G2, AC3S and BPS

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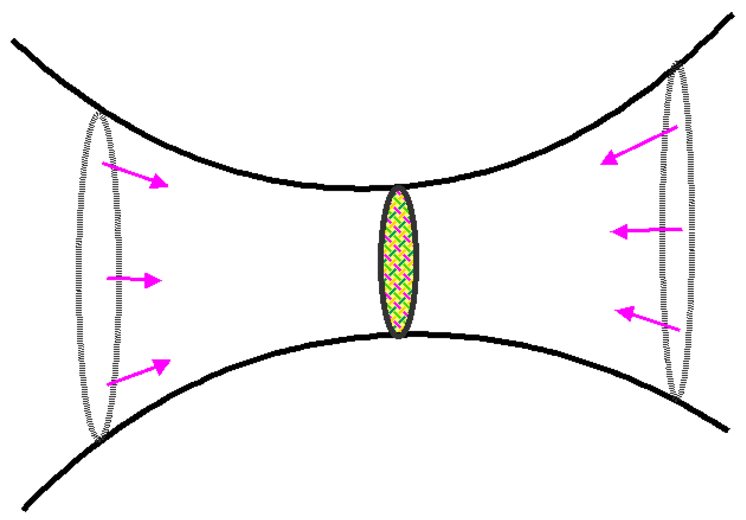
Physics Motivation

- When we compactify a supersymmetric theory, the internal manifold must have covariantly constant spinor in order to preserve SUSY. In seven-dimensions, the class of manifolds with this property are G_2 structure manifolds.
- Extended objects probe the internal geometry; SUSY tells us branes wrap calibrated cycles
- In the G_2 world, calibrated cycles can be 3d, associative cycles, or 4d, coassociative cycles.
- Their moduli is difficult to understand (esp. associatives), even locally, but physics demands global control.
- Perhaps these difficulties can be approached using other structures "almost contact three-structures".



Relation of ACM3S to calibrated cycles

- Recall: Subbundles of the tangent bundle are integrable if it is everywhere tangent to a submanifold.
- By Frobenius theorem, this is equivalent to involutivity of vector fields valued in the sub-bundle.
- If this happens everywhere, the space is foliated: Very strong property.
- A subbundle may not be integrable everywhere, but there may exist submanifold
- The vector fields of an ACM3S need not be closed subalgebra, but if it occurs the underlying submanifold is calibrated by the G_2-structure, a calibrated cycle
- Similarly, the orthogonal rank-4 bundle can only integrate to a coassociative cycle.
- This is the observation that relates these two concepts: AC3S express calibrated-ness in a different way!

The space of ACM3S

- At each tangent fibre, an ACM3S is (roughly) described by a triple of unit vectors. The space of appropriate triples is a homogeneous space $G_2/SU(2)$; so
- The space of all ACM3S is a space of sections of an associated fibre bundle, $\mathscr{C} = \Gamma_Y(TY \times_{G_2} G_2/SU(2))$.
- This space is itself a fibre bundle,

$$\begin{array}{ccc} \mathrm{Maps}(Y, SO(3)) & \longrightarrow & \mathscr{C} \\ & & \downarrow \\ & & \mathscr{S} \end{array}$$

- $\mathscr{S}$ is the space of trivial, rank-3 bundles: a space of sections of an associated bundle with $G_2/SO(4)$-fibres;
- The fibre is the space of trivialisations of a trivial rank 3-bundle, and hence a torsor for the mapping space $\mathrm{Maps}(Y, SO(3))$.

Conclusions and outlook

- The space of ACM3S is interesting and easy to describe.
- Linked with calibrated cycles and contains global information.
- How deep is the connection? Physics relevance?
- A natural energy functional with critical locus at ACM3S at volume minimising configurations?

Definitions

G_2 structure manifolds

- A G_2-structure manifold is a 7d manifold with tangent bundle's structure group reduced to $G_2 \subset SO(7)$.
- Such reduction exists whenever the manifold is orientable and spin-able and the moduli space is rich and interesting.
- Useful characterisation: fix a volume form, then a G_2 structure is equivalent to a choice of "positive" three-form $\varphi \in \Omega^3(Y)$.
- The three-form φ also encodes the torsion via its Levi-Civita covariant derivative; if this vanishes we have a G_2 manifold.
- There always exists a metric covariant derivative for which φ is covariantly constant.
- The three form + metric encode a vector cross-product $\times_\varphi$.

Almost contact metric three-structures

- An almost contact metric three-structure (ACM3S) on a G_2 structure manifold is encoded in a triplet of orthonormal vector fields, (R^1, R^2, R^3) such that $R^3 = R^1 \times_\varphi R^2$, [1];[3]
- Induces orthogonal decomposition of tangent bundle, $TY = \mathrm{span}\,(R^1, R^2, R^3) \oplus \mathrm{span}\,(R^1, R^2, R^3)^\perp$ and a trivialisation of the first summand.
- Results in a further reduction of tangent bundle structure group from G_2 to $SU(2)$.
- Covariant derivative preserving G_2 structure, need not preserve $SU(2)$.
- ACM3S always exist on G_2 structure manifolds, [2].

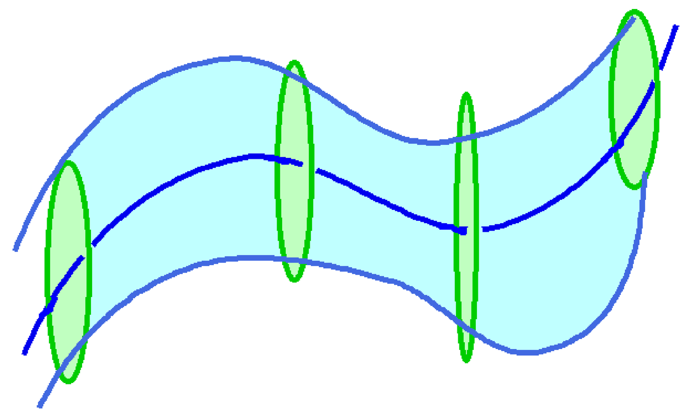
A non-compact example

Example modelled on a tubular neighbourhood of an associative submanifold of a G_2 manifold.

Interesting features:

- Simple to analyse topology of space of ACM3S;
- Test possibility of dependence on global structure, via boundary conditions.

The model: Let $X = S^3$, embedded as an associative cycle of a G_2 manifold. Let NX the normal bundle of X, viewed as embedded tubular neighbourhood.



Global structure dependence

1. There exists AC3S which is tangent to X; but
2. There are disconnected families of ACM3S $\implies$ not all can be deformed into TX;
3. Depends on the boundary of NX.

This is a manifestation of the global dependence in Y. In other words, if we restrict any global three-structure on Y to this neighbourhood of X, we have fixed boundary conditions. Have to check if we can locally deform the AC3S to be tangent to X.

Nontriviality of fibration

- In particular, the manifold is trivialisable, so the space of ACM3S is $\mathscr{C} \cong \mathrm{Maps}(S^3 \times \mathbb{R}^4, G_2/SU(2))$. Similar for base: $\mathscr{S} \cong \mathrm{Maps}(S^3 \times \mathbb{R}^4, G_2/SO(4))$.
- If the fibration is trivial, then
$$\pi_0(\mathscr{C}) \cong \pi_0(\mathrm{Maps}(NX, SO(3))) \times \pi_0(\mathscr{S}).$$
- But $\pi_0(\mathrm{Maps}(NX, SO(3)))) \cong \pi_3(SO(3)) = \mathbb{Z}$. Sequence chasing shows $|\pi_0(\mathscr{C})| < \infty$, so non-trivial fibration.

Bibliography

[1] Ying-yan Kuo. On almost contact 3-structure. *Tohoku Math. J. (2)*, 22(3):325–332, 1970.

[2] Emery Thomas. Vector fields on manifolds. *Bull. Amer. Math. Soc.*, 75(4):643–683, 07 1969.

[3] Albert J. Todd. An Almost Contact Structure on G_2-Manifolds. 2015.

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