ON SOME GEOMETRIC THETA SERIES AND THEIR BOUNDARY BEHAVIOUR

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Our goal in this partly expository note is to survey some recent developments around classical work of Kudla and Millson on the modularity of generating series of special cycles in locally symmetric spaces. We briefly review their construction [26, 27, 28] of special geometric theta series as well as the alternative approach developed in [16], using Quillen's notion of a superconnection, and related subsequent developments. We also discuss recent results of Bruinier–Zemel, Engel–Greer–Tayou and the author [10, 11, 18] on boundary contributions to generating series of special cycles on orthogonal Shimura varieties. Finally, we explore the boundary behaviour of Kudla-Millson type theta series on certain simple nilpotent orbits, providing a mild extension of the results of [18] to minimal degenerations of some non-classical variations of Hodge structure.

1. Special cycles and Kudla–Millson forms

1.1. Let us begin by reviewing the definition of special cycles due to Kudla and Millson. These cycles exist on locally symmetric spaces of orthogonal, unitary and quaternionic types, but for concreteness we restrict our discussion to the case of orthogonal groups attached to quadratic spaces defined over the field of rational numbers.

Let us fix a lattice L of finite rank n, endowed with a $\mathbb{Z}$-valued non–degenerate bilinear form

$$Q : L \times L \to \mathbb{Z}.$$

We assume that the lattice is even and indefinite, that is, the real quadratic vector space $V = L \otimes \mathbb{R}$ has signature (p, q) with $p > 0$ and $q > 0$. Let us fix an orientation of V once and for all. Let $\mathrm{Gr}(q, V)$ denote the Grassmannian of oriented q–dimensional subspaces of V and define

$$X = \{W \in \mathrm{Gr}(q, V) \mid Q|_W < 0\}$$

to be the subset of $\mathrm{Gr}(q, V)$ consisting of subspaces where the restriction of Q is negative–definite. Then X has two connected components and the group $G = \mathrm{O}(V)$ acts transitively on X. We fix a basepoint $x_0 \in X$ and write X^+ for the connected component containing x_0 and G^0 for the connected component of G; we obtain

$$X^+ \simeq G^0/K,$$

where $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$ is the stabiliser of x_0. Thus X^+ is a differentiable manifold of dimension pq that can be identified with the symmetric space of G^0.

We now consider quotients of X by arithmetic subgroups $\Gamma \leq G$. More precisely, let

$$L^\vee = \{v \in L \otimes \mathbb{Q} \mid Q(v, l) \in \mathbb{Z} \text{ for every } l \in L\} \supseteq L$$

be the dual lattice of L and define

$$\Gamma_L = \{\gamma \in \mathrm{SO}(L, Q) \mid \gamma|_{L^\vee/L} \equiv \mathrm{id}\}.$$

Let us fix a neat subgroup Γ of Γ_L of finite index; then the quotient

$$X_\Gamma := \Gamma \backslash X$$

is a locally symmetric space. When $p = 2$ or $q = 2$ it is in fact a complex quasi–projective variety, but for general (p, q) it is simply an (orientable) Riemannian manifold with no complex structure.

1.2. Let $v \in L^\vee$ be a vector with $Q(v, v) > 0$. Attached to v is a submanifold $X_v \subset X$ defined to consist of all negative–definite q-dimensional subspaces of V orthogonal to v:

$$X_v = \{W \in X \mid v \perp W\}. \tag{1}$$

The stabiliser G_v of v in G acts transitively on X_v, and so each connected component of X_v can be identified with the symmetric space of G_v. Let us write $\Gamma_v = \Gamma \cap G_v$. Then the inclusion $X_v \subset X$ induces a map

$$\iota_{\Gamma,v} : \Gamma_v \backslash X_v \to X_\Gamma$$

that is known to be proper. Moreover, one can always (cf. [28, Lemma 2.1]) find a subgroup $\Gamma' \leq \Gamma$ of finite index inducing a commutative diagram

$$\begin{tikzcd}
\Gamma'_v \backslash X_v \arrow[r] \arrow[d] & X_{\Gamma'} \arrow[d, "p_{\Gamma,\Gamma'}"] \\
\Gamma_v \backslash X_v \arrow[r] & X_\Gamma
\end{tikzcd}$$

where the vertical arrows are covering maps, the manifold $X_{\Gamma'}$ is naturally oriented and the top horizontal arrow is a proper embedding onto an oriented totally geodesic submanifold of $X_{\Gamma'}$ of codimension q. Integration of compactly supported differential forms on this manifold defines a functional

$$\mathrm{H}_c^{(p-1)q}(X_{\Gamma'}) \to \mathbb{R}, \quad \alpha \mapsto \int_{\Gamma'_v \backslash X_v} \iota^*_{\Gamma',v}\alpha.$$

By Poincaré duality this functional defines cohomology classes

$$Z(v)_{\Gamma'} \in \mathrm{H}^q(X_{\Gamma'})$$

and

$$Z(v)_\Gamma := \frac{1}{|\Gamma : \Gamma'|}(p_{\Gamma,\Gamma'})_*(Z(v)_{\Gamma'}) \in \mathrm{H}^q(X_\Gamma).$$

Note that for a non–zero vector v with $Q(v,v) \leq 0$ we have $X_v = \emptyset$ and so we set

$$Z(v)_\Gamma = 0.$$

The special cycles introduced by Kudla and Millson are linear combinations of the cycles $Z(v)_\Gamma$. Given a rational number $n \neq 0$ and an element $\mu \in L^\vee/L$, one can show that the number of orbits of Γ on the hyperboloid

$$L(n,\mu) = \{v \in \mu + L \mid Q(v,v) = 2n\}$$

is finite.

Definition 1.1. *Given a rational number n and $\mu \in L^\vee/L$, define*

$$Z(n,\mu)_\Gamma = \sum_{\substack{v \in L(n,\mu) \\ \bmod \Gamma}} Z(v)_\Gamma \in \mathrm{H}^q(X_\Gamma).$$

In their series of papers [26, 27, 28], Kudla and Millson show that the cycles $Z(n,\mu)_\Gamma$ can be arranged into generating series that transform like modular forms. More precisely, denote by $\mathbf{W}_\Gamma$ the tautological vector bundle of rank q over X_Γ. Then $\mathbf{W}_\Gamma$ is naturally oriented and we define

$$Z(0)_\Gamma = e(\mathbf{W}_\Gamma) \in \mathrm{H}^q(X_\Gamma)$$

to be its Euler class. Define

$$Z(\tau,\mu)_\Gamma = Z(0)_\Gamma \delta_{0,\mu} + \sum_{n \in Q(\mu,\mu)/2 + \mathbb{Z}} Z(n,\mu)_\Gamma \cdot q^n, \quad q = e^{2\pi i \tau}.$$

Let us write $\mathrm{Mp}_2(\mathbb{Z})$ for the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$ and denote by ρ_L the standard Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L^\vee/L]$ (cf. [8, §4]).

Theorem 1.1. [28] *The generating series*

$$Z(\tau)_\Gamma = \sum_{\mu \in L^\vee/L} Z(\tau,\mu)_\Gamma \otimes e^\mu$$

is a modular form of weight $(p+q)/2$ for $\mathrm{Mp}_2(\mathbb{Z})$ valued in $\mathrm{H}^q(X_\Gamma) \otimes \mathbb{C}[L^\vee/L]$.

1.3. Kudla and Millson prove Theorem 1.1 by explicitly constructing canonical differential forms Poincaré dual to special cycles. The key to their proof is the introduction of certain classes in the continuous cohomology group

$$\mathrm{H}^{nq}_{\mathrm{ct}}(G, \mathcal{S}(V^n)).$$

Here n is a positive integer and $\mathcal{S}(V^n)$ denotes the Schwartz space of V^n, which carries a natural action of G. Let us briefly review their construction as described in [28].

Denote by $\Omega^k(X)$ the smooth differential k-forms on X. By a theorem of van Est, the continuous cohomology groups $\mathrm{H}^k_{\mathrm{ct}}(G, \mathcal{S}(V^n))$ are the cohomology groups of a complex whose k-th term is

$$(\Omega^k(X) \otimes \mathcal{S}(V^n))^G.$$

That is, a class in $\mathrm{H}^k_{\mathrm{ct}}(G, \mathcal{S}(V^n))$ can be represented by a G-invariant closed differential k-form on X valued in $\mathcal{S}(V^n)$.

Let us write $K \subset G^0$ for the maximal compact subgroup of G^0 stabilising x_0 and let $\mathfrak{k}$ and $\mathfrak{g}$ denote the Lie algebras of K and G^0 respectively. We then have the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ where $\mathfrak{p}$ can be identified with the tangent space $T_{x_0}X$. Restriction to x_0 gives an identification

$$\Omega^k(X) \simeq (\wedge^k \mathfrak{p}^* \otimes \mathcal{C}^\infty(G))^K.$$

More generally, for any positive integer n we can identify

$$(\Omega^k(X) \otimes \mathcal{S}(V^n))^G \simeq (\wedge^k \mathfrak{p}^* \otimes \mathcal{S}(V^n))^K. \tag{2}$$

Let us write W for the negative–definite subspace of V of maximal dimension corresponding to our fixed basepoint x_0 and $W^\perp$ for its orthogonal complement. We obtain an orthogonal decomposition $V = W \oplus W^\perp$ and hence a positive–definite majorant Q_0 of Q defined by

$$Q_0(v, v) = Q(w, w) - Q(w^\perp, w^\perp)$$

for $v = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$. Associated with Q_0 is the Gaussian

$$\varphi_0^{(n)}(v_1, \dots, v_n) = e^{-\pi \Sigma_{j=1}^n Q_0(v_j, v_j)} \in \mathcal{S}(V^n)^K.$$

In [26], Kudla and Millson introduce the so–called Howe operator

$$\nabla : \wedge^\bullet \mathfrak{p}^* \otimes \mathcal{S}(V^n) \to \wedge^{\bullet + nq} \mathfrak{p}^* \otimes \mathcal{S}(V^n). \tag{3}$$

Using it they define the Kudla–Millson form

$$\varphi_{\mathrm{KM}}^{(n)} = \nabla \cdot \varphi_0^{(n)} \in (\wedge^{nq} \mathfrak{p}^* \otimes \mathcal{S}(V^n))^K.$$

Let us write G' for the metaplectic double cover of $\mathrm{Sp}_{2n}(\mathbb{R})$ and ω for the Weil representation of $G' \times G$ on $\mathcal{S}(V^n)$. The group G' has a maximal compact subgroup K' that maps to the standard maximal subgroup $\mathrm{U}(n) \subset \mathrm{Sp}_{2n}(\mathbb{R})$. The group K' has a character $\det^{1/2} : K' \to S^1$ whose square factors through the map to $\mathrm{U}(n)$ and defines the determinant character $\det : \mathrm{U}(n) \to S^1$.

Theorem 1.2 ([26]). (1) *The form* $\varphi_{\mathrm{KM}}^{(n)}$ *is closed.*

(2) *We have* $\varphi_{\mathrm{KM}}^{(n)} = \varphi_{\mathrm{KM}}^{(1)} \wedge \ldots \wedge \varphi_{\mathrm{KM}}^{(1)}$.

(3) *The group* K' *acts on* $\varphi_{\mathrm{KM}}^{(n)}$ *via the character* $\det^{(p+q)/2}$.

Part (1) of the theorem implies that φ_{KM} defines a class in $\mathrm{H}^{nq}_{\mathrm{ct}}(G, \mathcal{S}(V^n))$.

1.4. The forms $\varphi_{\mathrm{KM}}^{(n)}$ enjoy several additional useful properties; let us elaborate on two crucial ones. The first such property is that the class in $\mathrm{H}^{nq}_{\mathrm{ct}}(G, \mathcal{S}(V^n))$ defined by $\varphi_{\mathrm{KM}}^{(n)}$ is holomorphic. That is, let $\mathfrak{g}'$ and $\mathfrak{k}'$ be Lie algebras of G' and K' and write $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$ for the Cartan decomposition. The complex structure on the symmetric space for $\mathrm{Sp}_{2n}(\mathbb{R})$ induces a decomposition

$$\mathfrak{p}' \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

with $\mathfrak{p}^-$ corresponding to the anti–holomorphic tangent space of the basepoint. The Lie algebra $\mathfrak{g}^{'}$ acts on $\mathcal{S}(V^n)$ through the Weil representation, and hence also on the cohomology groups $\mathrm{H}^*_{\mathrm{ct}}(G,\mathcal{S}(V^n))$. The classes fixed by $\mathfrak{p}^-$ form the subspace

$$\mathrm{H}^{nq}_{\mathrm{ct}}(G,\mathcal{S}(V^n))^{\mathfrak{p}^-} \subset \mathrm{H}^{nq}_{\mathrm{ct}}(G,\mathcal{S}(V^n))$$

and are said to be holomorphic. Kudla and Millson show that indeed

$$\varphi^{(n)}_{\mathrm{KM}} \in \mathrm{H}^{nq}_{\mathrm{ct}}(G,\mathcal{S}(V^n))^{\mathfrak{p}^-}$$

(cf. [28, §8]).

The second crucial property of $\varphi^{(n)}_{\mathrm{KM}}$ is that it can be used to construct explicit differential forms that are Poincaré dual to special cycles. To do this, let us restrict our attention to $\varphi^{(1)}_{\mathrm{KM}}$, which we regard as a differential form

$$\varphi^{(1)}_{\mathrm{KM}} \in (\Omega^q(X)\otimes\mathcal{S}(V))^G$$

via the isomorphism (2). For a rational number n, a complex number $\tau = x+iy \in \mathbb{H}$ and $\mu \in L^\vee/L$, define

$$\Theta_{\mathrm{KM}}(\tau;n,\mu) = \sum_{v\in L(n,\mu)} \varphi^{(1)}_{\mathrm{KM}}(y^{1/2}v)e^{2\pi i n x}.$$

The sum converges rapidly to a smooth differential form on X. Moreover, the G-invariance of $\varphi^{(1)}_{\mathrm{KM}}$ implies that $\Theta_{\mathrm{KM}}(\tau;n,\mu)$ is invariant under Γ, that is

$$\Theta_{\mathrm{KM}}(\tau;n,\mu) \in \Omega^q(X)^\Gamma \simeq \Omega^q(X_\Gamma),$$

and so $\Theta_{\mathrm{KM}}(\tau;n,\mu)$ can be regarded as a closed differential q-form on X_Γ. The following property gives the crucial link between special cycles and Kudla–Millson forms.

Theorem 1.3. *Let α be a rapidly decreasing form on $\Gamma_v\backslash X$ of degree $(p-1)q$. For any positive rational number n we have*

$$\sum_{\substack{v\in L(n,\mu)\\ \mathrm{mod}\ \Gamma}} \int_{\Gamma_v\backslash X_v} \alpha\cdot q^n = \int_{\Gamma_v\backslash X} \alpha\wedge\Theta_{\mathrm{KM}}(\tau;n,\mu).$$

Writing $[\Theta_{\mathrm{KM}}(\tau,n,\mu)]$ for the cohomology class defined by $\Theta_{\mathrm{KM}}(\tau;n,\mu)$, this implies

$$[\Theta_{\mathrm{KM}}(\tau;n,\mu)] = Z(n,\mu)_\Gamma\cdot q^n.$$

The modularity of the generating series stated in Theorem 1.1 follows easily from this: for $\tau\in\mathbb{H}$ and $\mu\in L^\vee/L$, consider the theta series

$$\Theta_{\mathrm{KM}}(\tau,\mu) = \sum_{v\in\mu+L} \varphi^{(1)}_{\mathrm{KM}}(y^{1/2}v)e^{\pi i Q(v,v)x}.$$

The sum converges rapidly to a smooth closed differential form that is invariant under Γ, hence we have

$$\Theta_{\mathrm{KM}}(\tau,\mu) \in \Omega^q(X)^\Gamma \simeq \Omega^q(X_\Gamma).$$

As in the proof of modularity of classical theta series, an application of Poisson summation combined with Theorem 1.2 shows that $\sum_{\mu \in L^\vee/L} \Theta_{\mathrm{KM}}(\tau,\mu) \otimes e^\mu$ transforms under $\mathrm{Mp}_2(\mathbb{Z})$ like a (non-holomorphic) modular form of weight $(p+q)/2$ valued in ρ_L. The modularity statement in Theorem 1.1 follows from these facts by taking cohomology classes in the identity

$$\Theta_{\mathrm{KM}}(\tau,\mu) = \sum_{n \in \mathbb{Q}} \Theta_{\mathrm{KM}}(\tau; n, \mu).$$

2. The Mathai–Quillen formalism

The construction of the forms φ_{KM} outlined in the previous section relies on the introduction of the Howe operator (3). Moreover, the proofs of the main properties of these forms depend on computations using the explicit formulas for ∇. We now briefly review the author's alternative construction of the Kudla–Millson forms. In this approach the Howe operator does not appear, and the forms arise from a version of Chern–Weil theory applied to certain natural complexes defined over the symmetric space X. Moreover, this point of view allows to define some secondary invariants and to consider similar constructions for other symmetric spaces, as we will briefly see below.

Let us briefly recall the classical construction of characteristic classes of vector bundles due to Chern and Weil.

2.1. Let M be a differentiable manifold and let E be a complex vector bundle over M. Associated with E are certain characteristic classes, e.g. the Chern classes $c_k(E) \in \mathrm{H}^{2k}(M,\mathbb{Z}), \quad k = 0,1,2,\ldots$, and the Chern character

$$\mathrm{ch}(E) \in \mathrm{H}^{2*}(M,\mathbb{Q}).$$

These classes are functorial under pullbacks: if $f : M' \to M$ is a differentiable map, then

$$f^*(c_k(E)) = c_k(f^*E),$$

and a similar identity holds for $\mathrm{ch}(E)$.

The main content of Chern-Weil theory is that the characteristic classes above can be represented by characteristic differential forms once we fix a connection on E. That is, for each connection ∇ on E there exist closed differential forms

$$c_k(E,\nabla), \quad \mathrm{ch}(E,\nabla) \in \Omega^{2*}(M)$$

that represent the classes $c_k(E)$ and $\mathrm{ch}(E)$ in cohomology and are moreover functorial: for any $f : M' \to M$ we have

$$f^*(c_k(E,\nabla)) = c_k(f^*E, f^*\nabla).$$

The construction of the Chern–Weil forms proceeds as follows. Let us write $\Gamma(E)$ for the space of smooth sections of E. The connection is an additive map

$$\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for any $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(E)$. It can be extended uniquely to a map

$$\nabla : \Omega^k(X) \otimes \Gamma(E) \to \Omega^{k+1}(X) \otimes \Gamma(E), \quad k \geq 0,$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for any $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$. A simple computation shows that the composite

$$\nabla^2 := \nabla \circ \nabla : \Omega^k(M) \otimes \Gamma(E) \to \Omega^{k+2} \otimes \Gamma(E)$$

is $\mathcal{C}^\infty(M)$-linear, so that in fact ∇^2 lies in the subspace

$$\Omega^2(M) \otimes \mathrm{End}(E) \subset \mathrm{End}(\Omega^*(M) \otimes \Gamma(E)).$$

The quantity ∇^2 is called the curvature of the connection ∇. The Chern form of ∇ is defined to be

$$\mathrm{ch}(E, \nabla) = \mathrm{tr}(e^{\frac{i}{2\pi}\nabla^2}) = \sum_{k=0}^{\infty} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}\left(\frac{\nabla^{2k}}{k!}\right).$$

Note that the sum is finite since $\nabla^{2k} = 0$ as soon as $2k > \dim(M)$. Hence $\mathrm{ch}(E, \nabla)$ belongs to $\Omega^{2*}(M) = \oplus_{k \geq 0} \Omega^{2k}(M)$. One can show that $\mathrm{ch}(E, \nabla)$ is closed and that its cohomology class represents the Chern caracter $\mathrm{ch}(E)$ of E.

2.2. In [30], Quillen extends the above construction by introducing the notion of a superconnection. Let us explain this in a special case that relates to Kudla–Millson forms.

Consider again a complex vector bundle E of complex rank N over M endowed with a connection ∇. Suppose also that E carries a hermitian metric compatible with ∇: writing $\langle e, e' \rangle$ for the hermitian inner product of two local sections e, e' of E, we require that

$$d\langle e, e' \rangle = \langle \nabla e, e' \rangle + \langle e, \nabla e' \rangle.$$

We can then form the exterior algebra

$$\wedge E = \oplus_{k=0}^N \wedge^k E,$$

with its natural $\mathbb{Z}/2\mathbb{Z}$–grading given by $(\wedge E)^0 = \oplus_{k \text{ even}} \wedge^k E$ and $(\wedge E)^1 = \oplus_{k \text{ odd}} \wedge^k E$. The metric on E induces a metric on each summand $\wedge^k E$ and hence a metric on $\wedge E$ where different summands are orthogonal to each other; similarly one obtains a compatible connection $\nabla_{\wedge E}$ on $\wedge E$ as a sum of the connections on each summand $\wedge^k E$ induced by ∇. The Chern–Weil character

$$\mathrm{ch}(\wedge E, \nabla_{\wedge E}) = \mathrm{tr}(e^{\frac{i}{2\pi}\nabla^2_{\wedge E}})$$

represents the Chern class of $\wedge E$; in particular, its component

$$\mathrm{ch}_{2N}(\wedge E, \nabla) \in \Omega^{2N}(M)$$

in degree $2N$ represents the top Chern class $c_{2N}(E)$ (this follows from the identity

$$\mathrm{ch}(\wedge E) = c_{2N}(E)\mathrm{Td}(E),$$

where $\mathrm{Td} = 1 + \cdots \in \mathrm{H}^{2*}(M)$ denotes the Todd class).

Suppose now that we are given a section s of the dual bundle $E^\vee$. The section can be used to deform $\nabla_{\wedge E}$ into a superconnection ∇_s, as follows: contraction with s defines maps

$$s : \wedge^k E \to \wedge^{k-1} E, \quad k \geq 1$$

which induce an endomorphism of $\wedge E$ that we still denote by s. Let us write $s^* \in \mathrm{End}(\wedge E)$ for the adjoint of this endomorphism with respect to the metric on E. Note that $\nabla_{\wedge E}$, s and s^* can all be regarded as linear endomorphisms of the algebra

$$\Omega^*(M)\hat{\otimes}\Gamma(\wedge E).$$

Namely, $\nabla_{\wedge E}$ preserves each subspace $\Omega^*(X) \otimes \Gamma(\wedge^k E)$ $(k \geq 0)$ and the endomorphisms s and s^* preserve each subspace $\Omega^n(M) \otimes \Gamma(\wedge E)$ $(n \geq 0)$. The symbol $\hat{\otimes}$ denotes the usual tensor product, but where multiplication is subject to the Koszul rule of signs, so that

$$(\omega_1 \otimes s_1) \cdot (\omega_2 \otimes s_2) = (-1)^{\deg(\omega_2)\cdot\deg(s_1)}\omega_1\omega_2 \otimes s_1 s_2.$$

Let us define

$$\nabla_s = \nabla_{\wedge E} + i(s + s^*).$$

This is an odd endomorphism with respect to the natural $\mathbb{Z}/2\mathbb{Z}$–grading on $\Omega^*(M)\hat{\otimes}\Gamma(\wedge E)$. Moreover, it satisfies

$$\nabla_s(\omega \otimes s') = d\omega \otimes s' + (-1)^{\deg(\omega)}\omega \otimes \nabla_s(s').$$

Such an endomorphism is called a superconnection. As for a classical connection, one can show that its curvature ∇_s^2 is $\Omega^*(M)$–linear and so can be identified with an even element

$$\nabla_s^2 \in \Omega^*(M)\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathrm{End}(\Gamma(\wedge E)).$$

The supertrace on the bundle $\mathrm{End}(\wedge E)$ induces a functional

$$\mathrm{str} : \Omega^*(M)\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathrm{End}(\Gamma(\wedge E)) \to \Omega^*(M)$$

that can be used to define characteristic forms.

Definition 2.1. *The Chern character form of* $(\wedge E, \nabla_s)$ *is*

$$\mathrm{ch}(\wedge E, \nabla_s) = \mathrm{str}(e^{\nabla_s^2}) \in \Omega^{2*}(M).$$

The main properties of the Chern forms extend to $\mathrm{ch}(\wedge E, \nabla_s)$:

(i) The form $\mathrm{ch}(\wedge E, \nabla_s)$ is closed.

(ii) For any map $f : M' \to M$, we have

$$f^*(\mathrm{ch}(\wedge E, \nabla_s)) = \mathrm{ch}(\wedge f^* E, f^*\nabla_s).$$

(iii) Let us write $[\mathrm{ch}_{2k}(E, \nabla_s)]$ for the class in $\mathrm{H}^{2k}(M)$ defined by the component of $\mathrm{ch}(\wedge E, \nabla_s)$ of degree $2k$. Then

$$\sum_{k \geq 0} \left(\frac{i}{2\pi}\right)^k [\mathrm{ch}_{2k}(\wedge E, \nabla_s)] = \mathrm{ch}(\wedge E) \in \mathrm{H}^{2*}(M).$$

(iv) Suppose that E is the orthogonal sum of hermitian bundles E' and E'' and ∇ is the sum of metric connections ∇' on E' and ∇'' on E''. Let s' and s'' be the projections of s to the duals of E' and E'' respectively. Then

$$\mathrm{ch}(\wedge E, \nabla_s) = \mathrm{ch}(\wedge E', \nabla_{s'}) \wedge \mathrm{ch}(\wedge E'', \nabla_{s''}). \tag{4}$$

Of course, if s is the zero section, then the superconnection ∇_s becomes the connection $\nabla_{\wedge E}$ and so we recover the Chern-Weil form $\mathrm{ch}(\wedge E, \nabla)$. However, for a general section s, the forms $\mathrm{ch}(\wedge E, \nabla_s)$ and $\mathrm{ch}(\wedge E, \nabla)$ will look quite different. To see this, let us write

$$\begin{aligned}\nabla_s^2 &= (\nabla_{\wedge E} + i(s + s^*))^2 \\ &= \nabla_{\wedge E}^2 + i[\nabla_{\wedge E}, s + s^*] - |s|^2.\end{aligned}$$

The last summand corresponds to the operator of multiplication by $-|s|^2$ on $\Omega^*(M) \hat{\otimes}_{\mathcal{C}^\infty(M)} \Gamma(\wedge E)$, which commutes with the other two summands. Since $\exp(A + B) = \exp(A)\exp(B)$ for commuting operators A and B, we obtain the alternative expression

$$\mathrm{ch}(E, \nabla_s) = e^{-|s|^2} \cdot \mathrm{str}(e^{\nabla_{\wedge E}^2 + i[\nabla_{\wedge E}, s+s^*]}). \tag{5}$$

Note that the expression inside the supertrace is now a finite sum since the exponent has strictly positive differential-form degree. We conclude that for non–zero s the form $\mathrm{ch}(E, \nabla_s)$ is Gaussian–shaped and peaks at the locus $Z(s)$ where s vanishes. In fact, for a regular section s it can be shown that as t grows, the form $\mathrm{ch}_{2N}(E, \nabla_{ts})$ converges to the current of integration on the cycle $Z(s)$, i.e.

$$\lim_{t \to \infty} \mathrm{ch}_{2N}(E, \nabla_{ts}) \to \delta_{Z(s)}$$

(cf. [6, Thm. 3.2]).

A formula due to Mathai and Quillen [29] gives a more explicit expression for $\mathrm{ch}(E, \nabla_s)$. We will use this formula below in a special case: suppose that E is a holomorphic line bundle with hermitian metric. Let ∇ be the corresponding Chern connection and Ω be its curvature. Then the degree two component of $\mathrm{ch}(\wedge E, \nabla_s)$ is given by

$$\mathrm{ch}_2(E, \nabla_s) = e^{-|s|^2}(\Omega - i\frac{\partial |s|^2 \wedge \overline{\partial} |s|^2}{|s|^2}). \tag{6}$$

2.3. We now return to the setting of symmetric spaces. Let us first consider the hermitian symmetric space X associated with an orthogonal group $\mathrm{O}(V)$ for a real quadratic vector space (V,Q) of signature $(p,2)$ for some positive integer p.

In this case X is a complex manifold, more precisely a quadric in $\mathbb{P}(V_{\mathbb{C}})$, and the vector bundle $\mathcal{V}$ over X with constant fiber $V_{\mathbb{C}}$ admits a filtration

$$\mathcal{V}=\mathcal{F}^0\supset\mathcal{F}^1\supset\mathcal{F}^2$$

by holomorphic vector bundles. Here $\mathcal{F}^2$ is the pullback of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V_{\mathbb{C}})$, hence it is a line bundle that we denote by $\mathcal{L}$; the quadratic form Q vanishes on $\mathcal{L}$, and we define $\mathcal{F}^1\supset\mathcal{L}$ to be the annihilator of $\mathcal{L}$ in $\mathcal{V}$ under Q. The form Q induces a metric on $\mathcal{L}^\vee$ and an isomorphism

$$\mathcal{F}^0/\mathcal{F}^1\simeq\mathcal{L}^\vee.$$

In particular, to a vector $v\in V^n$ we can attach a section $s_v\in\Gamma((\mathcal{L}^{\oplus n})^\vee)$. Thus we have all the necessary ingredients to define a superconnection. Let us fix $n\in\mathbb{N}$ and write ∇ for the Chern connection on $\mathcal{L}^n$. For each $v\in V^n$ we define a superconnection ∇_v on $\wedge(\mathcal{L}^n)$ by

$$\nabla_v=\nabla_{\wedge\mathcal{L}^n}+i\sqrt{2\pi}(s_v+s_v^*).$$

The following theorem is proved in [16].

Theorem 2.1. *For $v\in V^n$ we have*

$$\varphi_{\mathrm{KM}}^{(n)}(v)=\left(\frac{i}{2\pi}\right)^n e^{-\pi Q(v,v)}\mathrm{ch}_{2n}(\wedge\mathcal{L}^n,\nabla_v).$$

The proof follows from a computation of both sides, using the explicit formulas for the Howe operator in [26][1]. Note that properties (1) and (2) in Theorem 1.2 follow immediately from this result and the general properties of Chern forms, and property (3) can be checked by a straightforward computation (cf. [16, Lemma 4.3]).

2.4. Consider now a different setting: let X be a differentiable manifold and $\pi:\mathbf{W}\to X$ be a smooth real vector bundle. Assume that $\mathbf{W}$ is oriented and is equipped with a euclidean metric and a compatible connection. For such a bundle, Mathai and Quillen [29] construct a canonical Thom form

$$U_{\mathrm{MQ}}(\mathbf{W})\in\Omega^q(\mathbf{W}),\qquad q=\mathrm{rk}\mathbf{W},$$

characterized by the following properties:

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[1] While the approach developed in [16] (and further in [9]) recovers all the fundamental properties of the Kudla–Millson forms φ_{KM}, the construction does not involve the Howe operator. It would be interesting to understand the relation between Quillen's formalism and this operator.

(1) For any map of vector bundles

$$\begin{tikzcd}
\mathbf{W}' \arrow[r, "f"] \arrow[d] & \mathbf{W} \arrow[d] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

preserving the metric, orientation and connection, we have

$$f^*(U_{\mathrm{MQ}}(\mathbf{W})) = U_{\mathrm{MQ}}(f^*\mathbf{W}).$$

(2) The form $U_{\mathrm{MQ}}(\mathbf{W})$ is closed and rapidly decreasing along the fibers of $\mathbf{W}$.

(3) For any $x \in X$, we have

$$\int_{\mathbf{W}_x} U_{\mathrm{MQ}}(\mathbf{W}) = 1.$$

(4) For $\mathbf{W} = \mathbb{R}^q$ with standard euclidean metric and orientation we have

$$U_{\mathrm{MQ}}(\mathbf{W}) = \pi^{-q/2} e^{-x_1^2-\cdots-x_q^2} dx_1 \wedge \cdots \wedge dx_q.$$

Let us now take X to be the symmetric space associated with an orthogonal group $\mathrm{O}(V)$ of a real quadratic vector space of general signature (p,q). There is then a tautological real vector bundle

$$\mathbf{W} \to X$$

of rank q whose fiber over the point $x = [W] \in X$ is the vector space W itself. The vector bundle $\mathbf{W}$ carries a natural orientation, euclidean metric and compatible connection. As in the case of signature $(p,2)$, the Mathai–Quillen construction allows one to recover the Kudla–Millson form. More precisely, note that a vector $v \in V$ defines a section $s_v \in \Gamma(\mathbf{W})$ whose zero locus is the special cycle X_v in (1). The following generalisation of Theorem 2.1 was proved by Romain Branchereau in his Ph.D. thesis.

Theorem 2.2. [9] *For any $v \in V$ we have*

$$\varphi_{\mathrm{KM}}^{(1)}(v) = 2^{-q/2} e^{-\pi Q(v,v)} s_v^*(U_{\mathrm{MQ}}(\mathbf{W})).$$

The Poincaré duality property in Proposition 1.3 follows from this description and the defining properties of the Thom form $U_{\mathrm{MQ}}(\mathbf{W})$.

2.5. The Mathai–Quillen construction also allows to define other characteristic forms arising as transgressions of the Chern character and Thom forms. One example is the $\partial\overline{\partial}$–transgression introduced by Bismut–Gillet–Soulé [5]. In our setting it leads to the following definition: assume that X is associated with $\mathrm{SO}(p,2)$ and let $N \in \mathrm{End}(\wedge(\mathcal{L}^{\oplus n}))$ be the operator acting on $\wedge^k(\mathcal{L}^{\oplus n})$ by multiplication by $-k$. For $v \in V^n$, define

$$\nu^{(n)}(v) = e^{-\pi Q(v,v)} \mathrm{str}(N e^{\nabla_v^2})_{2n-2}.$$

The results of [5] imply that the forms $\nu^{(n)}(v) \in \Omega^{2n-2}(X)$ and $\varphi_{\mathrm{KM}}^{(n)}(v) \in \Omega^{2n}(X)$ are related by the following differential equation. Let us assume for simplicity that $n = 1$ and denote by ω the Weil representation of $\mathfrak{g}' = \mathfrak{sl}_{2,\mathbb{R}}$ on $\mathcal{S}(V)$. Let $X_- \in \mathfrak{p}^-$ correspond to the differentation operator $-2i\partial_{\overline{\tau}}|_{\tau=i}$. Then

$$\omega(X_-)\varphi_{\mathrm{KM}}^{(1)}(v) = \partial\overline{\partial}\nu^{(1)}(v),$$

which recovers an identity first proved by Kudla [24, 25]. This equation implies that the class defined by $\varphi_{\mathrm{KM}}^{(n)}(v)$ in $\mathrm{H}^{nq}_{\mathrm{ct}}(G, \mathcal{S}(V^n))$ is holomorphic. The forms $\nu^{(n)}$ can be used to construct canonical Green currents for higher–dimensional special cycles [17] with properties predicted by Kudla's conjectures.

Coming back to the setting of real euclidean vector bundles $\pi : \mathbf{W} \to X$ equipped with a euclidean metric and compatible connection, Mathai and Quillen [29] also introduce a transgression $\tau_{\mathrm{MQ}}(\mathbf{W})$ of the Thom form $U_{\mathrm{MQ}}(\mathbf{W})$. Let $\iota : X \to \mathbf{W}$ be the zero section, so that $\mathbf{R}_{>0}$ acts on $\mathbf{W}\backslash\iota(X)$ by rescaling each fiber. Then $\tau_{\mathrm{MQ}}(\mathbf{W})$ is a functorial $(q-1)$-form defined on $\mathbf{W}\backslash\iota(X)$ that is basic with respect to this action and satisfies

$$d\tau_{\mathrm{MQ}}(\mathbf{W}) = \pi^*\iota^*U_{\mathrm{MQ}}(\mathbf{W}).$$

This form has many applications to the construction of secondary invariants of K-theoretic nature. For example, it was used by Bismut–Cheeger [7] in their classical work proving Hirzebruch's conjecture on the signature defect of Hilbert modular varieties. In a suitable equivariant setting it is the main character in work of Bergeron, Charollois and the author [3, 4] giving a topological approach to several constructions in the literature known as Eisenstein cocycles.

3. BOUNDARY CONTRIBUTIONS TO GENERATING SERIES OF SPECIAL CYCLES

3.1. The modularity of the generating series in Theorem 1.1 underlies some striking applications to enumerative geometry. Following [23], let us consider an example related to counting curves on families of polarized K3 surfaces.

Let $X \to S$ be a family of K3 surfaces parametrised by a smooth quasi–projective variety S. The integral cohomology $\mathrm{H}^2(X_s, \mathbb{Z})$ of the fibers, endowed with the symmetric bilinear form given by the cup product

$$\mathrm{H}^2(X_s, \mathbb{Z}) \times \mathrm{H}^2(X_s, \mathbb{Z}) \to \mathrm{H}^4(X_s, \mathbb{Z}) \simeq \mathbb{Z},$$

is an even unimodular lattice of signature $(3, 19)$. Let us fix a polarisation of this family; in particular, this gives a primitive class $\lambda_s \in \mathrm{Pic}(X_s) \subset \mathrm{H}^2(X_s, \mathbb{Z})$ on each fiber, of self–intersection $2d \in 2\mathbb{N}$, and a sublattice of primitive classes

$$\mathrm{H}^2_{\mathrm{prim}}(X_s) = \{\alpha \in \mathrm{H}^2(X_s, \mathbb{Z}) \mid \alpha \cdot \lambda_s = 0\} \subset \mathrm{H}^2(X_s, \mathbb{Z})$$

of signature $(2, 19)$. Let us assume for simplicity that the generic Picard rank of the family is one; in other words, away from a countable collection of subvarieties of S, the class in $\mathrm{H}^2(X_s,\mathbb{Z})$ defined by any algebraic curve in X_s is a multiple of λ_s. Then, for each positive integer n, the locus

$$Z(n)=\{s\in S\mid \exists\alpha\in \mathrm{Pic}(X_s)\cap \mathrm{H}^2_{\mathrm{prim}}(X_s),\ \alpha\cdot\alpha=-2n\}$$

is a proper closed analytic subset of S. Suppose now we are given a complete curve $C\subset S$. Then C intersects each $Z(n)$ at finitely many points, to which one can assign intersection multiplicities to define a non-negative integer $C\cdot Z(n)$. Let $\mathcal{L}$ be the line bundle over S with fiber $\mathcal{L}_s=\mathrm{H}^0(X_s,\Omega^2)$.

Theorem 3.1. *The series*

$$-\deg_C(\mathcal{L})+\sum_{n\geq 1}(C\cdot Z(n))\cdot q^n$$

is a modular form of weight 21/2.

This is a simple consequence of Theorem 1.1: the variation of Hodge structure $\mathbb{V}$ underlying $\mathrm{H}^2_{\mathrm{prim}}(X_s)$ induces a period mapping $\Phi_{\mathbb{V}}:S\to\Gamma\backslash\mathbb{D}$, where $\mathbb{D}$ is the hermitian symmetric domain associated with $\mathrm{SO}(19,2)$. Pulling back the generating series in Theorem 1.1 by $\Phi_{\mathbb{V}}$ and evaluating on the fundamental class $[C]\in\mathrm{H}_2(S)$ one obtains Theorem 3.1.

3.2. In order to generalize Theorem 3.1 to non-complete curves in S, one would like to extend Kudla and Millson's modularity Theorem 1.1 to cycles in an appropriate compactification of $\Gamma\backslash X$. There has been much work in this direction, e.g. the original results of Hirzebruch and Zagier [21] and work of Funke and Millson on Borel–Serre compactifications [14, 15].

Recently Bruinier and Zemel [10] have proved a result regarding modularity of generating series of special divisors on general toroidal compactifications of orthogonal Shimura varieties. They fix a quadratic vector space (V,Q) over $\mathbb{Q}$ of signature $(n,2)$ and a subgroup $\Gamma=\Gamma_L$, assumed to be neat, associated with an even lattice $L\subset V$; the quotient $X_\Gamma=\Gamma\backslash X$ is then a complex quasi–projective variety. Fix also a smooth toroidal compactification

$$X_\Gamma\subset X_\Gamma^{\mathrm{tor}}$$

with boundary a normal crossing divisor. By appropriately adding boundary components to the closure of the divisors $Z(m,\mu)$ in X_Γ^{tor}, they define divisor classes

$$Z^{\mathrm{tor}}(m,\mu)\in\mathrm{CH}^1(X_\Gamma^{\mathrm{tor}})\otimes\mathbb{R}$$

indexed by $\mu\in L^\vee/L$ and $m\in\frac{Q(\mu,\mu)}{2}+\mathbb{Z}$.

Theorem 3.2. [10] *Assume that n is greater than the Witt rank of L. Then the generating series*

$$-c_1(\mathcal{L})\delta_{0,\mu}e^\mu+\sum_\mu\sum_{m>0}Z^{\mathrm{tor}}(m,\mu)\cdot q^m\otimes e^\mu$$

is a modular form of weight $1+n/2$ *for* $\mathrm{Mp}_2(\mathbb{Z})$ *and representation* ρ_L*, valued in the group* $\mathrm{CH}^1(X_\Gamma^{\mathrm{tor}})\otimes\mathbb{R}$*.*

The proof in [10] proceeds by analyzing the boundary behaviour of certain automorphic Green functions associated with the divisors $Z(m,\mu)$. Very recently a similar result has been proved by Engel–Greer–Tayou [11] which improves on the above modularity result by removing the hypothesis on the Witt rank. Another approach has appeared in [18]. It is based on studying directly the singularities of the theta series Θ_{KM} using Schmid's analytic results on degenerations of Hodge structure rather than the theory of toroidal compactifications. This method should extend to variations with $h^{2,0} > 1$, as we explain next.

4. Minimal degenerations of certain VHS of weight two

In the preceding section we have restricted our attention to variations of Hodge structure (VHS) with $h^{2,0} = 1$. It would be desirable to obtain analogous results regarding Hodge loci in VHS with $h^{2,0} > 1$, coming for example from the primitive cohomology of polarized families of surfaces $X \to S$ parametrized by a complex smooth quasi–projective variety S. Natural examples arise from considering families parametrizing hyperplane sections of a Calabi–Yau threefold; recent work [1, 2] suggests that the naive generating series of Noether-Lefschetz classes is not modular, but that modularity can be restored by adding terms related to degenerate fibers. The resulting series is no longer holomorphic and is expected to be a mixed mock modular form (see [12]).

It is natural to tackle this problem using the work of Kudla and Millson. Let $\mathbb{V} \to S$ be a $\mathbb{Z}$-PVHS of weight two. Let $q = h^{2,0}$ and assume that

$$q \leq \dim(S).$$

The variation $\mathbb{V}$ defines a period map

$$\Phi_{\mathbb{V}} : S \to \Gamma\backslash\mathbb{D}$$

to an arithmetic quotient of a period domain classifying Hodge structures of type $(q, h^{1,1}, q)$. One can define a Kudla–Millson theta series

$$\Theta_{\mathrm{KM}}(\tau) \in \Omega^{q,q}(\Gamma\backslash\mathbb{D}) \otimes \mathbb{C}[L^\vee/L].$$

We set

$$\Theta_{\mathbb{V}}(\tau) = \Phi_{\mathbb{V}}^*(\Theta_{\mathrm{KM}}(\tau)) \in \Omega^{q,q}(S) \in \mathbb{C}[L^\vee/L].$$

More explicitly, we have

$$\Theta_{\mathbb{V}}(\tau) = \sum_{\mu \in L^\vee/L} \left(\sum_{v \in \mu + \mathcal{V}_{\mathbb{Z}}} \varphi_{\mathbb{V}}(y^{1/2}v) e^{\pi i x Q(v,v)} \right) e^{\mu}, \tag{7}$$

where

$$\varphi_{\mathbb{V}}(v) = e^{-\pi Q(v,v)} \mathrm{ch}_{2q}(\wedge \mathcal{F}^2, \nabla_v)$$

and $\mathcal{F}^2$ denotes the Hodge bundle. One hopes that integrating $\Theta_{\mathbb{V}}(\tau)$ over S gives rise to modular generating series related related to the Hodge loci of $\mathbb{V}$. Since $\Theta_{\mathbb{V}}(\tau)$ has singularities on the boundary of S, it is first necessary to show that these singularities are mild enough for $\Theta_{\mathbb{V}}$ to be integrable over S.

When S is a punctured Riemann surface and $q = 1$ this program was carried out in [18], which showed, extending previous work of Funke [13] that successfully handled the case when S is a modular curve, that $\Theta_{\mathbb{V}}(\tau)$ is integrable under very mild hypotheses on $\mathbb{V}$ and found that the Noether-Lefschetz degrees are the coefficients of a mixed mock modular form, with other contributions explicitly determined by the mixed Hodge structures at the punctures of S.

For arbitrary weight two variations $\mathbb{V}$ with $h^{2,0} > 1$, these results suggest that $\Theta_{\mathbb{V}}(\tau)$ may be integrable under very mild hypotheses on $\mathbb{V}$; or more precisely, that given a compactification $\overline{S}$ of S by a normal crossings divisor, $\Theta_{\mathbb{V}}(\tau)$ may extend to a current on $\overline{S}$. Below we offer some evidence for this hope by showing that a mild extension of some arguments in [18] allows to prove that $\Theta_{\mathbb{V}}(\tau)$ is indeed integrable for some of the simplest degenerations of variations with $h^{2,0} > 1$. We note that the hypotheses below are certainly too strong to be useful in variations arising in practice. We intend to return to the general case in later work.

4.1. We write $\Delta \subset \mathbb{C}$ for the open unit disk and $\Delta^* = \Delta \backslash \{0\}$. We fix a positive integer n and consider families of one–variable degenerations

$$\begin{tikzcd}\mathbb{V} \arrow[d] \\ \Delta^* \times \Delta^{n-1}\end{tikzcd}$$

with unipotent monodromy. We write N for the monodromy logarithm and $W_\bullet$ for the corresponding weight filtration of the local system $\mathcal{V}_{\mathbb{Q}}$ (we refer the reader to [20] for a summary of the results we will need regarding degeneration of Hodge structures). We use t and $w_1, \ldots, w_{n-1}$ for the coordinates on Δ^* and Δ^{n-1} respectively and let $t = e^{2\pi i z}$ with $z \in \mathbb{H}$. We write $\tilde{\mathcal{F}}^k$ for the canonical extension to Δ^n of the vector bundle $e^{-zN}\mathcal{F}^k$ and

$$\mathcal{F}^k_{\mathrm{lim}} = \tilde{\mathcal{F}}^k|_{t=0},$$

which gives a filtration of $\tilde{\mathcal{V}}|_{t=0}$ by holomorphic vector bundles. Then, for each $w \in \Delta^{n-1}$, the triple $(W_\bullet, Q, (\mathcal{F}^\bullet_{\mathrm{lim}})_w)$ defines a limiting mixed Hodge structure polarized by Q. We assume that

(i) $\mathbb{V} = \mathbb{V}^{\mathrm{nilp}}$ is the corresponding nilpotent orbit, that is,

$$\mathcal{F}^k = e^{zN}\mathcal{F}^k_{\mathrm{lim}}, \qquad k = 0, 1, 2.$$

(ii) The weight filtration splits over $\mathbb{R}$ so as to give rise to a direct sum of $\mathbb{R}$-VHS on Δ^{n-1}.

(iii) The degeneration corresponds to a minimal orbit in the period domain (cf. [19, Thm. 1.7]).

More precisely, (ii) means that we assume the existence of a splitting $W_n \otimes \mathbb{R} = \oplus_{k\leq n} V_k$ such that for each k the decreasing filtration

$$\mathcal{F}^\bullet_{k,\mathrm{lim}} = \mathcal{F}^\bullet_{\mathrm{lim}} \cap (V_k \otimes \mathcal{O}_{\Delta^{n-1}})$$

of $V_k \otimes \mathcal{O}_{\Delta^{n-1}}$ defines a VHS of weight k on Δ^{n-1}. We write $h^{p,q}_{\mathrm{lim}}$ for the Hodge numbers of these VHS. Then (iii) leaves two options::

- Type II: we have $N^2 = 0$,

  $$\begin{aligned} h^{1,1}_{\mathrm{lim}} &= \mathrm{rk}(\mathcal{V}_{\mathbb{Z}}) - 2 - 2q \\ h^{0,2}_{\mathrm{lim}} = h^{2,0}_{\mathrm{lim}} &= q - 1 \\ h^{1,0}_{\mathrm{lim}} = h^{0,1}_{\mathrm{lim}} = h^{1,2}_{\mathrm{lim}} = h^{2,1}_{\mathrm{lim}} &= 1 \end{aligned}$$

  and all other Hodge numbers are zero.
- Type III: we have $N^2 \neq 0$,

  $$\begin{aligned} h^{1,1}_{\mathrm{lim}} &= \mathrm{rk}(\mathcal{V}_{\mathbb{Z}}) - 2q \\ h^{0,2}_{\mathrm{lim}} = h^{2,0}_{\mathrm{lim}} &= q - 1 \\ h^{2,2}_{\mathrm{lim}} = h^{0,0}_{\mathrm{lim}} &= 1 \end{aligned}$$

  and all other Hodge numbers are zero.

(The terminology agrees with the standard one describing the semistable reduction of K3 surfaces when $q = 1$).

To estimate the size of differential forms we use the product of the standard metric on Δ^{n-1} and the Poincaré metric on Δ^*, so that for a volume form

$$\omega = f dw_1 d\overline{w}_1 \cdots dw_{n-1} d\overline{w}_{n-1} \frac{dt d\overline{t}}{|t|^2 \log|t|^2}, \qquad f \in \mathcal{C}^\infty(\Delta^* \times \Delta^{n-1})$$

we set

$$|\omega|_{t,w} = |f(t,w)|.$$

We say that α is nearly bounded if $|\alpha|_{t,w}$ is bounded in t locally uniformly in w; since the Poincaré metric has finite area, a nearly bounded form is locally integrable.

We will show that $\Theta_{\mathbb{V}}(\tau)_\mu \wedge \alpha$ is nearly bounded for any $\alpha \in \Omega^{n-q,n-q}(\Delta^n)$. The proof will use Schmid's analytic interpretation of the weight filtration: for a section v of $\mathcal{V}_{\mathbb{Q}}$, we have $v \in W_k - W_{k-1}$ if and only if its Hodge metric satisfies

$$\|v\|^2_{t.w} \sim (-\log|t|)^{k-2} \tag{8}$$

for t in any angular sector, locally uniformly in w.

4.2. As a first remark (see [18] for details), let us split the inner sum $\Theta_{\mathbb{V}}(\tau)_\mu$ over $\mu + \mathcal{V}_{\mathbb{Z}}$ in (7) into summands $\Theta'_{\mathbb{V}}(\tau)_\mu$ and $\Theta''_{\mathbb{V}}(\tau)_\mu$, where

$$\Theta'_{\mathbb{V}}(\tau)_\mu = \sum_{\substack{v\in\mu+\mathcal{V}_{\mathbb{Z}}\\ v\in W_2}} \varphi_{\mathbb{V}}(y^{1/2}v)e^{\pi ixQ(v,v)} \tag{9}$$

and $\Theta''_{\mathbb{V}}(\tau)_\mu$ given by a similar sum over $v \notin W_2$. Note that (5) shows that

$$\varphi_{\mathbb{V}}(v) = e^{-\pi Q(v,v)}\mathrm{ch}_{2\mathrm{q}}(\wedge\mathcal{F}^2, \nabla_v) = e^{-\pi\|v\|^2}p_{\mathbb{V}}(v),$$

where $\|v\|^2$ denotes the Hodge metric and $p_{\mathbb{V}}(v)$ is a polynomial in v with coefficients in curvature and connection forms associated with the hermitian connection ∇. Since these forms grow at most polynomially in $-\log|t|$ [22], the bound (8) easily shows that $\Theta''_{\mathbb{V}}(\tau)_\mu \wedge \alpha$ is nearly bounded. Thus it suffices to prove that $\Theta'_{\mathbb{V}}(\tau)_\mu \wedge \alpha$ is nearly bounded.

4.3. Setting

$$\mathcal{I}^{p,q} = \mathcal{F}^p_{\mathrm{lim}} \cap \overline{\mathcal{F}^q_{\mathrm{lim}}} \cap V_{p+q,\mathcal{O}},$$

we may then write

$$\mathcal{F}^2_{\mathrm{lim}} = \begin{cases} \mathcal{I}^{2,0} \oplus \mathcal{I}^{2,1}, & \text{(type II)}\\ \mathcal{I}^{2,0} \oplus \mathcal{I}^{2,2}, & \text{(type III).}\end{cases}$$

Since N is an operator of type $(-1,-1)$, we have $N\mathcal{I}^{2,0} = 0$. It follows that

$$\mathcal{F}^2 = e^{zN}\mathcal{F}^2_{\mathrm{lim}} = \mathcal{I}^{2,0} \oplus \mathcal{L}, \tag{10}$$

where $\mathcal{L}$ is the holomorphic line bundle given by

$$\mathcal{L} = \begin{cases} e^{zN}\mathcal{I}^{2,1}, & \text{(type II)}\\ e^{zN}\mathcal{I}^{2,2}, & \text{(type III).}\end{cases}$$

Lemma 4.1. *The decomposition $\mathcal{F}^2 = \mathcal{I}^{2,0} \oplus \mathcal{L}$ is orthogonal for the Hodge metric.*

Proof. For a general degeneration of Hodge structure we have $Q(W_1, W_2) = 0$ and $Q(F^1_{\mathrm{lim}}, F^2_{\mathrm{lim}}) = 0$. Let us assume that the degeneration is of type II and pick $v_1 \in \mathcal{I}^{2,1}_w$ and $v_0 \in \mathcal{I}^{2,0}_w$. Then $e^{zN}v_1 = v_1 + zNv_1$ and hence

$$Q(\overline{v_0}, e^{zN}v_1) = Q(\overline{v_0}, v_1) + zQ(\overline{v_0}, Nv_1).$$

We have $Q(\overline{v_0}, v_1) = Q(v_0, \overline{v_1}) = 0$ since $Q(\mathcal{I}^{2,0}, \mathcal{I}^{1,2}) = 0$, and $Q(\overline{v_0}, Nv_1) = -Q(N\overline{v_0}, w_1) = 0$ since N is of type $(-1,-1)$. The case of type III degenerations is similar. □

Using this Lemma we will factor the form $\varphi_{\mathbb{V}}(v)$. For $k \geq 2$, define

$$V_{k,\mathrm{prim}} = V_k \cap \ker(N^{k-1})$$

and $V_{(l)} = \oplus_{k\geq 0}N^kV_{l+2,\mathrm{prim}}$. Explicitly we have

$$V_{2,\mathrm{prim}} = V_2 \cap \ker(N)$$
$$V_{3,\mathrm{prim}} = V_3$$
$$V_{4,\mathrm{prim}} = V_4$$

and

$$\begin{aligned}V_{(0)} &= V_{2,\mathrm{prim}}\\ V_{(1)} &= V_1 \oplus V_3\\ V_{(2)} &= V_4 \oplus NV_4 \oplus N^2V_4.\end{aligned}$$

Then N preserves each subspace $V_{(l)}$ and we have

$$V_{\mathbb{R}} = V_{(0)} \oplus V_{(1)} \oplus V_{(2)}$$

([31, Lemma 6.4]). Moreover, the form Q and the filtration $\mathcal{F}^{\bullet}_{\mathrm{lim}}$ define on $\mathrm{Gr}^W_{2,\mathrm{prim}}\mathcal{V}_{\mathbb{Q}} = (W_2 \cap \ker(N))/W_1$ a $\mathbb{Q}$-PVHS on Δ^{n-1} that we denote by $\mathrm{Gr}^W_{2,\mathrm{prim}}\mathbb{V}$. Note also that $V_{(2)} = 0$ (resp. $V_{(1)} = 0$) for type II (resp. III). Thus for $v \in W_2 \otimes \mathbb{R}$ we can write $v = v_2 + v'$ for unique $v_2 \in V_{2,\mathrm{prim}}$ and $v' \in V_1$ (type II) or $v' \in NV_4 \oplus N^2V_4$ (type III).

Lemma 4.2. *For* $v \in W_2 \otimes \mathbb{R}$ *we have*

$$\varphi_{\mathbb{V}}(v) = \varphi_{\mathrm{Gr}^W_{2,\mathrm{prim}}\mathbb{V}}(v_2) \wedge e^{-\pi Q(v',v')}\mathrm{ch}_2(\wedge\mathcal{L}, \nabla_{v'}).$$

Proof. Since the bundles $\mathcal{I}^{2,0}$ and $\mathcal{L}$ are holomorphic and the decomposition $\mathcal{F}^2 = \mathcal{I}^{2,0} \oplus \mathcal{L}$ is orthogonal for the Hodge metric, the Chern connection on $\mathcal{F}^2$ is the sum of the Chern connections on $\mathcal{I}^{2,0}$ and $\mathcal{L}$. Next note that the sections $s_{v'}$ and s_{v_2} of $(\mathcal{F}^2)^\vee$ restrict to zero on $\overline{\mathcal{I}^{2,0}}$ and $\mathcal{L}$ respectively: this follows easily from $Q(\mathcal{F}^1_{\mathrm{lim}}, \mathcal{F}^2_{\mathrm{lim}}) = Q(\overline{\mathcal{F}^1_{\mathrm{lim}}}, \overline{\mathcal{F}^2_{\mathrm{lim}}}) = Q(W_1, W_2) = Q(W_0, W_3) = 0$. Thus the superconnection ∇_v is the sum of ∇_{v_2} and $\nabla_{v'}$. Then (4) implies

$$\mathrm{ch}(\wedge\mathcal{F}^2, \nabla_v) = \mathrm{ch}(\wedge\mathcal{I}^{2,0}, \nabla_{v_2}) \wedge \mathrm{ch}(\wedge\mathcal{L}, \nabla_{v'})$$

and the statement follows. □

Since the forms $\varphi_{\mathrm{Gr}^W_{2,\mathrm{prim}}\mathbb{V}}(v_2)$ extend to Δ^n, the lemma implies that to extend $\Theta'_{\mathbb{V}}(\tau)_\mu$ as a a current it suffices to prove the following.

Lemma 4.3. *The form*

$$\sum_{v \in W_1 \cap \mathcal{V}_{\mathbb{Z}}} \mathrm{ch}_2(\wedge\mathcal{L}, \nabla_v) \in \Omega^{1,1}(\Delta^* \times \Delta^{n-1})$$

defines a current on Δ^n.

Proof. The proof proceeds by computing the Chern forms $\mathrm{ch}(\wedge\mathcal{L}, \nabla_v)$ explicitly. Define $l = 1$ in case II and $l = 2$ in case III. Note that the line bundle $\mathcal{I}^{2,l}$ on Δ^{n-1} has a natural metric given by the form $i^l Q(\cdot, N^l\overline{(\cdot)})$; let us denote the metric on $\mathcal{I}^{2,l}_w$ by $h^{2,l}_w$, so that $h^{2,l}_w(v) = i^l Q(v, N^l\overline{v})$. In case II, for $v \in \mathcal{I}^{2,1}_w$ we have

$$-Q(e^{zN}v, \overline{e^{zN}v}) = -Q(v+zNv, \overline{v}+\overline{z}N\overline{v}) = (z-\overline{z})Q(v, N\overline{v}) = 2\mathrm{Im}(z)^l h^{2,l}_w(v).$$

A similar computation shows that the identity

$$-Q(e^{zN}v, \overline{e^{zN}v}) = 2\mathrm{Im}(z)^l h^{2,l}_w(v)$$

also holds in case III. Thus under the identification of $\mathcal{L}_{z,w}$ with $\mathcal{I}_w^{2,l}$ provided by $v \mapsto e^{zN}v$, the metric becomes rescaled by $2\mathrm{Im}(z)^l$ and so the curvature forms $\Omega_{\mathcal{L}}$ and $\Omega_{\mathcal{I}^{2,l}}$ are related by

$$\Omega_{\mathcal{L}} = \Omega_{\mathcal{I}^{2,l}} + l\partial\overline{\partial}\mathrm{Im}(z) = \Omega_{\mathcal{I}^2} + l\frac{i}{4\pi}\frac{dzd\overline{z}}{\mathrm{Im}(z)^2}.$$

Using the explicit formula (6) we may write

$$\mathrm{ch}_2(\wedge\mathcal{L}.\nabla_v) = e^{-2\pi h_w^{2,l}(v)/\mathrm{Im}(z)^l}(\alpha_0(v) + \alpha_1(v)\frac{dtd\overline{t}}{|t|^2(\log|t|)^2}),$$

where $\alpha_0(v)$ contains no $dtd\overline{t}$ term. Given a compactly supported form $\alpha \in \Omega_c^{n-1,n-1}(\Delta^n)$, we have $|\alpha_0(v) \wedge \alpha|_{t,w} \leq p(v)|t| \cdot (-\log|t|)$ for some polynomial p on $W_{1,\mathbb{R}}$. An application of Poisson summation then shows that

$$\left|\left(\sum_{v\in W_1\cap\mathcal{V}_{\mathbb{Z}}} e^{-2\pi h_w^{2,l}(v)/\mathrm{Im}(z)^l}\alpha_0(v)\right)\wedge\alpha\right|_{t,w} = O(t(\log|t|)^2).$$

The contribution from terms coming from $\alpha_1(v)$ was analysed in [18]. Writing $\|v\|_{t,w}^2 = 2h_w^{2,l}(v)/\mathrm{Im}(z)^l$ for the Hodge metric of $v \in W_1$, we have

$$e^{-2\pi h_w^{2,l}(v)/\mathrm{Im}(z)^l}\alpha_1(v) = \begin{cases} e^{-\pi\|v\|_{t,w}^2}(\pi\|v\|_{t,w}^2 - 1), & \text{type II,} \\ e^{-\pi\|v\|_{t,w}^2}(2\pi\|v\|_{t,w}^2 - 1), & \text{type III} \end{cases}$$

(cf. [18, (4.35), (4.46)]. The argument in op. cit. then shows that the contribution of α_1 to the theta series in the statement is rapidly decreasing. This finishes the proof. ☐

References

1. Sergei Alexandrov, Soheyla Feyzbakhsh, Albrecht Klemm, Boris Pioline, and Thorsten Schimannek, *Quantum geometry, stability and modularity*, Commun. Number Theory Phys. **18** (2024), no. 1, 49–151. MR 4761642
2. Sergei Alexandrov, Nava Gaddam, Jan Manschot, and Boris Pioline, *Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds*, Adv. Theor. Math. Phys. **27** (2023), no. 3, 683–744. MR 4834419
3. Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia, *Transgressions of the Euler class and Eisenstein cohomology of* $\mathrm{GL}_N(\mathbf{Z})$, Jpn. J. Math. **15** (2020), no. 2, 311–379. MR 4120422
4. Nicolas Bergeron, Pierre Charollois, and Luis E. García, *Cocycles de groupe pour* GL_n *et arrangements d'hyperplans*, CRM Monograph Series, vol. 39, American Mathematical Society, Providence, RI, [2023] ©2023. MR 4684242
5. J.-M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion*, Comm. Math. Phys. **115** (1988), no. 1, 49–78. MR 929146 (89g:58192a)
6. Jean-Michel Bismut, *Superconnection currents and complex immersions*, Invent. Math. **99** (1990), no. 1, 59–113. MR 1029391
7. Jean-Michel Bismut and Jeff Cheeger, *Transgressed Euler classes of* $\mathrm{SL}(2n,\mathbf{Z})$ *vector bundles, adiabatic limits of eta invariants and special values of L-functions*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 4, 335–391. MR 1186908

8. Richard E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562. MR 1625724
9. Romain Branchereau, *The Kudla-Millson form via the Mathai-Quillen formalism*, Canad. J. Math. **76** (2024), no. 5, 1638–1663. MR 4814534
10. Jan Hendrik Bruinier and Shaul Zemel, *Special cycles on toroidal compactifications of orthogonal Shimura varieties*, Math. Ann. **384** (2022), no. 1-2, 309–371. MR 4476226
11. Philip Engel, François Greer, and Salim Tayou, *Mixed mock modularity of special divisors*, 2023.
12. S. Feyzbakhsh and R. P. Thomas, *Curve counting and S-duality*, Épijournal Géom. Algébrique **7** (2023), Art. 15, 25. MR 4603486
13. Jens Funke, *Heegner divisors and nonholomorphic modular forms*, Compositio Math. **133** (2002), no. 3, 289–321. MR 1930980
14. Jens Funke and John Millson, *Spectacle cycles with coefficients and modular forms of half-integral weight*, Arithmetic geometry and automorphic forms, Adv. Lect. Math. (ALM), vol. 19, Int. Press, Somerville, MA, 2011, pp. 91–154. MR 2906907
15. ———, *Boundary behaviour of special cohomology classes arising from the Weil representation*, J. Inst. Math. Jussieu **12** (2013), no. 3, 571–634. MR 3062872
16. Luis E. Garcia, *Superconnections, theta series, and period domains*, Adv. Math. **329** (2018), 555–589. MR 3783423
17. Luis E. Garcia and Siddarth Sankaran, *Green forms and the arithmetic Siegel-Weil formula*, Invent. Math. **215** (2019), no. 3, 863–975. MR 3935034
18. Luis E. García, *Kudla-Millson forms and one-variable degenerations of Hodge structure*, Algebraic Geometry (to appear).
19. Mark Green, Phillip Griffiths, and Colleen Robles, *Extremal degenerations of polarized Hodge structures*, Hodge theory and L^2-analysis, Adv. Lect. Math. (ALM), vol. 39, Int. Press, Somerville, MA, 2017, pp. 321–376. MR 3751295
20. Richard Hain, *Periods of limit mixed Hodge structures*, Current developments in mathematics, 2002, Int. Press, Somerville, MA, 2003, pp. 113–133. MR 2059020
21. F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113. MR 453649
22. János Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 361–398. MR 946244
23. Stephen Kudla, *A note about special cycles on moduli spaces of K3 surfaces*, Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, Fields Inst. Commun., vol. 67, Springer, New York, 2013, pp. 411–427. MR 3156424
24. Stephen S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. (2) **146** (1997), no. 3, 545–646. MR 1491448
25. ———, *Integrals of Borcherds forms*, Compositio Math. **137** (2003), no. 3, 293–349. MR 1988501 (2005c:11052)
26. Stephen S. Kudla and John J. Millson, *The theta correspondence and harmonic forms. I*, Math. Ann. **274** (1986), no. 3, 353–378. MR 842618
27. ———, *The theta correspondence and harmonic forms. II*, Math. Ann. **277** (1987), no. 2, 267–314. MR 886423
28. ———, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. Publ. Math. (1990), no. 71, 121–172. MR 1079646
29. Varghese Mathai and Daniel Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), no. 1, 85–110. MR 836726
30. Daniel Quillen, *Superconnections and the Chern character*, Topology **24** (1985), no. 1, 89–95. MR 790678
31. Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR 382272

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