal pieces. Then there is a bounded linear operator F from the Banach space V into the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ such that the Cartan development of the image $F \circ \gamma$ has no logarithm at the terminal time θ, i.e., $Y^{F,\gamma}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.*

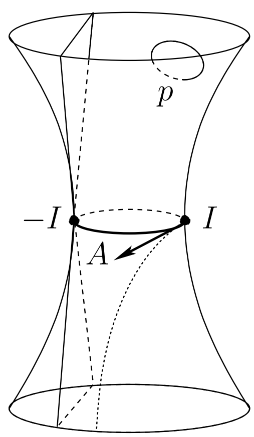
Proof. According to [12], $C \in \mathrm{SL}(2,\mathbb{R})$ is not an element of $\exp(\mathfrak{sl}(2,\mathbb{R}))$ if and only if

$$\operatorname{tr} C \leq -2 \qquad \text{and} \qquad C \neq -I.$$

Let us fix two matrices $A, B \in \mathfrak{sl}(2,\mathbb{R})$ such that $\exp(\pi A) = -I$, $A^2 = -I$, $B^2 = I$, and $AB = -BA$, for example

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The intuition is presented in the picture below. Namely, $\mathrm{SL}(2,\mathbb{R})$ is a 3-dimensional hyperboloid and we can think of it as a 2-dimensional hyperboloid with a circle attached to each point (as shown at the point p). The condition $\operatorname{tr} C = -2$ corresponds to the vertical hyperplane, and the complement of $\exp(\mathfrak{sl}(2,\mathbb{R}))$ consists of the points which are situated to the left of the hyperplane (including the hyperplane itself but excluding the single point $-I$). Our goal is to start at I and reach the compliment of $\exp(\mathfrak{sl}(2,\mathbb{R}))$ moving along a geodesic. We can begin to see from the picture that we can only do this if we move along the middle circle and come to the point $-I$ (which corresponds to the tangent matrix A satisfying the conditions below). Otherwise we would slide up or down faster than to the left and never cross the hyperplane (like the dotted line). Once we know how to reach $-I$, we will introduce a small perturbation generated by the motion in the direction B and we will be able to get slightly further left than $-I$.



$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

Change of coordinates: $\begin{cases} x = a + d, & u = b + c, \\ y = a - d, & v = b - c. \end{cases}$

$\mathrm{SL}(2,\mathbb{R})$ corresponds to the hyperboloid $x^2 - y^2 - u^2 + v^2 = 4$.

Denote $w^2 = y^2 + u^2$ to get the 2-dimensional hyperboloid $x^2 - w^2 + v^2 = 4$ in $\mathbb{R}^3$.

$\exp(\mathfrak{sl}(2,\mathbb{R})) = \{x > -2\} \cup \{(-2,0,0,0)\}$.

Let us now be precise. By definition of a 1-monotone path there exists $f \in V^*$ such that $\xi_1 = f \circ \gamma$ is strictly monotone. Multiplying f by a real number, if necessary, we can assume ξ_1 to be strictly increasing and $\xi_1(\theta) = 1$. Further, let $g \in V^*$ be not co-linear with f and such that $\xi_2 = g \circ \gamma$ is not identically zero (such a g exists as γ is not a concatenation of two equal pieces and so not a segment of a straight line). Moreover, we assume that $\xi_2(\theta) = 0$, which can be achieved by replacing g with $g + af$ with a suitable real number a.

Consider the family of bounded linear operators $F_h^{k,\lambda} : V \to \mathfrak{sl}(2,\mathbb{R})$ given by

$$F_h^{k,\lambda} = [(2k+1)\pi f + \lambda g]\, A + hgB,$$

where $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$ are arbitrary and h is a small real parameter.

Denote

$$\eta_1^{k,\lambda} = (2k+1)\pi\xi_1 + \lambda\xi_2 \qquad \text{and} \qquad \eta_2^{k,\lambda} = \xi_2.$$

Then the image path $X_h^{k,\lambda} = F_h^{k,\lambda} \circ \gamma$ in $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$X_h^{k,\lambda} = \eta_1^{k,\lambda} A + h\eta_2^{k,\lambda} B.$$

Denote by $Y_h^{k,\lambda}$ the development of $X_h^{k,\lambda}$ into the Lie group $\mathrm{SL}(2,\mathbb{R})$. Then, skipping the indices k and λ until the very end of the proof by abuse of notation, we can write down the equation for the development

$$dY_h = Y_h(A d\eta_1 + hB d\eta_2), \qquad Y_h(0) = I. \tag{3.1}$$

Let us represent the solution Y_h as a power series in h

$$Y_h = Z_0 + hZ_1 + h^2 Z_2 + \cdots . \tag{3.2}$$

Substituting this into (3.1) and comparing the coefficients at h^n for all n we obtain

$$\begin{aligned} dZ_0 &= Z_0 A d\eta_1 &&, \qquad Z_0(0) = I, \\ dZ_n &= Z_n A d\eta_1 + Z_{n-1} B d\eta_2 && \qquad Z_n(0) = 0, \qquad \text{for } n \geq 1. \end{aligned} \tag{3.3}$$

Let us look for solutions of these equations in the form $Z_n = C_n \exp(\eta_1 A)$. Obviously, $C_0 = I$. Further, substituting Z_n into (3.3) we get

$$dC_n \exp(\eta_1 A) = C_{n-1} \exp(\eta_1 A) B d\eta_2, \qquad C_n(0) = 0$$

and by Lemma 3.2 below this is equivalent to

$$dC_n = C_{n-1} \exp(2\eta_1 A) B d\eta_2, \qquad C_n(0) = 0.$$

Using $B^2 = I$ we obtain

$$\begin{aligned} C_1(t) &= \int_0^t \exp(2\eta_1(s)A) B d\eta_2(s), \\ C_2(t) &= \int_0^t \int_0^s \exp(2\eta_1(u)A) B \exp(2\eta_1(s)A) B d\eta_2(u) d\eta_2(s) \\ &= \int_0^t \int_0^s \exp(2(\eta_1(u) - \eta_1(s)A) d\eta_2(u) d\eta_2(s). \end{aligned}$$

It follows again from Lemma 3.2 that

$$\begin{aligned}
\operatorname{tr} C_1(t) &= \int_0^t \operatorname{tr}(\exp(2\eta_1(s)A)B)d\eta_2(s) = 0,\\
\operatorname{tr} C_2(t) &= \int_0^t \int_0^s \operatorname{tr}\exp(2(\eta_1(u) - \eta_1(s)A))d\eta_2(u)d\eta_2(s)\\
&= 2\int_0^t \int_0^s \cos(2(\eta_1(u) - \eta_1(s)))d\eta_2(u)d\eta_2(s)\\
&= 2\int_0^t \int_0^s (\cos(2\eta_1(u))\cos(2\eta_1(s))\\
&\qquad\qquad + \sin(2\eta_1(u))\sin(2\eta_1(s)))d\eta_2(u)d\eta_2(s)\\
&= 2\left(\int_0^t \cos(2\eta_1(s))d\eta_2(s)\right)^2 + 2\left(\int_0^t \cos(2\eta_1(s))d\eta_2(s)\right)^2\\
&= 2\left|\int_0^t e^{2i\eta_1(s)}d\eta_2(s)\right|^2 = 2a(k,\lambda)^2,
\end{aligned}$$

where $a(k,\lambda)$ is defined by the last equality. This implies

$$\begin{aligned}
\operatorname{tr} Z_0(\theta) &= \operatorname{tr}\exp(\eta_1(\theta)A) = \operatorname{tr}\exp((2k+1)\pi A) = -\operatorname{tr} I = -2,\\
\operatorname{tr} Z_1(\theta) &= \operatorname{tr}(C_1(\theta)\exp(\eta_1(\theta)A))\\
&= \operatorname{tr}(C_1(\theta)\exp((2k+1)\pi A)) = -\operatorname{tr} C_1(\theta) = 0,\\
\operatorname{tr} Z_2(\theta) &= \operatorname{tr}(C_2(\theta)\exp(\eta_1(\theta)A))\\
&= \operatorname{tr}(C_2(\theta)\exp((2k+1)\pi A)) = -\operatorname{tr} C_2(\theta) = -2a(k,\lambda)^2.
\end{aligned}$$

By Lemma 3.3 below there exist k and λ such that $c(k,\lambda) \neq 0$. It follows from the decomposition (3.2) that for such k and λ

$$\operatorname{tr} Y_h^{k,\lambda}(\theta) = \operatorname{tr} Z_0 + h\operatorname{tr} Z_1 + h^2 \operatorname{tr} Z_2 + o(h^2) = -2 - 2a(k,\lambda)^2h^2 + o(h^2),$$

and therefore there exists h such that

$$\operatorname{tr} Y_h^{k,\lambda}(\theta) < -2$$

and hence $Y_h^{k,\lambda}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.

Finally, we define $F = F_h^{k,\lambda}$ with k, λ, and h chosen above, which completes the proof. □

The next lemma describes certain features of the structure of $\mathfrak{sl}(2,\mathbb{R})$ which we have used in the proof of Proposition 3.1.

Lemma 3.2. *Let $A, B \in \mathfrak{sl}(2,\mathbb{R})$ be such that $AB = -BA$. Then*

(1) $\exp(A)B = B\exp(-A)$,
(2) $\operatorname{tr}(\exp(A)B) = 0$,
(3) $\operatorname{tr}(\exp(A)) = 2\cos\sqrt{\det A}$.

Proof. (1) This can be easily seen by

$$\exp(A)B = \sum_{n=0}^{\infty} \frac{A^n B}{n!} = \sum_{n=0}^{\infty} \frac{BA^n(-1)^n}{n!} = B\exp(-A).$$

(2) It follows from the cyclic invariance of the trace that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) = -\operatorname{tr}(AB)$$

and so $\operatorname{tr}(AB) = 0$. Since $\operatorname{tr} A = 0$ we have $A^2 = -(\det A)I$ and hence

$$\begin{aligned}\operatorname{tr}(\exp(A)B) &= \sum_{n=0}^{\infty} \frac{1}{n!}\operatorname{tr}(A^n B)\\ &= \sum_{k=0}^{\infty}\left[\frac{(-\det A)^k}{(2k)!}\operatorname{tr}(B) + \frac{(-\det A)^k}{(2k+1)!}\operatorname{tr}(AB)\right] = 0.\end{aligned}$$

(3) The eigenvalues of A are equal to $\pm\sqrt{-\det A}$ as $A \in \mathfrak{sl}(2,\mathbb{R})$. Hence

$$\begin{aligned}\operatorname{tr}(\exp(A)) &= \exp(\sqrt{-\det A}) + \exp(-\sqrt{-\det A})\\ &= 2\cosh(\sqrt{-\det A}) = 2\cos\sqrt{\det A},\end{aligned}$$

which completes the proof. □

The following lemma shows, using the Fourier series, that the only case when $a(k,\lambda) = 0$ for all k and λ is if the path γ was a concatenation of two equal pieces.

LEMMA 3.3. *There exist k and λ such that $a(k,\lambda) \neq 0$.*

Proof. Assume the statement is false. Since ξ_1 is strictly monotone and hence invertible we obtain

$$\begin{aligned}0 &= \int_0^\theta e^{2i\eta_1^{k,\lambda}(s)} d\eta_2^{k,\lambda}(s) = \int_0^\theta e^{2i(2k+1)\pi\xi_1(s)+2i\lambda\xi_2(s)} d\xi_2(s)\\ &= (2i\lambda)^{-1}\int_0^\theta e^{2i(2k+1)\pi\xi_1(s)} de^{2i\lambda\xi_2(s)}\\ &= -(2i\lambda)^{-1}\int_0^\theta e^{2i\lambda\xi_2(s)} de^{2i(2k+1)\pi\xi_1(s)}\\ &= -(2i\lambda)^{-1}\int_0^1 e^{2i\lambda(\xi_2\circ\xi_1^{-1})(t)} de^{2i(2k+1)\pi t}\end{aligned}$$

for all k and λ. By the property of the Fourier transform this means that

$$\int_0^1 (f\circ\xi_2\circ\xi_1^{-1})(t)de^{2i(2k+1)\pi t} = 0$$

for any continuous function f and all k.

Since the system of functions $(e^{2in\pi t})$ is complete on $[0,1]$ we obtain

$$(f\circ\xi_2\circ\xi_1^{-1})(s)=\sum_{k=0}^{\infty}c_ke^{4ik\pi t},$$

which implies

$$(f\circ\xi_2\circ\xi_1^{-1})(1/2+t)=(f\circ\xi_2\circ\xi_1^{-1})(t)$$

and therefore

$$(\xi_2\circ\xi_1^{-1})(1/2+t)=(\xi_2\circ\xi_1^{-1})(t).$$

This is equivalent to the condition that ξ and hence γ must be a concatenation of the same two pieces, which leads to a contradiction. $\square$

Remark 3.4. Notice that we used the monotonicity of ξ_1 only for proving Lemma 3.3. However, the statement of the lemma remains true for a much broader class of paths, and in Proposition 3.1 we can replace the condition on the path to be 1-monotone by the condition that there exists a two-dimensional projection of γ such that the corresponding $a(k,\lambda)$ is non-zero for some k and λ. The latter condition is much more general but much less clear.

4. The map $V\to\mathfrak{sl}(2,\mathbb{R})$ for non-double paths

In this section we first study unpaired paths in $\mathbb{R}^2$ and then generalise the results to non-double paths in V.

Let η be an unpaired path in $\mathbb{R}^2$. Denote by $P_0,\dots,P_n$ the corner points of η and by $u_i=P_i-P_{i-1}$ the edges. Let us reparametrise η by $t\in[0,n]$ in such a way that $\eta(i)=P_i$ and $\dot\eta(t)=u_i$ on $(i-1,i)$ for all i.

Let $G:\mathbb{R}^2\to\mathfrak{sl}(2,\mathbb{R})$ be a linear mapping. Denote by $X=G\circ\eta$ the image of η in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. For each λ, let us scale η and, correspondingly, X by λ and denote the development of the scaled path λX into the Lie group $\mathrm{SL}(2,\mathbb{R})$ by Y_λ (although both λX and Y_λ depend on G we will not write it explicitly by abuse of notation). Hence Y_λ solves the equation

$$dY_\lambda=\lambda Y_\lambda dX,\quad Y_\lambda(0)=I,$$

and the endpoint $Y^*_\lambda=Y_\lambda(n)$ of the development is given by

$$Y^*_\lambda=\exp(\lambda U_1)\cdots\exp(\lambda U_n),\tag{4.1}$$

where $U_i=G(u_i)$.

Our aim is to vary G and λ in such a way that $Y^*_\lambda\notin\exp(\mathfrak{sl}(2,\mathbb{R}))$. The following example shows that this is not possible for some piecewise linear paths and explains where the condition to be an unpaired path comes from.

Example 4.1. Let η be the concatenation of two identical paths $\tilde\eta$. Then, for all G and all λ, $Y^*_\lambda\in\exp(\mathfrak{sl}(2,\mathbb{R}))$.

Proof. For any choice of G and λ we have $Y_\lambda^* = (\tilde{Y}_\lambda^*)^2 = B^2$. As $B \in \mathrm{SL}(2,\mathbb{R})$ its characteristic polynomial is given by $t^2 - t\ \operatorname{tr} B + 1$ and so $B^2 = B\ \operatorname{tr} B - I$. Hence

$$\operatorname{tr} Y_\lambda^* = \operatorname{tr} B^2 = \operatorname{tr}(B\ \operatorname{tr} B - I) = (\operatorname{tr} B)^2 - 2.$$

If $\operatorname{tr} B = 0$ then $B^2 = -I$. If $\operatorname{tr} B \neq 0$ then the previous formula implies $\operatorname{tr} Y_\lambda^* > -2$. In both cases we get $Y_\lambda^* \in \exp(\mathfrak{sl}(2,\mathbb{R}))$ by [12]. □

The next proposition is an analogue of Proposition 3.1 for piecewise linear paths in $\mathbb{R}^2$. Also in this case we will construct a linear operator $\lambda G : \mathbb{R}^2 \to \mathfrak{sl}(2,\mathbb{R})$ such that Y_λ^* has no logarithm. But in contrast to the proof of Proposition 3.1, where we used small perturbations in order to get $\operatorname{tr} Y_\lambda^*$ just less than -2, we will use large perturbations in order to obtain $\operatorname{tr} Y_\lambda^* \approx -\infty$.

Proposition 4.2. *If η is an unpaired piecewise linear path in $\mathbb{R}^2$ then there exist a linear operator G from $\mathbb{R}^2$ to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and a scaling λ such that the Cartan development of the image $\lambda G \circ \gamma$ has no logarithm at the terminal time θ, i.e., $\operatorname{tr} Y_\lambda^* < -2$ and so $Y_\lambda^* \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.*

Proof. Consider

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

(in the sequel we will make no distinction between $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^3$). For every $A \in \mathfrak{sl}(2,\mathbb{R})$ its characteristic polynomial is given by $t^2 + \det A = 0$. This means that A has two real eigenvalues if $\det A = -a^2 - bc < 0$ and has two purely imaginary eigenvalues if $\det A = -a^2 - bc > 0$. The condition $\det A = 0$ defines a cone C_0 in $\mathbb{R}^3$ and the matrices with real (respectively, purely imaginary) eigenvalues correspond to its exterior C_{ext} (respectively, interior C_{int}).

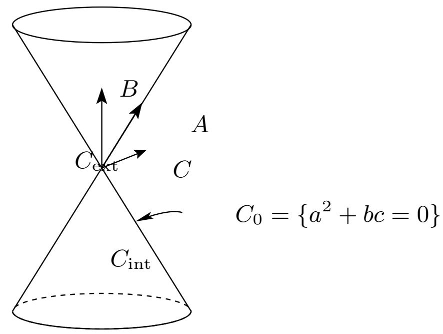
Let $v \in \mathbb{R}^2$ be a non-zero vector such that there exists an odd number of edges u_j of η such that $u_j = \pm v$ (the existence of v follows from η being an unpaired path). Denote by K the set of indices of all the edges u_k which are co-linear with v. Further, fix any $\tilde{v} \in \mathbb{R}^2$ that is not co-linear with v.

The idea is to map $\mathbb{R}^2$ into $\mathfrak{sl}(2,\mathbb{R})$ in such a way that only u_k, $k \in K$, are mapped into the interior of the cone and the images of all the other u_i lie outside the cone. Moreover, the map should be sufficiently close to a map for which one could easily compute Y_λ^*.

Choose $A, B, C \in \mathfrak{sl}(2,\mathbb{R})$ in such a way that $A \in C_0$, B lies on the axis of C_0 and has an acute angle with A, and C is orthogonal to B and lies in the tangent plane to C_0 at the point A, i.e.,

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -(b+c) & 2a \\ 2a & b+c \end{pmatrix}$$

where $a^2 + bc = 0$ and $b > c$. Moreover, it will be convenient for the further computation if we choose them in such a way that $b - c = 1$.



Fix $\rho > 1$ and consider the family G_s of linear maps from $\mathbb{R}^2$ to $\mathfrak{sl}(2, \mathbb{R})$ (indexed by $s \in \mathbb{R}$) such that

$$G_s(v) = A + s^{2\rho} B \quad \text{and} \quad G_s(\tilde{v}) = sC.$$

As $(v, \tilde{v})$ is a basis for $\mathbb{R}^2$ one has

$$u_k = x_k v + y_k \tilde{v}$$

with some coefficients x_k, y_k. As above, denote $U_k(s) = G_s(u_k) = x_k A + x_k s^{2\rho} B + y_k s C$ and compute

$$\tag{4.2} \begin{aligned} \det U_k(s) &= \det \begin{pmatrix} x_k a - y_k s(b+c) & x_k b + x_k s^{2\rho} + 2a y_k s \\ x_k c - x_k s^{2\rho} + 2a y_k s & -x_k a + y_k s(b+c) \end{pmatrix} \\ &= -y_k^2 s^2 + x_k^2 s^{2\rho} + x_k^2 s^{4\rho}. \end{aligned}$$

This means that for small s one has $U_k(s) \in C_{\text{int}}$ for $k \in K$ and $U_k(s) \in C_{\text{ext}}$ for $k \notin K$. Indeed, for $k \notin K$ we have $y_k \neq 0$ and the leading term in (4.2) is $-y_k^2 s^2 < 0$. On the other hand, for $k \in K$ $y_k = 0$ and so the leading term $x_k^2 s^{2\rho} > 0$.

Further, according to [12] we know how to compute exp of any element of $\mathfrak{sl}(2, \mathbb{R})$, which gives

$$\exp(\lambda U) = \cos(\lambda \sqrt{\det U}) \left(I + \frac{\tan(\lambda \sqrt{\det U})}{\sqrt{\det U}} U \right)$$

for any $U \in \mathfrak{sl}(2, \mathbb{R})$ and any real λ. Let us take

$$\lambda(s) = \lambda_0 s^{-\rho},$$

where λ_0 will be chosen later, and compute the asymptotic of $\exp(\lambda(s) U_k(s))$ with respect to s separately for the cases $k \notin K$ and $k \in K$.

First, consider $k \notin K$. We have

$$\sqrt{\det U_k(s)} = iy_k s(1+o(1))$$

and so

$$\tan\left(\lambda(s)\sqrt{\det U_k(s)}\right) = i\tanh\left(\lambda_0 s^{1-\rho} y_k(1+o(1))\right) = i + o(1).$$

Denote

$$h_k(s) = \cos\left(\lambda(s)\sqrt{\det U_k(s)}\right) = \cosh\left(\lambda_0 s^{1-\rho} y_k(1+o(1))\right) \tag{4.3}$$

and notice that $h_k(s) \to \infty$ faster than any power of s. This implies

$$\begin{aligned}\exp(\lambda(s)U_k(s)) &= h_k(s)\left[I + \frac{1+o(1)}{y_k s}(x_k(A+s^{2\rho}B) + y_k sC)\right]\\ &= \frac{h_k(s)}{s}\left[\frac{x_k}{y_k}A + (C+I)s + \frac{x_k}{y_k}Bs^{2\rho}\right](1+o(1)).\end{aligned}$$

Consider now $k \in K$. We have

$$\sqrt{\det U_k(s)} = (x_k^2 s^{4\rho} + x_k^2 s^{2\rho})^{1/2} = x_k s^\rho\left(1 + \frac{s^{2\rho}}{2} + o(s^{2\rho})\right)$$

and so

$$\begin{aligned}\cos\left(\lambda(s)\sqrt{\det U_k(s)}\right) &= \cos(\lambda_0 x_k)\cos\left(\frac{\lambda_0 x_k s^{2\rho}}{2} + o(s^{2\rho})\right)\\ &\qquad - \sin(\lambda_0 x_k)\sin\left(\frac{\lambda_0 x_k s^{2\rho}}{2} + o(s^{2\rho})\right)\\ &= \cos(\lambda_0 x_k) - \frac{\sin(\lambda_0 x_k)\lambda_0 x_k}{2}s^{2\rho} + o(s^{2\rho}),\\ \sin\left(\lambda(s)\sqrt{\det U_k(s)}\right) &= \sin(\lambda_0 x_k)\cos\left(\frac{\lambda_0 x_k s^{2\rho}}{2} + o(s^{2\rho})\right)\\ &\qquad + \cos(\lambda_0 x_k)\sin\left(\frac{\lambda_0 x_k s^{2\rho}}{2} + o(s^{2\rho})\right)\\ &= \sin(\lambda_0 x_k) + \frac{\cos(\lambda_0 x_k)\lambda_0 x_k}{2}s^{2\rho} + o(s^{2\rho}).\end{aligned}$$

Now we obtain

$$\begin{aligned}\exp(\lambda(s)U_k(s)) &= \left[\cos(\lambda_0 x_k) - \frac{\sin(\lambda_0 x_k)\lambda_0 x_k}{2}s^{2\rho} + o(s^{2\rho})\right] I \\ &\quad + \left[\sin(\lambda_0 x_k) + \frac{\cos(\lambda_0 x_k)\lambda_0 x_k}{2}s^{2\rho} + o(s^{2\rho})\right](A + s^{2\rho}B) \\ &\quad \times \frac{s^{-\rho}}{x_k}\left[1 - \frac{s^{2\rho}}{2q^2} + o(s^{2\rho})\right] x_k \\ &= s^{-\rho}\left[A\sin(\lambda_0 x_k) + I\cos(\lambda_0 x_k)s^{\rho}\right. \\ &\qquad\qquad \left. + B\sin(\lambda_0 x_k)s^{2\rho}\right](1 + o(1)).\end{aligned}$$

Denote

$$M_k(s) = \begin{cases} z_k A + (C+I)s + z_k B s^{2\rho}, & k \notin K, \\ A\sin(\lambda_0 x_k) + I\cos(\lambda_0 x_k)s^{\rho} + B\sin(\lambda_0 x_k)s^{2\rho}, & k \in K, \end{cases}$$

where $z_k = x_k/y_k$. Taking into account the relations

$$\begin{gathered} A^2 = 0, \quad B^2 = -I, \quad C^2 = I, \\ AC = -CA, \quad BC = -CB, \quad AB + BA = I, \end{gathered} \tag{4.4}$$

we will show using Lemma 4.3 that the leading term of $\operatorname{tr}(M_1(s)\cdots M_n(s))$ is of order $r = \rho|K| + |K^c|$.

Indeed, the product $M_1(s)\cdots M_n(s)$ is a sum of monomials of A, $(C+I)$, and B, which are picked from the matrices $M_k(s)$. By part (3) of Lemma 4.3, if we have picked A m times we have to pick B at least m times. If $2m > |K|$ then the order of such a monomial will be at least $2\rho m + n - m$ which is greater then $\rho|K| + |K^c|$ as

$$\begin{aligned}(2\rho m + n - m) - (\rho|K| + |K^c|) &= \rho(2m - |K|) + |K| - m \\ &> 2m - |K| + |K| - m = m > 0.\end{aligned}$$

If $2m \le |K|$ and we have picked at least one A or B from $M_k(s)$ with $k \notin K$ then the order will be at least $2\rho m + \rho + n - 2m - 1 > 2\rho m + n - |K| = r$. This implies that A and B may only be picked from $M_k(s)$ with $k \in K$. But in this case condition (4) of Lemma 4.3 is fulfilled since our path was generic and so any two matrices $M_i(s)$ and $M_j(s)$ with $i, j \in K$ are separated by a matrix $M_k(s)$ with $k \notin K$, and so the trace of such a monomial is equal to zero. This means that we are not allowed to pick A from any of $M_i(s)$ in order to get the leading term. Hence

$$\operatorname{tr}(M_1(s)\cdots M_n(s)) = s^r \prod_{k\in K}\cos(\lambda_0 x_k)\operatorname{tr}(C+I)^{|K^c|} + o(s^r)$$

and combining everything together we obtain

$$\operatorname{tr}(Y_\lambda^*) = 2^{|K^c|}\prod_{k\in K}\cos(\lambda_0 x_k)\prod_{k\notin K} h_k(s) + o(1). \tag{4.5}$$

Since η is not an unpaired path and by our choice of the direction v there is a nonempty subset $K' \subset K$ such that

$$\operatorname{sign} \prod_{k \in K} \cos(\lambda_0 x_k) = \operatorname{sign} \prod_{k \in K'} \cos(\lambda_0 |x_k|)$$

and all $|x_k|$ are different. Hence there exists λ_0 such that

$$\prod_{k \in K} \cos(\lambda_0 x_k) < 0. \tag{4.6}$$

One can take any $\lambda_0 \in (\pi/2x', \pi/2x'')$, where

$$x' = \max\{|x_i|, i \in K'\} \quad \text{and} \quad x'' = \begin{cases} \max\{|x_i| \neq x', i \in K'\} & \text{if } |K'| > 1, \\ x'' = x'/2 & \text{if } |K'| = 1. \end{cases}$$

Finally, combining (4.3), (4.5) and (4.6) together and letting s go to zero we obtain $\operatorname{tr}(Y^*_\lambda) \to -\infty$ as a function of s. Taking s sufficiently small we obtain the linear map G_s and the scaling $\lambda(s)$ such that $\operatorname{tr}(Y^*_\lambda) < -2$ and so $Y^*_\lambda \notin \exp(\mathfrak{sl}(2, \mathbb{R}))$. □

The next lemma describes the features of the structure of $\mathfrak{sl}(2, \mathbb{R})$ which we have used in the proof of Proposition 4.2.

Lemma 4.3. *Suppose $A, B, C \in \mathfrak{sl}(2, \mathbb{R})$ satisfy the relations* (4.4). *Then the following hold.*

(1) $\operatorname{tr}(AC) = \operatorname{tr}(CA) = 0$.
(2) $(I \pm C)^k = 2^{k-1}(I \pm C)$, $(I - C)^i(I + C)^j = 0$, *and* $\operatorname{tr}(I \pm C)^k = 2^k$ *for all* $i, j, k > 0$.
(3) *If D is a monomial in A, B, C such that A appears in D m times and B appears in D less than m times then* $\operatorname{tr} D = 0$.
(4) *If D is a monomial in $A, B, C + I$ such that there are no A and B which are neighbours of each other (in the cyclic sense, i.e., the first and the last element are regarded as neighbours) and A appears at least once, then* $\operatorname{tr} D = 0$.

Proof. (1) Obviously follows from $AC = -CA$ as $\operatorname{tr}(AC) = \operatorname{tr}(CA) = -\operatorname{tr}(AC)$.

(2) Notice that $(I \pm C)^2 = I \pm 2C + C^2 = 2(I \pm C)$ and by induction we get the first and the last formula. Further,

$$(I - C)^i(I + C)^j = 2^{i+j-2}(I - C)(I + C) = 2^{i+j-2}(I - C^2) = 0.$$

(3) Since A appears in D more often than B we have either $D = D_1 AC^k AD_2$ or $D = C^i AD_3 AC^j$, where D_1, D_2, D_3 are monomials in A, B, C. Using $AC = -CA$ and $A^2 = 0$ we get

$$\operatorname{tr} D = \operatorname{tr}(D_1 AC^k AD_2) = (-1)^k \operatorname{tr}(D_1 C^k A^2 D_2) = 0$$

or

$$\operatorname{tr} D = \operatorname{tr}(C^i A D_3 A C^j) = \operatorname{tr}(A C^{i+j} A D_3) = (-1)^{i+j} \operatorname{tr}(A^2 C^{i+j} D_3) = 0.$$

(4) If B does not appear in D then the statement follows from (4). If there is at least one B then either $D = D_1(C+I)^i A(C+I)^k B(C+I)^j D_2$ or $D = D_2(C+I)^i B(C+I)^k A(C+I)^j D_2$, where D_1, D_2 are monomials in $A, B, (C+I)$, $k > 0$, and $i + j \neq 0$. Without loss of generality consider the first case and $i > 0$. Using (2) we obtain

$$\begin{aligned} \operatorname{tr} D &= \operatorname{tr}(D_1(C+I)^i A(C+I)^k B(C+I)^j D_2) \\ &= \operatorname{tr}(D_1 A(I-C)^i (C+I)^k B(C+I)^j D_2) = 0. \end{aligned}$$

The remaining cases are analogous. □

Now we can pass on to non-double paths in V.

PROPOSITION 4.4. *Let $\gamma : [0,\theta] \to V$ be a piecewise linear non-double path starting at zero. Then there exists a bounded linear operator F from the Banach space V to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ such that the Cartan development of the image $F \circ \gamma$ has no logarithm at the terminal time θ, i.e., $Y^{F,\gamma}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.*

Proof. By the definition of a non-double path there are two bounded linear functionals $f, g \in V^*$ such that $\nu = (f \circ \gamma, g \circ \gamma)$ is equivalent to an unpaired path η. By Proposition 4.2 there is a linear mapping $G : \mathbb{R}^2 \to \mathfrak{sl}(2,\mathbb{R})$ and $\lambda \in \mathbb{R}$ such that

$$\operatorname{tr} Y^{\lambda G,\eta}(\theta) < -2.$$

Hence the same is true for ν as the trace of the endpoint of the development is invariant under cyclic permutations of the linear pieces, adding and removing equal pieces traversed in opposite directions, and joining up co-linear pieces. This means that $Y^{\lambda G,\nu}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.

Finally, we define F as a composition of the two-dimensional projection $(f,g) : V \to \mathbb{R}^2$ and $\lambda G : \mathbb{R}^2 \to \mathfrak{sl}(2,\mathbb{R})$. Obviously,

$$Y^{F,\gamma}(\theta) = Y^{\lambda G,\nu}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R})),$$

and the statement is proved. □

5. Finiteness of the radius of convergence

In this section we prove the finiteness of the injective radius $R_\varepsilon(\gamma)$ which will immediately imply the finiteness of $R(\gamma)$.

Following [13], we define the injective norm $\varepsilon(\cdot)$ on the algebraic tensor product $U \otimes W$ of two Banach spaces U and W by

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) \right| : \varphi \in B_{U^*}, \psi \in B_{W^*} \right\},$$

where $\sum x_i \otimes y_i$ is any representation of u and B_{U^*}, B_{W^*} denote the unit balls in the dual spaces U^* and W^*, respectively. We denote by $U \otimes_\varepsilon W$ the tensor product $U \otimes W$ with the injective norm and by $U \widehat{\otimes}_\varepsilon W$ its completion.

Let us introduce a norm on $M_{2\times 2}$ given by

$$\|G\| = \max_{1\le i,j\le 2} |g_{ij}|, \qquad \text{where } G = (g_{ij}).$$

Let $A : U \to M_{2\times 2}$ and $B : W \to M_{2\times 2}$ be two bounded linear operators. We define the operator $A * B : U \otimes W \to M_{2\times 2}$ by

$$(A * B)(u) = \sum A(x_i)B(y_i),$$

which is obviously independent of the particular representation $\sum x_i \otimes y_i$ chosen for u.

Lemma 5.1. *$A * B$ is a bounded linear operator from $U \widehat{\otimes}_\varepsilon W$ to $M_{2\times 2}$ and $\|A * B\| \le 2\|A\|\,\|B\|$.*

Proof. Denote by a_{ij}, b_{ij}, and c_{ij} the linear functionals on U, W, and $U \otimes W$ representing the operators A, B, and $A * B$ respectively. Obviously, they are bounded and $\|a_{ij}\| \le \|A\|$ and $\|b_{ij}\| \le \|B\|$ for all i, j. By Proposition 3.1 from [13] $a_{ij} \otimes b_{kl}$ is a bounded linear functional on $U \widehat{\otimes}_\varepsilon W$ and $\|a_{ij} \otimes b_{kl}\| = \|a_{ij}\|\,\|b_{kl}\|$. Now we have

$$c_{ij} = \sum_{k=1}^{2} a_{ik} \otimes b_{kj}$$

and so they are bounded linear functionals on $U \widehat{\otimes}_\varepsilon W$ with the norm satisfying $\|c_{ij}\| \le 2\|A\|\,\|B\|$. Now the statement follows from the definition of the norm on $M_{2\times 2}$. □

Now, using our specified Banach space V, we can construct the sequence of Banach spaces $V^{\widehat{\otimes}_\varepsilon n}$ with the injective cross-norms $\varepsilon(\cdot)$. The associativity of this construction has been proved in [7]. We denote by T_ε (respectively, L_ε) the tensor algebra (respectively, the tensor Lie algebra) corresponding to the injective tensor products and by $\widehat{T}_\varepsilon$ (respectively, $\widehat{L}_\varepsilon$) its completion with respect to the augmentation ideal.

Analogously, for a bounded linear operator $A : V \to M_{2\times 2}$ we obtain the sequence of bounded operators $A^{*n} : V^{\widehat{\otimes}_\varepsilon n} \to M_{2\times 2}$ with $\|A^{*n}\| \le 2^{n-1}\|A\|^n$. For convenience we denote $A^{*0} = I$ and $A^{*1} = A$.

Let us extend A to a linear operator A^∞ on the linear subspace T_A of $\widehat{T}$ defined by

$$T_A = \left\{ u = \sum_{n=0}^{\infty} u_n : u_n \in V^{\widehat{\otimes}_\varepsilon n} \text{ for all } n \text{ and } \sum_{n=0}^{\infty} \|A^{*n}(u_n)\| < \infty \right\} \tag{5.1}$$

by taking the limit

$$A^\infty(u) = \sum_{n=0}^{\infty} A^{*n}(u_n). \tag{5.2}$$

LEMMA 5.2. *For any bounded linear mapping A one has $\mathcal{S}(\gamma) \in T_A$.*

Proof. This follows from the factorial decay of the iterated integrals and the geometric growth of the norms of A^{*n}. In fact, denote by $l : [0, \theta] \to \mathbb{R}$ the function such that $l(t)$ is the length of the path γ up to time t. Since $\varepsilon(\cdot)$ is a cross-norm we have

$$\begin{aligned}
\varepsilon(\mathcal{S}^{(n)}(\gamma)) &\le \int \cdots \int_{0<u_1<\cdots<u_n<\theta} \varepsilon(d\gamma(u_1)) \cdots \varepsilon(d\gamma(u_n)) \\
&= \int \cdots \int_{0<u_1<\cdots<u_n<\theta} \|d\gamma(u_1)\| \cdots \|d\gamma(u_n)\| \\
&= \int \cdots \int_{0<u_1<\cdots<u_n<\theta} dl(u_1) \cdots dl(u_n) = \frac{l(\gamma)^n}{n!}.
\end{aligned}$$

Hence

$$\|A^{*n}(\mathcal{S}^{(n)}(\gamma))\| \le \|A^{*n}\| \varepsilon(\mathcal{S}^{(n)}(\gamma)) \le \frac{(2\|A\| l(\gamma))^n}{2n!}$$

and so the series in (5.1) converges. $\square$

LEMMA 5.3. *Let $A : V \to \mathfrak{sl}(2, \mathbb{R}) \subset M_{2\times 2}$ be a bounded linear operator and γ be such that $R_\varepsilon(\gamma)$ is infinite. Then $\mathcal{LS}(\gamma) \in T_A$ and $A^\infty(\mathcal{LS}(\gamma)) \in \mathfrak{sl}(2, \mathbb{R})$.*

Proof. As $\mathcal{LS}(\gamma)$ is an entire function with respect to $\varepsilon(\cdot)$, for any $\rho > 0$ one has eventually $\varepsilon(\mathcal{LS}^{(n)}(\gamma)) < \rho^n$. For $\rho < 1/(3\|A\|)$ one eventually has

$$\|A^{*n}(\mathcal{LS}^{(n)}(\gamma))\| \le \|A^{*n}\| \varepsilon(\mathcal{LS}^{(n)}(\gamma)) \le 2^{n-1} \|A\|^n \rho^n < (2/3)^n$$

and therefore the series in (5.1) converges and so $\mathcal{LS}(\gamma) \in T_A$.

To prove the second statement notice that A^{*n} maps the n-th component of the tensor Lie algebra L_ε to the n-th commutator of $\mathfrak{sl}(2, \mathbb{R})$, which lies in $\mathfrak{sl}(2, \mathbb{R})$ as it is a Lie algebra. Therefore the limit is in $\mathfrak{sl}(2, \mathbb{R})$ as well. $\square$

LEMMA 5.4. *If $u \in T_A$ and $\exp(u) \in T_A$ then $(\exp \circ A^\infty)(u) = (A^\infty \circ \exp)(u)$, for any bounded linear mapping A.*

Proof. This follows from the absolute convergence of the series (5.2). □

Now, let γ be a continuous path of bounded variation and assume that there is a linear mapping $F : V \to \mathfrak{sl}(2,\mathbb{R})$ such that $Y^{F,\gamma}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.

It is well known that the solution of the equation (2.5) can be written as a convergent series of iterated integrals of the driving path

$$\begin{aligned} Y^{F,\gamma}(\theta) &= I + \int_{0<u_1<\theta} dX^{F,\gamma}(u_1) + \iint_{0<u_1<u_2<\theta} dX^{F,\gamma}(u_1)dX^{F,\gamma}(u_2) + \cdots \\ &= I + \int_{0<u_1<\theta} F(d\gamma(u_1)) + \iint_{0<u_1<u_2<\theta} F(d\gamma(u_1))F(d\gamma(u_2)) + \cdots \\ &= F^{*0} + F^{*1}\int_{0<u_1<\theta} d\gamma(u_1) + F^{*2}\iint_{0<u_1<u_2<\theta} d\gamma(u_1)d\gamma(u_2) + \cdots \\ &= F^{\infty}(\mathcal{S}(\gamma)). \end{aligned}$$

Using the last three lemmata and the relation $\exp(\mathcal{LS}(\gamma)) = \mathcal{S}(\gamma)$ we obtain that the assumption $R_\varepsilon(\gamma) = \infty$ implies

$$\begin{aligned} Y^{F,\gamma}(\theta) &= F^{\infty}(\mathcal{S}(\gamma)) = (F^{\infty} \circ \exp)(\mathcal{LS}(\gamma)) \\ &= \exp(F^{\infty}(\mathcal{LS}(\gamma))) \in \exp(\mathfrak{sl}(2,\mathbb{R})). \end{aligned}$$

This contradiction is crucial for proving our two main theorems.

5.1. The main results.

THEOREM 5.5. *Let γ be a 1-monotone path that is not a segment of a straight line. Then the radius of convergence $R(\gamma)$ of the logarithmic signature of γ is finite and so $\mathcal{LS}(\gamma)$ is not an entire function.*

Proof. If γ is not a segment of a straight line then there exists $k \geq 1$ and a 1-monotone path γ_0 (different from a segment of a straight line) such that γ is a concatenation of k copies of γ_0, and γ_0 is not a concatenation of two equal paths.

Using the observation that for any two paths α and β one has $\mathcal{S}(\alpha \cup \beta) = \mathcal{S}(\alpha)\widehat{\otimes}_\varepsilon\mathcal{S}(\beta)$ (see [2]), we obtain $\mathcal{S}(\gamma) = \mathcal{S}(\gamma_0)^{\widehat{\otimes}_\varepsilon k}$ and hence $\mathcal{LS}(\gamma) = k\mathcal{LS}(\gamma_0)$. This means that $R_\varepsilon(\gamma) = R_\varepsilon(\gamma_0)$. Further, $\mathcal{S}(\gamma) = \mathcal{S}(\gamma - x)$ for every $x \in V$ and therefore $R_\varepsilon(\gamma) = R_\varepsilon(\gamma - \gamma(0))$. Hence it is sufficient to show the finiteness of $R_\varepsilon(\gamma)$ for a 1-monotone path γ that starts at zero and is not a concatenation of the same two pieces.

Such a γ satisfies the conditions of Proposition 3.1 and so there is a bounded linear operator $F : V \to \mathfrak{sl}(2,\mathbb{R})$ such that $Y^{F,\gamma}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$. As we have seen above this implies $R_\varepsilon(\gamma) < \infty$ and the inequality (2.6) completes the proof. □

THEOREM 5.6. *Let γ be a non-double piecewise linear path in V. Then the radius of convergence $R(\gamma)$ of the logarithmic signature of γ is finite and so $\mathcal{LS}(\gamma)$ is not an entire function.*

Proof. As in the previous theorem, we can assume that γ starts at zero. By Proposition 4.4 there is a linear mapping $F : V \to \mathfrak{sl}(2,\mathbb{R})$ such that $Y^{F,\gamma}(\theta) \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$. As above, this implies the finiteness of $R(\gamma)$. $\square$

6. Lower bound for the radius of convergence

In this section we find a lower bound for the radius of convergence of the logarithmic signature for any continuous path of bounded variation in V.

By definition of the log we have

$$\begin{aligned}\mathcal{LS}^{(n)}(\gamma) &= \log^{(n)}(\mathcal{S}(\gamma))\\ &= \sum_{k=1}^{n} \frac{(-1)^k}{k} \sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\ldots,i_k\le n}} \mathcal{S}^{(i_1)}(\gamma)\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{S}^{(i_k)}(\gamma).\end{aligned}$$

Recall that in Lemma 5.2 we proved the factorial decay of the iterated integrals with respect to the injective norm. However, we only used the fact that it is a cross-norm corresponding to the original norm on V and so the same is true for the cross-norms $\|\cdot\|$. Thus,

$$\|\mathcal{S}^{(m)}(\gamma)\| \le \frac{l(\gamma)^m}{m!},$$

where $l(\gamma)$ is the length of γ, and we obtain

$$\begin{aligned}\|\mathcal{LS}^{(n)}(\gamma)\| &\le \sum_{k=1}^{n} \frac{1}{k} \sum_{\substack{j_1+\cdots+j_k=n-k\\ 1\le j_1,\ldots,j_k\le n-1}} \frac{l(\gamma)^n}{(j_1+1)!\cdots(j_k+1)!}\\ &\le \sum_{k=1}^{n} \sum_{\substack{j_1+\cdots+j_k=n-k\\ 1\le j_1,\ldots,j_k\le n-k}} \frac{l(\gamma)^n}{j_1!\cdots j_k!} = l(\gamma)^n \sum_{k=1}^{n} \frac{k^{n-k}}{(n-k)!}.\end{aligned}$$

This implies

$$\begin{aligned}R(\gamma)^{-1} &\le l(\gamma) \limsup_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k^{n-k}}{(n-k)!}\right)^{1/n} \qquad (6.1)\\ &= l(\gamma) \limsup_{n\to\infty} \max_{1\le k\le n} \left(\frac{k^{n-k}}{(n-k)!}\right)^{1/n}\\ &= l(\gamma) \limsup_{n\to\infty} \left(\frac{k(n)^{n-k(n)}}{(n-k(n))!}\right)^{1/n},\end{aligned}$$

where $k(n)$ is an index where the maximum is attained. Any subsequence of $(n-k(n))$ contains a subsubsequence such that it is either bounded or goes to infinity. Thus, it is sufficient to find an upper bound for (6.1) assuming first that the $n-k(n)$ are bounded by a number M and then that $n-k(n)\to\infty$.

If $n-k(n)\leq M$ then

$$R(\gamma)^{-1}\leq l(\gamma)\lim_{n\to\infty}k(n)^{M/n}=1. \tag{6.2}$$

Consider now the case $n-k(n)\to\infty$. Denote $q(n)=k(n)/n$. By the Stirling formula we obtain

(6.3)

$$\begin{aligned}\limsup_{n\to\infty}\left(\frac{k(n)^{n-k(n)}}{(n-k(n))!}\right)^{1/n} &= \limsup_{n\to\infty}\frac{(ek)^{1-k(n)/n}}{(n-k(n))^{1-k(n)/n}}\\ &= \limsup_{n\to\infty}\frac{e^{1-q(n)}}{(1/q(n)-1)^{1-q(n)}}\\ &= \limsup_{n\to\infty}\exp\left((1-q(n))(1-\log(1/q(n)-1))\right)\end{aligned}$$

Consider the real function

$$f(q)=(1-q)(1-\log(1/q-1))$$

defined on $(0,1)$. It attains its maximum at the point q_0 such that

$$1-1/q_0=\log(1/q_0-1). \tag{6.4}$$

Let $p_0=(1/q_0-1)^{-1}$. Then p_0 is the solution of $p_0\log p_0=1$. Using (6.4) we obtain

$$\exp f(q)\leq\exp f(q_0)=\exp(1/q_0-1)=(1/q_0-1)^{-1}=p_0$$

and using (6.1) and (6.3) we get $R^{-1}\leq l(\gamma)p_0$. Combining this with (6.2) we finally obtain

$$R(\gamma)\geq l(\gamma)^{-1}p_0^{-1}=l(\gamma)^{-1}p,$$

where p is such that $\log p=-p$.

We have proved the following theorem.

THEOREM 6.1. *Let γ be a continuous path of bounded variation in V. Then there is a lower bound for the radius of convergence of the logarithmic signature of γ*

$$R(\gamma)\geq l(\gamma)^{-1}p,$$

where $l(\gamma)$ is the length of the path and p is such that $\log p=-p$.

7. Non-exponential diffeomorphisms

In this section we illustrate the importance of the above results discussing examples of diffeomorphisms which do or do not have logarithms depending on the radius of convergence of their driving control.

Let $\mathbb{R}^2$ be our target space. Consider a linear controlled system

$$dy_t = A(y_t)d\gamma_t = A_1(y_t)d\gamma_t^1 + A_2(y_t)d\gamma_t^2, \tag{7.1}$$

where A_1 and A_2 are two linear vector fields on $\mathbb{R}^2$ given by

$$A_1(y) = \begin{pmatrix} (\ln 2)/\pi & 0 \\ 0 & 0 \end{pmatrix} y \qquad \text{and} \qquad A_2(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y,$$

and γ is a two-dimensional piecewise linear path containing two linear pieces

$$\gamma_t^1 = \begin{cases} t, & t \in [0,\pi], \\ \pi, & t \in (\pi, 2\pi] \end{cases} \qquad \text{and} \qquad \gamma_t^2 = \begin{cases} 0, & t \in [0,\pi], \\ t-\pi, & t \in (\pi, 2\pi]. \end{cases}$$

The controlled system (7.1) generates a flow of diffeomorphisms of $\mathbb{R}^2$, and we are particularly interested in the diffeomorphism φ which is the value of the flow after time 2π. Since the control γ is piecewise linear, φ can be computed as the composition of the flows along its two linear pieces, which are obviously exponents of the corresponding vector fields

$$\varphi(y) = \exp(\pi A_2)\exp(\pi A_1)y = -I \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} y = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} y = \Phi y, \tag{7.2}$$

where Φ is defined by the left hand side of the equality. Thus, φ is a linear operator on $\mathbb{R}^2$, and the crucial fact is that Φ cannot be expressed as the exponent of a 2×2 matrix.

We will show that, although φ is produced by a linear differential system, it has no logarithm, that is, there is no vector field B on $\mathbb{R}^2$ such that $\varphi(x) = y(\tau, x)$ for all x, where

$$\frac{dy}{dt}(t,x) = B(y(t,x)) \qquad \text{for all } (x,t) \in \mathbb{R}^2 \times [0,\tau] \tag{7.3}$$

with the initial conditions $y(0,x) = x$.

First of all, it is obvious that φ has no linear logarithm. Indeed, if B is a linear operator on $\mathbb{R}^2$ then $y(T,x) = \exp(\tau B)x$ and so it cannot be equal to $\varphi(x)$ as Φ has no logarithm.

But it turns out that φ does not even have a nonlinear logarithm and cannot be produced by any flow on $\mathbb{R}^2$. In order to show this assume that such a vector field B exists. Let us choose a point $(a, 0)$ on the x-axis such that $|a| > 1$. Using the fact that we know the behavior of the flow for the time increments equal to τ, we obtain that the integral curve starting at a will pass through $-2a$ at time τ and cross the y-axis at some point b at time

$\theta < \tau$. Hence the integral curve will pass through b at times $\theta + 2n\tau$ and through $-b$ at times $\theta + (2n+1)\tau$, and through the points $(-2)^n a_1$ at times $n\tau$. But this is obviously impossible as the integral curve cannot be closed and approach infinity at the same time.

This is a good illustration of what happens if $\|A\| > R(\gamma)$ because otherwise $A^\infty(\mathcal{LS}(\gamma))$ would converge and give a linear logarithm for the flow generated by (7.1). Let us now decrease the norm of A in some obvious way so that it would lie inside the circle of convergence of the logarithmic signature and see that in that case the logarithm will exist.

Consider δA for some $\delta < 1$ instead of A and denote by φ_δ the diffeomorphism produced by the system

$$dy_t = \delta A(y_t) d\gamma_t$$

at time 2π. Similarly to (7.2), we obtain

$$\varphi_\delta(y) = \exp(\pi\delta A_2)\exp(\pi\delta A_1) y = \begin{pmatrix} 2^\delta \cos(\pi\delta) & \sin(\pi\delta) \\ -2^\delta \sin(\pi\delta) & \cos(\pi\delta) \end{pmatrix} y = \Phi_\delta y.$$

It is easy to see that for $\delta < 1/2$ the matrix Φ_δ has a logarithm as its eigenvalues are positive. So, any δA with $\delta < 1/2$ would have a linear logarithm.

References

[1] R. F. Bass, B. M. Hambly, and T. J. Lyons, *Extending the Wong-Zakai theorem to reversible Markov processes*, J. Eur. Math. Soc. (JEMS) **4** (2002), 237–269. MR 1924401 (2003f:60128)

[2] K.-T. Chen, *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula*, Ann. of Math. (2) **65** (1957), 163–178. MR 0085251 (19,12a)

[3] W.-L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117** (1939), 98–105. MR 0001880 (1,313d)

[4] L. Coutin and Z. Qian, *Stochastic analysis, rough path analysis and fractional Brownian motions*, Probab. Theory Related Fields **122** (2002), 108–140. MR 1883719 (2003c:60066)

[5] T. Fawcett, *Problems in stochastic analysis. Connections between rough paths and non-commutative harmonic analysis*, DPhil Thesis, Oxford University, 2003.

[6] B. M. Hambly and T. J. Lyons, *Uniqueness for the signature of a path of bounded variation and continuous analogues for the free group*, preprint.

[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **1955** (1955), 140. MR 0075539 (17,763c)

[8] T. J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), 215–310. MR 1654527 (2000c:60089)

[9] T. J. Lyons and Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2002. MR 2036784 (2005f:93001)

[10] T. J. Lyons and N. Sidorova, *Convergence of minimisers with increasing degree and the reconstruction of the reduced path with given signature*, preprint.

[11] P. K. Rashevski, *About connecting two points of complete nonholonomic space by admissible curve*, Uch Zapiski ped. inst. Libknekhta **2** (1938), 83–94.

[12] W. Rossmann, *Lie groups*, Oxford Graduate Texts in Mathematics, vol. 5, Oxford University Press, Oxford, 2002. MR 1889121 (2003f:22001)

[13] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2002. MR 1888309 (2003f:46030)

T.J. Lyons, Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, United Kingdom

E-mail address: tlyons@maths.ox.ac.uk

N. Sidorova, Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, United Kingdom

Current address: Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, United Kingdom

E-mail address: n.sidorova@bath.ac.uk