Three-term arithmetic progressions

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Abstract

In this paper, we will explore the history of three-term arithmetic progressions. In particular, we will explore the improvements to the bounds on the density of subsets not containing non-trivial three-term arithmetic progressions. We start by looking at the first bound given by Bourgain. Then, we will see an improvement by Bloom. We finish by looking at a very recent breakthrough by Kelley and Meka, achieving the bound

$$|A| \leq N \exp(-\Omega((\log N)^{1/11})).$$

Contents

1 Introduction

A three term arithmetic progression is a sequence of elements of the form $a, a+d, a+2d$. If $d \neq 0$, we call this non-trivial. In this project, we will be looking at the history of results of the density of subsets of $[N] = \{1, \ldots, N\}$ which do not contain any non-trivial three-term arithmetic progressions (3APs). We will focus on improving the upper bound of the density of these sets. For the lower bound, Behrend, in 1946 [2], proved the following theorem, which is essentially still the best known lower bound.

Theorem 1.1. *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \geq N \exp(-c\sqrt{\log N}).$$

The first result for the upper bound of this density was given by Roth [9], and states:

Theorem 1.2 (Roth)**.** *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| = o(N).$$

The specific theorem we will prove, which in turn proves Roth's theorem, was proved by Bourgain in 1999 [4].

Theorem 1.3 (Bourgain)**.** *Consider a subset A of $[N]$. If A has no non-trivial three-term arithmetic progressions, then we have*

$$|A| \ll \left(\frac{\log\log N}{\log N}\right)^{1/2} N.$$

We shall outline the proof of this theorem, by first diverging by looking at subsets of $\mathbb{F}_p^n$. Then, we will see how we can adapt this proof to the above theorem.

We will next look at an improvement to Roth's theorem, by Thomas Bloom [3], which states

Theorem 1.4. *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \ll \frac{(\log\log N)^4}{\log N} N.$$

We finish by discussing a very recent breakthrough in the bound, by Kelley and Meka [8], who proved the following theorem.

Theorem 1.5. *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \leq N \exp(-\Omega((\log N)^{1/11})).$$

We will explore the new ideas of Kelley and Meka through the lens of the summary by Bloom and Sisask [5].

2 Background Information

We start by defining many of the tools we will use to prove our bounds on the density of any set which does not contain non-trivial 3APs. We will define the Fourier transform. Next, we define our convolutions, and see how we can map our statement of the number of three-term arithmetic progressions into a statement of a Fourier transform of convolutions. Lastly, we will define Bohr sets, which are analogous to subspaces of a vector space.

2.1 The Fourier Transform

We consider a general group G, and a function $f : G \to \mathbb{C}$. We want to define a function with respect to f which takes as its argument an element of

$$\hat{G} = \{\gamma : G \to \mathbb{C}^{\times} | \gamma \text{ is a continuous homomorphism}\},$$

the dual group of G, and outputs a value in $\mathbb{C}$.

Definition 2.1. [4, p.22] Consider a group G. Let $f : G \to \mathbb{C}$. The Fourier transform of f is the function $\hat{f} : \hat{G} \to \mathbb{C}$ with

$$\hat{f}(\gamma) = \langle f, \gamma \rangle = \sum_{x \in G} f(x)\overline{\gamma(x)} = \sum_{x} f(x)\gamma(-x).$$

Let us look at some simple properties of the Fourier transform. Any group G has, as an element of its dual group, the identity $\mathbb{1}$ which sends every element to 1. Given an arbitrary function $f : G \to \mathbb{C}$, what is its Fourier transform evaluated at $\mathbb{1}$?

$$\hat{f}(\mathbb{1}) = \langle f, \mathbb{1} \rangle = \sum_{x \in G} f(x)\overline{\mathbb{1}(x)} = \sum_{x} f(x).$$

So it is the sum of all the values of f.

Now if f itself is a homomorphism, we can take its Fourier transform and evaluate it at itself:

$$\hat{f}(f) = \langle f, f \rangle = \sum_{x \in G} f(x)\overline{f(x)} = \sum_{x} |f(x)|^2,$$

and hence $\langle f, f \rangle = ||f||_2^2$.

Another very important property of Fourier transforms is Parseval's identity. We will use this lemma many times throughout this paper. The proof is just a simple application of definitions, and can be found in [4, p.22].

Lemma 2.1 (Parseval's identity)**.** Given any $f, g : G \to \mathbb{C}$, we have

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

Specifically, $||f||_2 = ||\hat{f}||_2$ for any function $f : g \to \mathbb{C}$.

2.2 Convolutions

Now given two functions $f, g : G \to \mathbb{C}$, we want to define two types of convolution.

Definition 2.2. [4, p.4] The *convolution* of functions f and g is the function $f * g : G \to \mathbb{C}$ defined by

$$f * g(x) = \sum_{y \in G} f(y)g(x - y) = \sum_{y+z=x} f(y)g(z).$$

Definition 2.3. [4, p.4] The *difference convolution* of functions f and g is the function $f \circ g : G \to \mathbb{C}$ defined by

$$f \circ g(x) = \sum_{z \in G} f(x + z)\overline{g(z)} = \sum_{y-z=x} f(y)\overline{g(z)}.$$

Now we can combine our two convolutions with our Fourier transform, to get the following helpful proposition.

Proposition 2.1. [4, p.22] (Diagonalising convolution). Given functions $f, g : G \to \mathbb{C}$,

$$\widehat{f * g} = \hat{f} \cdot \hat{g},$$

and

$$\widehat{f \circ g} = \hat{f} \cdot \bar{\hat{g}}.$$

2.3 Bohr sets

When we consider the group $G = \mathbb{F}_p^n$, we can consider subspaces of $\mathbb{F}_p^n$. This has the nice property that these subspaces are closed under addition and multiplication by a factor. We would like to be able to generalise this to any group. To do this, we define Bohr sets.

Definition 2.4. [4, p.28] Let $\Gamma \subset \hat{G}$ and let $\rho \in [0, 2]$. The Bohr set with frequency set Γ and width ρ is defined as the set

$$B = \mathrm{Bohr}(\Gamma, \rho) = \{x \in G : |1 - \gamma(x)| \leq \rho \ \forall \gamma \in \Gamma\}.$$

Now take $\lambda > 0$. We write B_λ for the set $\mathrm{Bohr}(\Gamma, \lambda\rho)$.

The next lemma gives us some useful bounds on Bohr sets. A proof of this lemma can be found in [4, pp.30-31]. It is a simple argument using the Dirichlet box principle.

Lemma 2.2. Let $B = \mathrm{Bohr}(\Gamma, \rho)$ with $\rho \in (0, 1]$. Then

$$|B| \geq (\rho/8)^d N.$$

We also have that

$$|B_{1/2}| \geq 8^{-d}|B|.$$

So in particular, for any $0 < \delta < 1$,

$$|B_\delta| \geq (\delta/2)^{3d}|B|.$$

Now in general, these sets will not be closed under addition. But a certain type of Bohr set, called a regular Bohr set, can be considered very close to being closed under addition.

Definition 2.5. [4, p.32] Consider a Bohr set B. We call B *regular* if, for all $0 \leq \delta \leq 1/200d$, we have

$$|B_{1+\delta}| \leq (1 + 200d\delta)|B|,$$

and

$$|B_{1-\delta}| \geq (1 - 200d\delta)|B|.$$

Now not every Bohr set is regular. But we can make every Bohr set regular by scaling by some appropriate factor, as the next lemma tells us. A proof of this lemma can be found in [4, p.32].

Lemma 2.3 (Bourgain's regularity lemma)**.** Given any Bohr set B, there is a $\lambda \in [1/2, 1]$ such that B_λ is regular.

Now the next lemma shows us how we will use regularity of Bohr sets, applied to the Fourier transform of a function. The idea is that, if we apply a convolution by a small subset, then we can obtain the original Fourier transform with small error.

Lemma 2.4. [4, p.34] Let B be a regular Bohr set of rank d. Let $B' \subset B_\delta$ with $0 < \delta \leq 1/200d$. If f is any function supported on B with $|f(x)| \leq M$ for all $x \in B$, then

$$\langle f, 1_B * 1_{B'} \rangle = \langle f, 1_B \rangle |B'| + O(\delta d M |B||B'|).$$

So in particular, if $A \subset B$, and if we take $f = 1_A$, then

$$\langle 1_A, 1_B * 1_{B'} \rangle = |A||B'| + O(\delta d |B||B'|).$$

2.4 ϵ-control and the spectrum

Definition 2.6. [3, p.646] Let B be a subset of G, and let Γ be a subset of the dual group $\hat{G}$. Then, for $\epsilon \in [0, 2]$, B has *ϵ-control* of Γ if

$$|1 - \gamma(x)| \leq \epsilon,$$

for all $x \in B$ and $\gamma \in \Gamma$.

Now our overall strategy for each of our proofs is to show that there are few points at which the Fourier transform of the function is large. So we define the following set

Definition 2.7. [3, p.646] Given a function $f : G \to \mathbb{C}$, and a $\eta \in [0, 1]$, we define the *spectrum* as the set

$$\Delta_\eta(f) = \{\gamma \in \hat{G} : |\hat{f}(\gamma)| \geq \eta ||f||_1\}.$$

We also define the *level spectrum* as the set

$$\tilde{\Delta}_\eta(f) = \{\gamma \in \hat{G} : \eta ||f||_1 \leq |\hat{f}(\gamma)| < 2\eta ||f||_1\}.$$

2.5 Additive energy

We next define a function which takes a function $\omega : \hat{G} \to \mathbb{R}_+$, an integer $m \geq 1$, and a set $\Gamma \subset \hat{G}$. Then it outputs, weighted by the function ω, the size of the set

$$\{\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_m \in \hat{G} \mid (\gamma_1 + \ldots + \gamma_m) - (\gamma'_1 + \ldots + \gamma'_m) \in \Gamma\}.$$

Definition 2.8. The *additive energy* of a subset $\Gamma \subset \hat{G}$, function $\omega : \hat{G} \to \mathbb{R}_+$, and integer $m \geq 1$ is the sum

$$E_{2m}(\omega, \Gamma) = \sum_{\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_m} \omega(\gamma_1) \ldots \omega(\gamma'_m) \Gamma \left(\sum_{i=1}^{m} \gamma_i - \sum_{j=1}^{m} \gamma'_i \right).$$

The *restricted energy* of $\Gamma \subset \hat{G}$ and integers $t_1, t_2 \geq 1$ is the sum

$$E^{\#}_{t_1,t_2}(\omega, \Gamma) = \sum_{\substack{\Delta_1 \in \binom{\hat{G}}{t_1}, \Delta_2 \in \binom{\hat{G}}{t_2} \\ \Delta_1 \cap \Delta_2 = \emptyset}} \prod_{\gamma \in \Delta_1 \cap \Delta_2} \omega(\gamma) \Gamma \left(\sum_{\gamma \in \Delta_1} \gamma - \sum_{\gamma' \in \Delta_2} \gamma' \right),$$

where $\binom{\hat{G}}{t_i}$ symbolises the set of all subsets of $\hat{G}$ of size t_i.

In particular, we want to use the additive energy when ω is actually the characteristic function of some set S.

We want to create a bound on the additive energy with respect to the restricted energy. The following lemma is just a combinatorial argument, using linearity of expectation and the Cauchy-Schwarz inequality. A full proof can be found in [3, pp.647-648].

Lemma 2.5. Given any function $\omega : \hat{G} \to \mathbb{R}_+$, integer $m \geq 1$, and subset $\Gamma \subset \hat{G}$, we have the inequality

$$E_{2m}(\omega, \Gamma) \leq 2^{4m}(m!)^2||\omega||_2^{2m} \sum_{0 \leq t_1, t_2 \leq m} \frac{||\omega||_2^{-t_1-t_2}}{((m-t_1)!(m-t_2)!)^{1/2}} \sup_\lambda E^{\#}_{t_1,t_2}(\omega, \Gamma + \lambda).$$

Another helpful lemma bounds the additive energy from below. A proof of this lemma can be found in [3].

Lemma 2.6. Let $\epsilon \in [0, 1]$. Consider a subset $B \subset G$ and $\eta \in [0, 1]$. Let $f : B \to \mathbb{C}$ and $\omega : \hat{G} \to \mathbb{R}_+$. Suppose that f and ω are supported on $\Delta_\eta(f)$. Then, for all integers $m \geq 1$, we have

$$E_{2m}(\omega, \Delta_\epsilon(B)) \geq ||\omega||_1^{2m} \left(\left(\eta \frac{||f||_1}{||f||_{2m/(2m-1)}|B|^{1/2m}} \right)^{2m} - \epsilon \right).$$

2.6 Covering and dimension

We now need to define coverings and dimensions of subsets of $\hat{G}$. We define the notation $\langle \Lambda \rangle$ as the set

$$\langle \Lambda \rangle = \left\{ \sum_{\lambda \in \Lambda} \epsilon_\lambda \lambda \mid \epsilon_\lambda \in \{-1, 0, 1\} \right\},$$

with $\langle \emptyset \rangle = \{0\}$.

Definition 2.9. A subset $S \subset \hat{G}$ is *d-covered* by some subset $\Gamma \subset \hat{G}$ if there exists $\Lambda \subset \hat{G}$ with maximum size d such that

$$S \subset \Gamma - \Gamma + \langle \Lambda \rangle.$$

Definition 2.10. A subset $\Delta \subset \hat{G}$ is Γ-*dissociated* if, for any $k \geq 1$ and $\lambda \in \hat{G}$, there is not more than 2^k pairs of disjoint subsets $\Delta_1, \Delta_2 \subset \Delta$ such that $|\Delta_1 \cup \Delta_2| = k$ and

$$\sum_{\gamma \in \Delta_1} \gamma - \sum_{\gamma' \in \Delta_2} \gamma' \in \Gamma + \lambda.$$

Definition 2.11. A subset $S \subset \hat{G}$ has Γ-*dimension* d if d is the size of the largest Γ-dissociated subset of S.

Finally, to help with notation, we denote by $\mathcal{L}(\alpha)$ the value $2 + \lceil \log(1/\alpha) \rceil$.

3 Roth's theorem

3.1 Roth's theorem in $\mathbb{F}_p^n$

We will first prove an upper bound for the size of the largest subset of $\mathbb{F}_p^n$ with no non-trivial three-term arithmetic progressions. The ideas in the proof can then be adapted to prove Bourgain's Theorem.

First we note what happens if $p = 2$. In this case, the progression $x, x+d, x+2d$ just becomes $x, x+d, x$. Now this sequence exists for any subset $A \subset \mathbb{F}_2^n$ provided $|A| > 1$. So we only consider primes $p > 2$. So let us fix this prime $p \geq 3$.

3.1.1 Outline of proof

Assume we have a subset $A \subset \mathbb{F}_p^n$, and we only know its size. So its density is $\alpha = |A|/p^n$. Now we want to be able to conclude that, as long as α is sufficiently large, A must contain a 3AP. Two important cases can be first considered.

1. If A has lots of structure, then A contains many 3APs. For instance, if A is a subspace, then, for any two distinct elements x and $x + d$, $x + 2d$ must also appear in A.
2. Now if A is a random set, we expect the number of 3APs to be $\alpha^3 p^{2n}$. The number of trivial 3APs is $|A| = \alpha p^n$, so if $\alpha^3 p^{2n} \gg \alpha p^n$, i.e $\alpha \gg p^{-n/2}$, we are certain to have a non-trivial 3AP.

We now need to introduce the idea of a density increment. Suppose we have a subset A of $\mathbb{F}_p^n$, and suppose that A has no non-trivial 3APs. Our aim is to show that its density $\alpha = |A|/p^n$ is small. Now we either have that

1. A is random, so A has $\gg \alpha^3 p^{2n}$ many 3APs, and so
   (a) A is very small, i.e. $\alpha \ll p^{-n/2}$, or
   (b) A contains non-trivial 3APs,

   or,
2. A is not evenly distributed across different cosets, so we can find a large coset of a subspace $W \leq \mathbb{F}_p^n$ where the density of A increases.

But now we should look at A intersected with this coset. We can translate this coset so that it becomes a subspace. As 3APs are translation invariant, the translation of A also contains no non-trivial 3APs. Now we have obtained a large subset of W without 3APs. So we now have a vector space and a set containing no non-trivial 3APs. So we are back to our original scenario.

We now apply the density increment again. We cannot continue to fall into the second case forever, as the density increases by some factor each time, and the density can trivially never exceed 1. So at some point, we must conclude that the density of A is small, or A contains non-trivial three-term arithmetic progressions.

3.1.2 Meshulam

We want to prove the following theorem of Meshulam.

Theorem 3.1 (Meshulam)**.** *If $A \subset \mathbb{F}_p^n$ has no non-trivial 3APs, then we have*

$$|A| \ll_p p^n/n.$$

Note that, if we write $N = p^n$ for the size of the group, then the upper bound is

$$|A| \ll N/\log N.$$

We use the following lemma to prove this theorem. This gives us the density increment argument we are looking for. A proof of this lemma can be found in [4, pp.25-28].

Lemma 3.1. Consider an n-dimensional vector space V over $\mathbb{F}_p$, and let $A \subset V$ be a subset of density $\alpha = |A|/p^n$. Suppose that the only 3APs in A are trivial. Then either

1. $|A| \leq (2p^n)^{1/2}$; or
2. there exists a subspace $V' \leq V$ which has codimension 1, and there is an $x \in V$ such that

$$\frac{|(A-x) \cap V'|}{|V'|} \geq (1 + 1/4\alpha)\alpha.$$

Let us see how we can apply this lemma to prove Meshulam's theorem.

of Meshulam's theorem. Consider a set $A \in \mathbb{F}_p^n$ of density $\alpha > 0$, which has no non-trivial 3APs. We want to show that $\alpha \ll p^n/n$. We can assume that $\alpha > p^{-n/4}$, otherwise we are done. Now we only need to prove the bound for large n, as for small n we can use the bound $|A| \leq p^n$, and fix the constant in $|A| \ll_p p^n/n$.

Now we let $k \geq 0$ be the maximal value for which the following holds: There exists sets $A_0, ..., A_k$, and vector spaces $V_0, ..., V_k$ such that

1. $A_0 = A$, $V_0 = \mathbb{F}_p^n$;
2. For all i, $A_i \subset V_i$;
3. For all i, A_i has no non-trivial 3APs;
4. Let $\alpha_i = |A_i|/|V_i|$. Then $\alpha_{i+1} \geq (1 + \alpha_i/4)\alpha_i$;
5. For all i, $|V_{i+1}| \geq |V_i|/p$.

Now we want to explore how big k can be. By induction, we have that

$$\alpha_i \geq (1 + \alpha/4)^i \alpha \geq (1 + i\alpha/4)\alpha.$$

Now the bound of α_i after $\lceil 4/\alpha \rceil$ steps is $\alpha_i \geq 2\alpha$. Then after another $\lceil 4/2\alpha \rceil$ the bound is $\alpha_i \geq 4\alpha$. This process continues, and so after

$$\sum_{i=0}^{r} \lceil 4/2^i\alpha \rceil$$

steps, we have that the density $\alpha_i \geq 2^r\alpha$. Now the density must be less than or equal to 1. So this forces $r \ll \log(1/\alpha)$. Hence,

$$k \leq \sum_{i=0}^{O(\log(1/\alpha))} \lceil 4/2^i\alpha \rceil \ll \sum_{i=0}^{\infty} (4/2^i\alpha) + O(\log(1/\alpha)) \ll \alpha^{-1}.$$

Hence, we can in particular assume that $k \leq n/10$, otherwise $\alpha \ll 1/n$ as required.

Now we want to apply the lemma to $A_k \subset V_k$. We have chosen k to be maximal, so the second case of the lemma cannot hold - if it did hold, then we can pick $V_{k+1} = V'$ and $A_{k+1} = A - x$, and so k would not be maximal. So we must have that the first case holds, so

$$|A_k| = \alpha_k |V_k| \ll |V_k|^{1/2}.$$

Therefore, we have

$$p^{-n/4} \leq \alpha \leq \alpha_k \ll |V_k|^{-1/2}.$$

Now, by induction, we know that $|V_k| \geq p^{n-k} \geq p^{9n/10}$, and so

$$p^{-n/4} \ll p^{-9n/20}.$$

Now this is a contradiction for large enough n. □

3.2 Bourgain's bound

In the previous subsection, we have continually used the property that $\mathbb{F}_p^n$ is a vector space, and so contains many subspaces. In particular, we use that there is a large enough subspace such that $\gamma(x) = 1$. We would like to prove a bound on $\mathbb{Z}/N\mathbb{Z}$, where this statement no longer holds. This is where we introduce Bohr sets. These are analogous to subspaces for more general abelian groups.

We would now like to prove the following theorem.

Theorem 3.2 (Bourgain)**.** *Consider a subset A of $\{1, \dots, N\}$. If A has no non-trivial three-term arithmetic progressions, then we have*

$$|A| \ll \left(\frac{\log \log N}{\log N}\right)^{1/2} N.$$

Immediately we have a problem - $\{1, \dots, N\}$ is not a group, whereas all our theory has been developed with finite abelian groups. Therefore, we would actually like to prove the following theorem:

Theorem 3.3. *Assume G is a finite abelian group. Let G have odd order N. Let A be a subset of G. If A has no non-trivial three-term arithmetic progressions, then we have*

$$|A| \ll \left(\frac{\log \log N}{\log N}\right)^{1/2} N.$$

We can now very easily deduce Theorem 3.2 from Theorem 3.3, by passing to a larger group of order $2N - 1$.

Proof of Theorem 3.2 from Theorem 3.3. Let $M = 2N - 1$. Let A be a subset of $\{1, \dots, N\}$ with no non-trivial three-term arithmetic progressions. Now suppose there exists $x, y, z \in A$ distinct such that $x + y \equiv 2z \mod N$. So, modulo M, A did contain a non-trivial 3AP. Now, by definition of A, x, y and z lie between 1 and N. So hence,

$$-M < -2N + 2 \leq x + y - 2z \leq 2N - 2 < M.$$

So if $x + y - 2z \equiv 0 \mod M$, then $x + y - 2z = 0$. But this contradicts our assumption that A contains no non-trivial 3APs. So we conclude that, viewing A as a subset of $\mathbb{Z}/M\mathbb{Z}$, A still contains no non-trivial three-term arithmetic progressions. Hence, we can apply Theorem 3.3 to conclude

$$|A| \ll \left(\frac{\log \log M}{\log M}\right)^{1/2} M \ll \left(\frac{\log \log N}{\log N}\right)^{1/2} N.$$

□

So we are just left to prove Theorem 3.3. To do this, we use a density increment lemma. The idea is that this lemma will incrementally either give us the bound on the size of A that we desire, or will give us a Bohr set such that A has larger relative density.

Lemma 3.2. Assume we have a regular Bohr set B, with rank d and width ρ. Let A be a subset of B with density α. Assume that A has no non-trivial 3APs. Then there exists a constant $c > 0$ such that

1. either $|A| \ll (d/\alpha)^{O(d)} |B|^{1/2}$, or

2. there exists another regular Bohr set B', which is a subset of B, with rank $\leq d+1$ and width $\gg \rho(\alpha/d)^{O(1)}$, and x such that

$$\frac{|(A-x)\cap B'|}{|B'|} \geq (1+c\alpha)\alpha.$$

Now let us see how this density increment lemma proves Theorem 3.3. Compare this proof to the proof of Meshulam, Theorem 3.1. Here, we are still defining a maximal sequence of sets, but instead of having associated vector spaces, we have associated Bohr sets.

Proof of Theorem 3.3. Let A be a subset of a finite abelian group G, with density $\alpha > 0$. Assume A has no non-trivial 3APs. Assume also that $\alpha \geq 1/\log N$. Otherwise, our claim immediately follows.

We construct a maximal sequence $A_1, \ldots, A_k$ of subsets of G, with associated Bohr sets $B_1 \ldots, B_k$. Let d_i be the rank of B_i, and let ρ_i be the width of B_i. These subsets must satisfy

1. $A_0 = A$ and $B_0 = G$, so $d_0 = 1$ and $\rho_0 = 1$.
2. B_i contains A_i.
3. A_i only has trivial 3APs.
4. Let α_i be the density of A_i in B_i. Let c be the constant from Lemma 3.2. Then

$$\alpha_{i+1} \geq (1+c\alpha_i)\alpha_i.$$

5. $d_i \leq i+1$.
6. $\rho_{i+1} \gg (\alpha/d_i)^{O(1)}\rho_i$.

Now, just as in the proof of Theorem 3.1, we can conclude that $k \gg \alpha^{-1}$. So, in particular, $k+1 \gg \alpha^{-1}$. Applying Lemma 3.2 to $A_k \subset B_k$, if the second case held, then we would have a contradiction to the maximality of k. So the first case holds, and hence

$$\frac{1}{\log N} \leq \alpha \leq \alpha_k \ll (d_k/\alpha)^{O(d_k)}|B_k|^{-1/2} \ll (1/\alpha)^{O(\alpha^{-1})}|B_k|^{-1/2}.$$

Now, as $k+1 \gg \alpha^{-1}$, we have $d_i \ll \alpha^{-1}$. Therefore,

$$\rho_{i+1} \gg \alpha^{O(1)}\rho_i,$$

and hence $\rho_k \gg \alpha^{O(d_k)} \gg \alpha^{O(\alpha^{-1})}$. Now, by using our helpful bounds from Lemma 2.2, we have

$$|B_k| \geq (\rho_k/8)^{d_k} N \gg \alpha^{O(\alpha^{-2})} N.$$

By combining these inequalities, we get

$$\frac{1}{\log N} \ll \alpha^{-O(\alpha^{-1})}|B_k|^{-1/2} \ll \alpha^{-O(\alpha^{-2})}N^{-1/2}.$$

Therefore,

$$\alpha^{-2}\log(1/\alpha) \ll \log N.$$

Now remember that we have made the assumption that $\alpha \geq 1/\log N$, so we have $\log(1/\alpha) \ll \log\log N$. Hence,

$$\alpha^{-2} \gg \frac{\log N}{\log\log N},$$

and so

$$\alpha \ll \left(\frac{\log\log N}{\log N}\right)^{1/2}.$$

□

So now our job becomes proving Lemma 3.2. Note that this lemma is very similar to Lemma 3.1. However, we need to take care when dealing with Bohr sets which have different width.

To prove Lemma 3.2, we need two other lemmas. Firstly, we need a lemma which tells us that, given a regular Bohr set, we either have a density increment, or we can bound the number of 3APs in our set from below, if, in the Fourier transform $\langle 1_A * 1_A, 1_{2\cdot A}\rangle$, we replace $2 \cdot A$ with $2 \cdot B_\delta$, for δ small.

Lemma 3.3. Consider a regular Bohr set B with rank d and width ρ. Let $\delta \leq c_0\alpha/d$ for some constant $c_0 > 0$ sufficiently small such that B_δ is regular. Consider a subset $A \subset B$ with relative density $\alpha = |A|/|B|$. Then we either have

1. a lot of progressions relative to $2 \cdot B_\delta$. In particular,

$$\langle 1_A * 1_A, 1_{2\cdot B_\delta}\rangle \geq \frac{1}{2}\alpha^2|B||B_\delta|,$$

   or

2. we have our desired density increment. So, in particular, there exists a regular Bohr set B' with rank at most d, and width $\gg \delta^2\rho$, and an $x \in G$ such that

$$\frac{|(A-x)\cap B'|}{|B'|} \geq (1 + 1/256)\alpha.$$

The proof decomposes A into elements with large Fourier coefficients and elements with small Fourier coefficients. We can conclude that this second set is large, and so there are many elements of A with small Fourier coefficients. We then use the regularity of B_δ to find lots of elements in B whose Fourier coefficient is small. A full proof of this lemma can be found in [4, pp.37-38].

This lemma is suggesting that we need to work with two different Bohr sets at the same time. Here we are counting 3APs with two elements from B, and the third element from B_δ. So if we are looking at a subset $A \subset B$, we will need to be counting the 3APs where two elements lie in A, and our third element lies in $A \cap B_\delta$. But we need to make sure that $A \cap B_\delta$ contains many elements. Our second lemma tells us that we can always find a translate of A which is either dense enough in two narrowed copies of B_δ, or we have a density increment. The proof is just an application of the definition of regularity, using Lemma 2.4. A full proof can be found in [4, p.39].

Lemma 3.4. Consider a regular Bohr set B with rank d and width ρ. Consider a subset $A \subset B$ with relative density $\alpha = |A|/|B|$. Let $B', B'' \subset B_\delta$ for $\delta = c_0\alpha\epsilon/d$, for some sufficiently small fixed constant $c_0 > 0$. Then we either have

1. a large enough density in both B' and B''. In particular,

$$|(A-x)\cap B'| \geq (1-2\epsilon)\alpha|B'| \text{ and } |(A-x)\cap B''| \geq (1-2\epsilon)\alpha|B''|,$$

   or

2. a density increment in either B' or B''. So

$$\max\left(\frac{|(A-x)\cap B'|}{|B'|}, \frac{|(A-x)\cap B''|}{|B''|}\right) \geq (1+\epsilon)\alpha.$$

We are now finally able to explain the proof of Lemma 3.2. This proof considers two retractions of our Bohr set B. Let us define $B^{(1)} = B_{\delta_1}$ and $B^{(2)} = (B^{(1)})_{\delta_2}$, with $\delta_i = c_i\alpha^2/d$, with c_1, c_2 sufficiently small such that $B^{(1)}, B^{(2)}$ are regular. Now we can apply Lemma 3.4 with $B^{(1)}$ and $B^{(2)}$ acting as B' and B'', and with $\epsilon = c\alpha$, with c a small constant which we can later fix. Now if the second case of Lemma 3.4 holds, we are done as we have our desired density increment. Otherwise, if case one holds, we can find an $x \in G$ such that $A_1 = (A-x)\cap B^{(1)}$ and $A_2 = (A-x)\cap B^{(2)}$ both have large enough density. Now we have assumed that A has no non-trivial 3APs, so neither does A_1 or A_2. So, in particular,

$$\langle 1_{A_1} * 1_{A_1}, 1_{2\cdot A_2}\rangle = |A_2|.$$

Then applying Lemma 3.3, we again conclude that either we have the density increment that we desire, or we are able to find some character γ with large Fourier transform. Now instead of considering cosets, as in Lemma 3.1, we average over the translates $x+B'$ with $x \in B^{(2)}$. Using regualarity, we can bound

$$\langle f'_A, \theta\gamma + 1\rangle \gg \alpha|A_2|.$$

Now we use the fact that, although $\gamma(x)$ is not constant on the translates $B'+x$, it is approximately constant. We deduce that there exists some $x \in B^{(2)}$ such that

$$|A_2 \cap (B'+x)| - \alpha_2|B'| \gg \alpha\alpha_2|B'|.$$

So there must exist a constant $c > 0$ such that

$$\frac{|(A_2-x)\cap B'|}{|B'|} \geq (1+c\alpha)(1-2\epsilon)\alpha.$$

Choosing $\epsilon = c\alpha/8$ gives us our desired result as A_2 is a subset of a translate of A. A full proof with technical details can be found in [4, pp.40-42].

4 Bloom's improvement

Our next challenge is to prove the following statement. This was proved by Thomas Bloom in 2014 [3].

Theorem 4.1. *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \ll \frac{(\log\log N)^4}{\log N} N.$$

4.1 Outline of Bloom's improvement

The proof of Theorem 4.1 follows a very similar format to the proof of Bourgain. The improvement comes from forming a stronger density increment. This in turn follows from an improvement in Chang's lemma. Chang's lemma states:

Theorem 4.2 (Chang)**.** *Let A be a subset of G with density α. Then the spectrum $\Delta_\eta(A)$ is d-covered, for some d satisfying*

$$d \ll \eta^{-2}\log(1/\alpha).$$

To prove Chang's lemma, and the improvement of the statement, we need to look at the additive energy of large spectra. We then want to prove a connection between the dimension of a set and its additive energy. We are considering a set $S \subset \hat{G}$, and taking as ω the characteristic function of this set. Then we first bound the restricted energy of Γ by the Γ-dimension of S. We then use Lemma 2.5 which bounds the additive energy by the supremum of a set of restricted energies. This gives us the relation between the additive energy and the dimension. We then prove a technical lemma which gives a connection between covering and dimension.

These ideas together tell us that either the additive energy is small, or there is a finite set $\Delta \subset \hat{G}$ such that

$$\sum_{\gamma\in\Delta}\omega(\gamma) \geq n/d||\omega||_1,$$

which is $2d$-covered by Γ.

We next look at the structure found in the spectra. Shkredov [10] proved a useful lower bound on the additive energy of the large spectrum of a function $f : G \to \mathbb{C}$. Lemma 2.6 is a generalisation of this bound.

Together, these allow us to prove a generalised form of Chang's lemma.

4.2 Improvement of Chang's lemma

We need to use Chang's lemma to achieve a covering of the spectrum. However, when proving Theorem 4.1, we need a stronger theorem which tells us that there is a subset of the spectrum which is covered by an even smaller d.

Theorem 4.3. *Let A be a subset of G with density α. Then there is a subset $\Delta' \subset \Delta_\eta(A)$ with size $|\Delta'| \gg \eta|\Delta_\eta(A)|$ which is d-covered, for some d satisfying*

$$d \ll \eta^{-1}\log(1/\alpha).$$

Firstly we bound the restricted energy by the dimension. This is done by applying the definition of Γ-dimension to split S into two disjoint sets. Here, we are using S to both symbolise a subset of $\hat{G}$, and also to symbolise the characteristic function of this subset. So $S(\gamma) = 1$ if $\gamma \in S$, and $S(\gamma) = 0$ otherwise.

Lemma 4.1. Consider a set $S \subset \hat{G}$ with Γ-dimension $|S| - k$. Then for all $0 \leq t_1, t_2 \leq m$, we have

$$E^{\#}_{t_1,t_2}(S,\Gamma) \leq 4^{k+m}.$$

Proof. By definition of Γ-dimension, there is a partition of S into $S_0 \sqcup S_1$, where S_1 has size k and S_0 is Γ-dissociated. Now, as we are taking our weight function to be the characteristic function of S, the restricted energy becomes

$$E^{\#}_{t_1,t_2}(S,\Gamma) = \sum_{\substack{\Delta_1\in\binom{S}{t_1},\Delta_2\in\binom{S}{t_2}\\ \Delta_1\cap\Delta_2=\emptyset}} \Gamma\left(\sum_{\gamma\in\Delta_1}\gamma - \sum_{\gamma'\in\Delta_2}\gamma'\right).$$

Now we want to consider the contributions of S_1 to Δ_1 and Δ_2. So we sum over the possible sizes of $\Delta_1 \cap S_1$ and $\Delta_2 \cap S_1$. This gives us an upper bound of

$$E^{\#}_{t_1,t_2}(S,\Gamma) \leq \sum_{\substack{0 \leq r_1 \leq t_1 \\ 0 \leq r_2 \leq t_2}} \binom{k}{r_1}\binom{k}{r_2} \sup_{\lambda} \sum_{\substack{\Delta_1 \in \binom{S_0}{t_1 - r_1}, \Delta_2 \in \binom{S_0}{t_2 - r_2} \\ \Delta_1 \cap \Delta_2 = \emptyset}} \Gamma\left(\sum_{\gamma \in \Delta_1} \gamma - \sum_{\gamma' \in \Delta_2} \gamma' + \lambda\right).$$

Now S_0 is Γ-dissociated, so the inner sum is bounded by $2^{t_1+t_2}$. This gives us our result. □

We are left to prove a lemma which gives a connection between covering and dimension.

Lemma 4.2. Consider a symmetric set $\Gamma \subset \hat{G}$. Let $\Delta \subset \hat{G}$ have Γ-dimension r. Then there is a partition $\hat{G} = \Lambda_0 \sqcup \Lambda_1$ such that Λ_0 is $2r$-covered by Γ, and, given any $\gamma \in \Lambda_1$, $\Delta \cup \{\gamma\}$ has Γ-dimension $r+1$.

Proof. By definition of Γ-dimension, we can decompose Δ into $\Delta_0 \sqcup \Delta_1$, where Δ_0 is Γ-dissociated and the size of Δ_0 is r. We define a new set $\Delta' \subset \hat{G}$ to be all elements γ in $\hat{G}$ such that $\Delta_0 \cup \{\gamma\}$ is not Γ-dissociated.

We then define

$$\Lambda_0 = \Delta' \cup \Delta_0, \ \Lambda_1 = \hat{G} \backslash \Lambda_0,$$

and we claim that these two sets give us the claimed decomposition of $\hat{G}$ in the lemma. First we will show that, given any $\gamma \in \Lambda_1$, $\Delta \cup \{\gamma\}$ has Γ-dimension $r+1$. By definition, $\Delta_0 \cup \{\gamma\}$ is Γ-dissociated, and so, as $|\Delta_0 \cup \{\gamma\}| = r+1$, we have that $\Delta \cup \{\gamma\}$ has Γ-dimension at least $r+1$, as required.

Now we need to show that Λ_0 is $2r$-covered by Γ. Now clearly $\Delta_0 \subset \Gamma - \Gamma + \langle \Delta_0 \rangle + \langle \Delta_0 \rangle$. So it suffices to show that

$$\Delta' \subset \Gamma - \Gamma + \langle \Delta_0 \rangle + \langle \Delta_0 \rangle.$$

So let $\gamma \in \Delta'$. By definition of Δ', $\Delta_0 \cup \{\gamma\}$ is not Γ-dissociated. So there must be a $k \geq 1$ and $\lambda \in \hat{G}$ such that there are at least $2^k + 1$ triples of the form $(\epsilon, \Delta_1', \Delta_2')$ with $\epsilon \in \{-1, 0, 1\}$, $\Delta_1, \Delta_2' \subset \Delta_0$ disjoint, and $|\Delta_1' \cup \Delta_2'| + |\epsilon| = k$, such that

$$\epsilon\gamma + \sum_{\gamma_1' \in \Delta_1'} \gamma_1' - \sum_{\gamma_2' \in \Delta_2'} \gamma_2' \in \Gamma + \lambda.$$

Now let us consider all the cases for ϵ.

1. $\epsilon = 0$ for all triples. Then Δ_0 is not Γ-dissociated, which is a contradiction.

2. There is at least one triple with $\epsilon = 0$ and at least one triple with $\epsilon = 1$. Assume those triples are $(0, \Delta_1', \Delta_2')$ and $(1, \Delta_1'', \Delta_2'')$. Then we have that

$$\sum_{\gamma_1' \in \Delta_1'} \gamma_1' - \sum_{\gamma_2' \in \Delta_2'} \gamma_2' \in \Gamma + \lambda,$$

and we have that

$$\gamma + \sum_{\gamma_1'' \in \Delta_1''} \gamma_1'' - \sum_{\gamma_2'' \in \Delta_2''} \gamma_2'' \in \Gamma + \lambda.$$

Hence,

$$\gamma + \sum_{\gamma_1'' \in \Delta_1''} \gamma_1'' - \sum_{\gamma_2'' \in \Delta_2''} \gamma_2'' - \sum_{\gamma_1' \in \Delta_1'} \gamma_1' + \sum_{\gamma_2' \in \Delta_2'} \gamma_2' \in \Gamma + \lambda - (\Gamma + \lambda) = \Gamma - \Gamma.$$

Therefore, it follows that $\gamma \in \Gamma - \Gamma + \langle \Delta_0 \rangle + \langle \Delta_0 \rangle$.

3. There is at least one triple with $\epsilon = 0$ and at least one triple with $\epsilon = -1$. The argument is almost identical to the previous case.

4. $\epsilon \in \{-1, 1\}$ for all triples. Firstly, if $k = 1$, then Δ_1' and Δ_2' are empty. So there cannot be at least 3 distinct triples. Then, for $k > 1$, the pigeonhole principle requires that there be at least $2^{k-1} + 1$ triples with the same value of ϵ. So for this fixed ϵ, consider the translate $\Gamma + \lambda - \epsilon\gamma$. So there are at least $2^{k-1} + 1$ pairs Δ_1' and Δ_2' such that

$$\sum_{\gamma_1' \in \Delta_1'} \gamma_1' - \sum_{\gamma_2' \in \Delta_2'} \gamma_2' \in \Gamma + \lambda - \epsilon\gamma.$$

   This contradicts the fact that Δ_0 is Γ-dissociated.

So we can conclude that $\gamma \in \Gamma - \Gamma + \langle \Delta_0 \rangle + \langle \Delta_0 \rangle$, and hence that Λ_0 is $2r$-covered by Γ. □

Now these two lemmas, combined with Lemma 2.5, prove the following theorem. This theorem proves the idea that if we have a set such that every large subset is not $2d$-covered by Γ for some d, then the additive energy of Γ must be small with respect to d. The proof of this theorem is an adaptation of the proof given in Bateman and Katz [1].

Theorem 4.4. *Consider a symmetric set $\Gamma \subset \hat{G}$ and a weight function $\omega : \hat{G} \to \mathbb{R}_+$. Let $m \geq 2$ and $d \geq n \geq 2$ be constants such that $m \leq d/4$ and $||\omega||_2 \leq m^{1/2} d^{-1} ||\omega||_1$. Then either*

(i) there exists a finite set $\Delta \subset \hat{G}$ such that

$$\sum_{\gamma \in \Delta} \omega(\gamma) \geq n/d ||\omega||_1,$$

such that Δ is $2d$-covered by Γ; or

(ii) our additive energy is small. In particular,

$$E_{2m}(\omega, \Gamma) \leq 2^{13m+6n} m^{2m} d^{-2m} ||\omega||_1^{2m}.$$

The proof of this theorem uses random sampling. We consider S as a set of some maximum size chosen at random, with each element being selected with probability given by our weight function ω. We then prove that, given any $k \geq 0$, the random set S has Γ-dimension at least $d - k$ with probability at most $n^k/k!$. Assuming we are not in case 1 of the theorem, we can bound the probability that any k distinct elements of $\hat{G}$ are contained in S. Applying linearity of expectation, we can bound the restricted energy from below. Finally, we use Lemma 2.5 to bound the additive energy.

We will only prove this theorem for the special case that $\Gamma = \{0\}$ and w is the indicator function of some set $T \subset \hat{G}$. The proof of the general case can be found in [3, pp.649-650]. In this specific case, our statement becomes:

Theorem 4.5. *Let $T \subset \hat{G}$. Let $m \geq 2$ and $d \geq n \geq 2$ be constants such that $m \leq d/4$ and $|T|^{1/2} \geq m^{-1/2} d$. Then either*

(i) there exists a finite set $\Delta \subset \hat{G}$ such that

$$|T \cap \Delta| \geq n/d|T|,$$

such that there is a set $\Lambda \subset \hat{G}$ with $|\Lambda| \leq d$ and $T \subset \langle \Lambda \rangle$; or *(ii) our additive energy is small. In particular,*

$$|\{\gamma_1, \dots, \gamma_m, \gamma_1', \dots, \gamma_m' \in T \mid \sum \gamma_i = \sum \gamma_i'\}| \le 2^{13m+6n} m^{2m} d^{-2m} |T|^{2m}.$$

Proof. We first normalise our indicator function $\omega = 1_T$ so that $||\omega||_1 = 1$ without loss of generality. Then we have that $||\omega||_2 = |T|^{-1/2}$. Let us assume that case 1 does not hold. So, we can assume that, given any subset $\Delta \subset \hat{G}$ which is $2d$-covered by $\{0\}$, we have that

$$|T \cap \Delta| \le nd^{-1}.$$

So we now pick a random set S of size at most d by selecting d elements of T. Suppose we have selected d' elements of S. Call this set S'. Let S' have Γ-dimension r. Now we can apply Lemma 4.2 to partition $\hat{G}$ into $\Lambda_0 \sqcup \Lambda_1$. Then for all $\gamma \in \Lambda_1$, $S' \cup \{\gamma\}$ has Γ-dimension at least $r + 1$. So we now have

$$\mathbb{P}(\dim(S' \cup \{\gamma\}) \le \dim(S')) \le |T \cap \Lambda_0| \le n/d,$$

as Λ_0 is $2d$-covered by $\{0\}$. Now we note that dimension is non-decreasing and the dimension of the empty set is zero. So we conclude that the probability that S has dimension $d - k$ is at most

$$\binom{d}{k} n^k d^{-k} \le n^k / k!.$$

Now we apply Lemma 4.1 to conclude that for all $\lambda \in \hat{G}$ and integers $t_1, t_2 \le m$, we have

$$\mathbb{E} E^{\#}_{t_1,t_2}(S, \lambda) \le 4^m \sum_{k=0}^{\infty} \frac{(4n)^k}{k!} = 4^m e^{4n}.$$

Given any distinct $\gamma_1, \dots, \gamma_k \in \hat{G}$,

$$\mathbb{P}(\gamma_1, \dots, \gamma_k \in S) \ge k! \binom{d}{k} \omega(\gamma_1) \dots \omega(\gamma_k) \left(1 - \sum_{i=1}^{k} \omega(\gamma_i)\right)^{d-k}.$$

Now, by Cauchy-Schwarz, we conclude that

$$\sum_{i=1}^{k} \omega(\gamma_i) \le (2m)^{1/2} |T|^{-1/2} \le 2md^{-1} \le \frac{1}{2}.$$

Using this fact, combined with the fact that $k \le d/2$ so $k!\binom{d}{k} \ge (d/2)^k$, we deduce that

$$\mathbb{P}(\gamma_1, \dots, \gamma_k \in S) \ge 2^{-5m} d^k \omega(\gamma_1) \dots \omega(\gamma_k).$$

Now, assuming $t_1 + t_2 \le m$, by linearity of expectation we have

$$\mathbb{E} E^{\#}_{t_1,t_2}(S, \lambda) \ge 2^{-5m} d^{t_1+t_2} E^{\#}_{t_1,t_2}(\omega, \lambda).$$

So we conclude that, for any $\lambda \in \hat{G}$ and $0 \le t_1, t_2 \le m$, we have

$$E^{\#}_{t_1,t_2}(\omega, \lambda) \le 2^{7m} e^{4n} d^{-t_1-t_2}.$$

Now we apply Lemma 2.5 to bound our additive energy, giving

$$|\{\gamma_1,\dots,\gamma_m,\gamma_1',\dots,\gamma_m' \in T \mid \sum \gamma_i = \sum \gamma_i'\}| \le 2^{11m} e^{4n} m! |T|^m \left(\sum_{0\le t\le m} \frac{(m!)^{1/2}(|T|^{1/2}d^{-1})^t}{((m-t)!)^{1/2}} \right)^2 .$$

We can then bound the sum by Cauchy-Schwarz, to achieve

$$|\{\gamma_1,\dots,\gamma_m,\gamma_1',\dots,\gamma_m' \in T \mid \sum \gamma_i = \sum \gamma_i'\}| \le 2^{13m} e^{4n} m^{2m} d^{-2m} |T|^{2m}.$$

Bounding e^4 by 2^6 gives the required bound. □

Finally, we can prove Theorem 4.3. We actually are able to prove a stronger statement which considers any set B, a general weight function ω, and a constant ϵ. To recover Theorem 4.3, we take $B = G$, consider the weight function to be the characteristic function of $\Delta_\eta(f)$, and let $\epsilon \to 0$.

Theorem 4.6. *Consider a function $f : B \to \mathbb{C}$. Let $\alpha = ||f||_1/||f||_\infty |B|$. Consider a weight function $\omega : \hat{G} \to \mathbb{R}_+$. Assume that ω is supported on $\Delta_\eta(f)$. Let $0 \le \epsilon \le \exp(-8\mathcal{L}(\eta)\mathcal{L}(\alpha))$. Then there exists a subset $\Delta' \subset \Delta_\eta(f)$ such that*

$$\sum_{\gamma\in\Delta'} \omega(\gamma) \ge 2^{-12}\eta ||\omega||_1,$$

such that Δ' is $2^{14}\mathcal{L}(\alpha)\eta^{-1}$-covered by $\Delta_\epsilon(B)$.

Proof. Again we assume that $||\omega||_1 = 1$ without loss of generality. Let us first consider when $||\omega||_2 \ge 2^{-12}\mathcal{L}(\alpha)^{-1/2}\eta$. Similarly to the proof of Theorem 4.4, we take Δ' to be a random subset of $\hat{G}$ with each element γ of $\hat{G}$ being chosen independently with probability $2^{13}\eta^{-1}\mathcal{L}(\alpha)\omega(\gamma)$. Now we apply Chernoff's inequality to conclude that $|\Delta'| \le 2^{14}\eta^{-1}\mathcal{L}(\alpha)$ with probability at least 7/8. Also, we have

$$\mathbb{E} \sum_{\gamma\in\Delta'} \omega(\gamma) \ge 2^{13}\eta^{-1}\mathcal{L}(\alpha)||\omega||_2^2 \ge 2^{-11}\eta.$$

We can then apply Markov's inequality to conclude that $\sum_{\gamma\in\Delta'}\omega(\gamma) \ge 2^{-12}\eta$ with probability at least 1/2. So there must exist a set $\Delta' \subset \Delta_\eta(f)$ such that

$$\sum_{\gamma\in\Delta'} \omega(\gamma) \ge 2^{-12}\eta ||\omega||_1,$$

such that Δ' is $2^{14}\mathcal{L}(\alpha)\eta^{-1}$-covered by $\Delta_\epsilon(B)$.

Now let us consider when $||\omega||_2 < 2^{-12}\mathcal{L}(\alpha)^{-1/2}\eta$. Let $n = m = \mathcal{L}(\alpha)$ and $d = \lfloor 2^{12}\eta^{-1}m \rfloor$. We apply Lemma 2.6 to conclude

$$E_{2m}(\omega, \Delta_\epsilon(B)) \ge \left(\eta \frac{||f||_1}{||f||_{2m/(2m-1)}|B|^{1/2m}} \right)^{2m} - \epsilon.$$

We can then apply the bound $||f||_{2m/(2m-1)} \le ||f||_\infty^{1/2m} ||f||_1^{1-1/2m}$ to conclude that, if $\epsilon \le \eta^{2m}\alpha/2$, then

$$E_{2m}(\omega, \Delta_\epsilon(B)) \ge \eta^{2m}\alpha/2.$$

Now applying Lemma 4.3, we either find a set Δ' such that Δ' is $2d$-covered by $\Delta_\epsilon(B)$, and

$$\sum_{\gamma \in \Delta'} \omega(\gamma) \geq 2^{-12}\eta||\omega||_1,$$

or

$$\eta^{2m}\alpha \leq 2^{19m+1}m^{2m}d^{-2m},$$

and we would conclude that

$$d \leq 2^{10}m\eta^{-1}\alpha^{-1/2m}.$$

This is a contradiction to our choice of d and m. □

4.3 More density increments

Our aim is to achieve a better density increment than the one in Section 3.2. We do this by using the next three lemmas, whose proofs can be found in [3], and using the improvement to Chang's lemma.

This first lemma extracts information of the balanced function, converting it to a density increment.

Lemma 4.3. Consider a function $f : B \to [0, 1]$ and let $\mathbf{f} = f - \alpha B$, where $\alpha = ||f||_1/|B|$. Assume that the following inequality holds:

$$\sum_{\gamma \in \Gamma} |\hat{\mathbf{f}}(\gamma)|^2 \geq \nu\alpha||f||_1 N.$$

Also assume that we have a symmetric set B' satisfying

$$|\hat{B}'(\gamma)| \geq 2^{-1}|B'|,$$

for all $\gamma \in \Gamma$. Further assume that

$$|(2B' + B)\backslash B| \geq 2^{-4}\nu\alpha|B|.$$

Then we have

$$||f * B'||_\infty \geq (1 + 2^{-3}\nu)\alpha|B'|.$$

The idea of the proof is to convert the inequality into an inequality of the L^2-norm of the balanced function. Then, using the hypotheses, we can convert this information into a bound of the L^2-norm of the original function. This then leads to the desired bound of the L^∞-norm.

This next lemma tells us that if we have a set which controls a few elements, and has small dimension, then this set controls all elements.

Lemma 4.4. Let $\Delta \subset \hat{G}$ be d-covered by Γ. Then there exists a $\Lambda \subset \hat{G}$, of size at most d, such that if we have a set B with $(4d)^{-1}$ control of Λ and $\Gamma(1/8)$, then we have

$$|\hat{B}(\gamma)| \geq 2^{-1}|B|,$$

for all $\gamma \in \Delta$.

The proof is just a simple application of the definitions of covering and control, along with the triangle inequality.

Our last lemma gives us the control we want to use Lemma 4.4.

Lemma 4.5. Let $c > 0$ and B and B' be sets such that

$$|(B+B')\backslash B| \leq c\epsilon|B|.$$

Then B' has $2c$-control of $\Delta_\epsilon(B)$.

We can now state and prove our density increment theorem.

Theorem 4.7. *Given any two sets $B, B' \subset G$, let A be a subset of B with relative density α, and let $f : B' \to [-1,1]$ with density $\tau = ||f||_1/|B'|$. Define the function $\mathbf{A}(x) = A(x) - \alpha B(x)$. Assume that we have*

$$\sum_\gamma |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2 = \nu\alpha||f||_1|A|N. \tag{1}$$

Then there exists a set $\Lambda \subset \hat{G}$ with size

$$d \leq 2^{16}\mathcal{L}(\tau)(\nu\alpha)^{-1},$$

such that, if we have a symmetric set $B'' \subset G$, and B'' has $(4d)^{-1}$-control of Λ,

$$|(2B''+B)\backslash B| \leq 2^{-17}\nu\alpha|B|, \tag{2}$$

and

$$|(B''+B)\backslash B| \leq 2^{-4}\exp(-2^4\mathcal{L}(\tau)\mathcal{L}(\nu\alpha))|B'|, \tag{3}$$

then there exists an $x \in G$ such that

$$|(A-x)\cap B''| \geq (1+2^{-16}\nu)\alpha|B''|.$$

Proof. Let $\eta = \nu\alpha/2$ and consider $\Delta = \Delta_\eta(f)$. Considering the sum over all elements $\gamma \notin \Delta$,

$$\sum_{\gamma\notin\Delta} |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2 \leq 2^{-1}\nu\alpha||f||_1||\mathbf{A}||_2^2N \leq 2^{-1}\nu\alpha||f||_1|A|N.$$

Therefore, by equation 1, we have

$$\sum_{\gamma\notin\Delta} |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2 \geq 2^{-1}\nu\alpha||f||_1|A|N.$$

Define a weight function

$$\omega(\gamma) = |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2,$$

and a constant

$$\epsilon = \exp(-2^4\mathcal{L}(\tau)\mathcal{L}(\nu\alpha)) \leq \exp(-8\mathcal{L}(\tau)\mathcal{L}(\eta)).$$

Now we want to split Δ into a dyadic decomposition with $\Delta_i = \tilde{\Delta}_{2^i\eta}(f)$. Now applying Theorem 4.6, we obtain a set Δ_i' for each i. We then have

$$\begin{aligned} 2^{-12}\sum_{\gamma\in\Delta_i} |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2 &\leq (2^i\eta)^{-1}\sum_{\gamma\in\Delta_i'} |\hat{f}(\gamma)||\hat{\mathbf{A}}(\gamma)|^2 \\ &\leq 2||f||_1\sum_{\gamma\in\Delta_i'} |\hat{\mathbf{A}}(\gamma)|^2. \end{aligned}$$

Next, we sum over all i, defining $\Delta' = \bigcup_i \Delta'_i$, to get

$$\sum_{\gamma\in\Delta'} |\hat{\mathbf{A}}(\gamma)|^2 = \sum_i \sum_{\gamma\in\Delta'_i} |\hat{\mathbf{A}}(\gamma)|^2 \geq 2^{-13}\nu\alpha|A|N.$$

Now by Theorem 4.6, each Δ'_i is $2^{14}\mathcal{L}(\tau)(2^i\eta)^{-1}$-covered by $\Delta_\epsilon(B')$. So hence, for some d satisfying

$$d \leq 2^{14}\mathcal{L}(\tau)\sum_i (2^i\eta)^{-1} \leq 2^{16}\mathcal{L}(\tau)(\nu\alpha)^{-1},$$

Δ' is d-covered by $\Delta_\epsilon(B')$. Now by equation 3, and the definition of ϵ, $|(B''+B')\backslash B'| \leq 2^{-4}\epsilon|B'|$. So applying Lemma 4.5, B'' has 2^{-3}-control of $\Delta_\epsilon(B')$. Then applying Lemma 4.4, we can find a Λ of size $d \leq 2^{16}\mathcal{L}(\tau)(\nu\alpha)^{-1}$, such that, if B'' has $(4d)^{-1}$-control of Λ, then

$$|\hat{B''}(\gamma)| \geq 2^{-1}|B''|,$$

for all $\gamma \in \Delta'$. Now applying equation 2 to Lemma 4.3, we conclude that

$$|(A-x)\cap B''| \geq (1+2^{-16}\nu)\alpha|B''|.$$

□

We note that the number of solutions to the equation $x_1 + x_2 - 2x_3 = 0$ is given by $\langle A * A, 2A\rangle$. So we can now apply our previous theorem to prove another density increment theorem.

Theorem 4.8. *Let $B' \subset B \subset G$, and suppose that $A_1 \subset B'$, $A_2, A_3 \subset B$, with relative densities α_i. Define $\alpha = 2^{-1}\min(2^{-5}, \alpha_1, \alpha_2, \alpha_3)$. Let us assume that B and B' satisfy*

$$|(B'+B)\backslash B| \leq 2^{-2}\alpha|B|.$$

Then we either have that

$$\langle A_1 * A_2, A_3\rangle \geq 2^{-2}\alpha_1\alpha_2\alpha_3|B||B'|, \tag{4}$$

or we can find a set $\Lambda \subset \hat{G}$ with size

$$d \leq 2^{-19}\mathcal{L}(\alpha)\alpha^{-1}$$

such that, if we have a symmetric set B'' having $(4d)^{-1}$-control of Λ, with

$$|(2B''+B)\backslash B| \leq 2^{-19}\alpha|B|$$

and

$$|(B''+B)\backslash B| \leq 2^{-4}\exp(-2^5\mathcal{L}(\alpha)^2)|B'|,$$

then we can find an $x \in G$ and $i \in \{2,3\}$ such that

$$|(A_i - x)\cap B''| \geq (1+2^{-18})\alpha_i|B''|.$$

Proof. Let us define $\mathbf{A}_1 = A_1 - \alpha_1 B'$, $\mathbf{A}_2 = A_2 - \alpha_2 B$ and $\mathbf{A}_3 = A_3 - \alpha_3 B$. Then we have

$$\sum_x A_1 * \mathbf{A}_2(x)\mathbf{A}_3(x) = \sum_x A_1 * A_2(x)A_3(x) - \alpha_2\sum_{x\in B} A_3 * A_1(x)$$
$$- \alpha_3 \sum x \in B A_1 * A_2(x) + \alpha_2\alpha_3 \sum_{x\in B} B * A_1(x).$$

Let us suppose that some sets A and A' satisfy $A \subset B$ and $A' \subset B'$. Then we conclude that

$$\left| \sum_{x \in B} A' * A(x) - |A'||A| \right| \leq |A'||(B'+B)\backslash B| \leq 2^{-2}|A'||A|.$$

So in our case we get

$$\sum_x A_1 * \mathbf{A}_2(x)\mathbf{A}_3(x) \leq \sum_x A_1 * A_2(x)A_3(x) - 2^{-1}\alpha_1\alpha_2\alpha_3|B||B'|.$$

Let us assume that equation 4 is not true. Then, applying the triangle inequality, we get

$$\sum_\gamma |\widehat{A_1}(\gamma)||\widehat{\mathbf{A}_2}(\gamma)||\widehat{\mathbf{A}_3}(\gamma)| \geq 2^{-2}\alpha_1\alpha_2\alpha_3|B||B'|N.$$

Next, we use the Cauchy-Schwarz inequality. We can then conclude that

$$\left(\sum_\gamma |\widehat{A_1}(\gamma)||\widehat{\mathbf{A}_2}(\gamma)|^2 \right) \left(\sum_\gamma |\widehat{A_1}(\gamma)||\widehat{\mathbf{A}_3}(\gamma)|^2 \right) \geq 2^{-4}|A_1|^2\alpha_2^2\alpha_3^2|B|^2N^2.$$

So there must be an $i \in \{2, 3\}$ with

$$\sum_\gamma |\widehat{A_1}(\gamma)||\widehat{\mathbf{A}_i}(\gamma)|^2 \geq 2^{-2}|A_1|\alpha_i^2|B|N.$$

Applying Theorem 4.7, we conclude that either equation 4 holds, or we can find an $x \in G$ and $i \in \{2, 3\}$ such that

$$|(A_i - x) \cap B''| \geq (1 + 2^{-18})\alpha_i|B''|.$$

□

Our final step in producing a density increment is to reduce down to Bohr sets, and use the specific properties of Bohr sets to produce a stronger density increment.

Theorem 4.9. *Suppose $B \subset \mathbb{Z}_N$ is a regular Bohr set of rank $d \leq \exp(c\mathcal{L}(\alpha)^2)$. Suppose $A_1, A_2 \subset B$ and $A_3 \subset B(\delta)$, with relative densities α_i. Then there exists a constant $c > 0$ such that the following holds. Let $\alpha = \min(c, \alpha_1, \alpha_2, \alpha_3)$ and suppose $B(\delta)$ is regular with $cd^{-1}\alpha/4 \leq \delta \leq cd^{-1}\alpha$. Then we either have that*

$$\langle A_1 * A_2, A_3 \rangle \gg \alpha_1\alpha_2\alpha_3|B||B(\delta)|, \tag{5}$$

or we can find a regular Bohr set B' with rank

$$\mathrm{rk}(B') \leq d + O(\mathcal{L}(\alpha)\alpha^{-1}) \tag{6}$$

and with

$$|B'| \geq \exp(-o(\mathcal{L}(\alpha)^2(d + \mathcal{L}(\alpha)\alpha^{-1})))|B|, \tag{7}$$

such that there is $x \in \mathbb{Z}_N$ and $i \in \{1, 2\}$ with

$$|(A_i - x) \cap B'| \geq (1 + c)\alpha_i|B'|.$$

The proof bounds the size of $(B(\delta) + B)\backslash B'$ for δ sufficiently small using the regularity of B. Then we can apply Theorem 4.8 to conclude that either condition 5 holds or we can find a subset $\Lambda \subset \hat{G}$, of small size relative to α, such that if B' has $(4l)^{-1}$-control of Λ, and $(2B' + B)\backslash B$ and $(B' + B(\delta))\backslash B(\delta)$ are bounded as in Theorem 4.8, then we can find an $x \in \mathbb{Z}_N$ and $i \in \{1, 2\}$ such that

$$|(A_i - x) \cap B'| \geq (1 + 2^{-18})\alpha_i |B'|.$$

We then define another Bohr set B' in terms of the function $1/4l$. This Bohr set has rank $d + O(\mathcal{L}(\alpha)\alpha^{-1})$ and has $(4l)^{-1}$-control of Λ. Provided we pick our constants in such a way that our bounds of Theorem 4.8 are satisfied, we can conclude by regularity of $B(\delta)$ that equations 6 and 7 are satisfied. A full proof with all the technical details can be found in [3, pp.659-660].

4.4 Proof of theorem

We can now use our final density increment theorem to prove Theorem 4.1.

Theorem 4.10. *Let N be a sufficiently large prime, and A any subset of $\mathbb{Z}_N$. Let $Y(A) = \langle A*A, 2A \rangle$ be the number of three-term arithmetic progressions in A. Then*

$$Y(A) \geq \exp(-O_{\mathbf{c}}(\mathcal{L}(\alpha)^4 \alpha^{-1}))N^2.$$

The proof is very similar to the proof of Theorem 3.3. The idea is to define a sequence of maximal Bohr sets with some properties. Then, applying Theorem 4.9, we either conclude that $|A|$ is large, as required, or there is another Bohr set which satisfies the properties required, and so contradicts the maximality of our sequence. Details can be found in [3, pp.660-661].

Now Theorem 4.1 immediately follows from this theorem, by considering $\{1, \ldots, N\}$ as a subset of $\mathbb{Z}_{N'}$, where $N' \ll N$ is a prime which is large enough such that, if x_1, x_2, x_3 is a solution to $x_1 + x_2 - 2x_3 = 0$ within $\mathbb{Z}_N$, then it is also a solution within $\{1, \ldots, N\}$.

5 Kelley and Meka

In this section, we quickly overview the very recent breakthrough by Kelley and Meka. Using an unbalancing argument, where we can derive a bound on the size of a function based on the size of its balance function, we can prove the following theorem.

Theorem 5.1 (Kelley-Meka)**.** *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \leq N \exp(-\Omega((\log N)^{1/11})).$$

Just as in Section 3, Bloom and Sisask first prove a related statement in $\mathbb{F}_p^n$.

Theorem 5.2 (Kelley-Meka)**.** *Let q be an odd prime and $A \subseteq \mathbb{F}_p^n$. If A has no non-trivial three-term arithmetic progressions, then*

$$|A| \ll q^{n - \Omega(n^{1/9})}.$$

In this paper, we will quickly overview the proof of this theorem. To prove Theorem 5.1, a very similar argument is made, replacing subspaces with Bohr sets.

To prove Theorem 5.2, we prove the following unbalancing lemma.

Lemma 5.1. Consider a function $f = g \circ g$, where $g : G \to \mathbb{R}$ is an arbitrary function. Assume that $||f||_p \geq 1/4$ for some p. Then there exists some $p' \ll p$ such that $||f+1||_{p'} \geq 1 + 1/8$.

So in particular, we can apply this to the balance function $f = \mu_A \circ \mu_A - 1 = (\mu_A - 1) \circ (\mu_A - 1)$ to conclude that if $||\mu_A \circ \mu_A - 1||_p \geq 1/4$, then $||\mu_A \circ \mu_A||_{p'} \geq 1 + 1/8$.

Proof. Firstly, without loss of generality, we can assume that p is an odd integer greater than 4. We now want to show that $\mathbb{E}(g \circ g)^p$ is non-negative. This is a simple combinatorial argument. By definition,

$$\mathbb{E}(g \circ g)^p = \mathbb{E}_x(\mathbb{E}_y(g(x+y)g(y)))^p.$$

Reparametrising, we get

$$\mathbb{E}(g \circ g)^p = \mathbb{E}_{x,x'}(\mathbb{E}_y(g(x+y)g(x'+y)))^p.$$

Now we can expand the power and the expectations, to get

$$\frac{1}{N^{p+2}} \sum_{x,x'} \sum_{y_1,\ldots,y_p} g(x+y_1)\cdots g(x+y_p)g(x'+y_1)\cdots g(x'+y_p).$$

Interchanging the sums, we get

$$\frac{1}{N^p} \sum_{y_1,\ldots,y_p} \left(\frac{1}{N^2} \sum_{x,x'} g(x+y_1)\cdots g(x+y_p)g(x'+y_1)\cdots g(x'+y_p) \right).$$

This simplifies to

$$\mathbb{E}_{y_1,\ldots,y_p}(\mathbb{E}_x(g(x+y_1)\cdots g(x+y_p)))^2,$$

which is clearly non-negative. Now we note that $2\max(x,0) = x + |x|$, and so

$$2\langle \max(f,0), f^{p-1}\rangle = \frac{1}{N}\mathbb{E}(f^p) + \langle |f|, f^{p-1}\rangle \geq ||f||_p^p \geq (1/4)^p.$$

Let us define two sets

$$P = \{x \mid f(x) \geq 0,\}$$

and

$$T = \{x \in P \mid f(x) \geq \frac{3}{16}\}.$$

Then we note that our bound above gives $\langle 1_P, f^p\rangle \geq \frac{1}{2} \cdot (\frac{1}{4})^p$, and $\langle 1_{P\setminus T}, f^p\rangle \leq \frac{1}{4} \cdot \epsilon^p$. So, applying the Cauchy-Schwarz inequality, we have

$$\frac{1}{|T|^{1/2}} ||f||_{2p}^p \geq \langle 1_T, f^p\rangle \geq \frac{1}{4} \cdot \left(\frac{1}{4}\right)^p.$$

Now, by the triangle inequality,

$$||f||_{2p} \leq 1 + ||f+1||_{2p}.$$

So we can assume that $||f||_{2p} \leq 3$, otherwise we would be done with $p' = 2p$. So therefore, we can conclude that $\frac{1}{|T|} \geq (\frac{1}{40})^{2p}$. So now, for any $p' \geq 1$, we have

$$||f+1||_{p'} \geq \langle 1_T, |f+1|^{p'}\rangle^{1/p'} \geq (1 + \frac{3}{16})(\frac{1}{40})^{2p/p'}.$$

So we just choose p' to be a suffiently large multiple of p. □

Now we have proven this, we can use Holder's inequality to prove that, if $\langle \mu_A * \mu_A, \mu_C \rangle \leq 1/2$, then $||\mu_A \circ \mu_A||_p \geq 1/4$ for some p, and the lemma can then be applied in this case.

Then, via dependent random choice, we can conclude that if $||\mu_A \circ \mu_A||_{p'} \geq 1 + 1/8$, that we can find sets $A_1, A_2 \subseteq A$ with large enough density relative to p', such that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle \geq 1 - 1/32,$$

where $S = \{x \mid \mu_A \circ \mu_A > 1 + 1/16\}$.

We next can make an almost-periodicity statement. It tells us that if we have sets $A_1, A_2, S \subseteq \mathbb{F}_p^n$, we can find a subspace, with small dimension relative to the densities of A_1, A_2, S, such that

$$|\langle \mu_V * \mu_{A_2} \circ \mu_{A_2}, 1_S \rangle - \langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle| \leq 1/100.$$

Finally, we can obtain our density increment. This gives us a subspace V, of small codimension relative to the density of A, such that $A \cup V$ has large density.

6 Conclusion

In this project, we have looked at some of the history of three-term arithmetic progressions. Defining the Fourier transform, we have seen how it is a powerful tool in additive combinatorics. We have looked at the specific results of Bourgain and Bloom, and quickly looked at the very new result by Kelley and Meka. For each of these theorems, we have used new techniques to strengthen our density increment. This in turn reduces our upper bound on the size of a set containing no non-trivial three-term arithmetic progressions.

There are a few other notable results within the field of bounds for 3APs. For instance, Bloom and Sisask, in 2020, managed to break the logarithmic barrier for this problem [6], proving the following theorem.

Theorem 6.1. *Let A be a subset of $[N]$. Assume that A contains no non-trivial three-term arithmetic progressions. Then*

$$|A| \ll \frac{1}{(\log N)^{1+c}} N,$$

where c is an absolute constant.

Bateman and Katz also made a significant contribution to the field. Our proof of Theorem 4.4 is an adaptation of the argument made in Section 5 of [1]. Here, Bateman and Katz introduce the idea of random sampling to bound the additive energy.

There have also been large breakthroughs when we consider k-term arithmetic progressions, for a general k. In particular, Szemerédi's theorem states

Theorem 6.2. *Given any $\delta > 0$ and $k \geq 1$, there exists $N \ll_{\delta,k} 1$ such that if we have a subset $A \subset \{1, \ldots, N\}$ with size $|A| \geq \delta N$, then A must contain a non-trivial k-term arithmetic progression.*

The most effective method for proving Szemerédi's theorem was given by Gowers [7]. If $r_k(N)$ is the size of the largest subset of $\{1, \ldots, N\}$ containing no non-trivial three-term arithmetic progressions, then Gowers proved the following:

Theorem 6.3. *For every $k \in \mathbb{N}$, there exists a constant $c_k > 0$, only depending on k, such that*

$$r_k(N) \ll \frac{N}{(\log \log N)^{c_k}}.$$

Gowers still used a density increment argument, as we have used in this paper many times. However, for three-term arithmetic progressions, we have been using the usual character $\gamma(n) = \mathrm{e}(rn/N)$. To prove the more general statement of Gowers, he had to introduce the idea of using polynomial characters.

References

[1] Michael Bateman and Nets Hawk Katz. New bounds on cap sets. *J. Amer. Math. Soc.*, 25(2):585–613, 2012.

[2] F. A. Behrend. On sets of integers which contain no three terms in arithmetical progression. *Proc. Nat. Acad. Sci. U.S.A.*, 32:331–332, 1946.

[3] Thomas Bloom. A quantitative improvement for Roth's theorem on arithmetic progressions. *Journal of the London Mathematical Society*, 93(3):643–663, 2016.

[4] Thomas Bloom. Additive combinatorics. In *Lecture notes*. University of Cambridge, 2021.

[5] Thomas Bloom and Olof Sisask. The Kelley–Meka bounds for sets free of three-term arithmetic progressions, 2023.

[6] Thomas F. Bloom and Olof Sisask. Breaking the logarithmic barrier in Roth's theorem on arithmetic progressions, 2020.

[7] W. T. Gowers. A new proof of szemerédi's theorem. *Geometric & Functional Analysis GAFA*, 11(3):465–588, 2001.

[8] Zander Kelley and Raghu Meka. Strong bounds for 3-progressions, 2023.

[9] K. F. Roth. On certain sets of integers. *J. London Math. Soc.*, 28:104–109, 1953.

[10] I. D. Shkredov. On sets of large trigonometric sums. *Izv. Ross. Akad. Nauk Ser. Mat.*, 72(1):161–182, 2008.